Solution to Assignment #4

Ex 7.3.1 (a) $\quad u(x,y,t) = h(t)\,\phi(x,y)$

$$\phi(x,y) = f(x)\,g(y)$$

$$\Rightarrow \quad \frac{dh}{dt} = -\lambda k h,$$

$$\begin{cases} \dfrac{d^2f}{dx^2} = -\mu f \\ f(0)=0 \\ f(L)=0 \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{d^2g}{dy^2} = -(\lambda-\mu) g \\ g(0)=0 \\ g(H)=0 \end{cases}$$

$$\mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n=1,2,3,\dots$$

$$f_n(x) = \sin\frac{n\pi x}{L}$$

$$\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad m=1,2,3,\cdots$$

$$g_{nm}(y) = \sin\frac{m\pi y}{H}$$

Thus, $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$ and $\phi_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$.

$$\frac{dh}{dt} = -\lambda_{nm} k h \Rightarrow h_{nm}(t) = e^{-\lambda_{nm} k t}$$

Hence, $u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} e^{-\lambda_{nm} k t}$

I.C.: $u(x,y,0) = f(x,y) \Rightarrow f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$

By applying formulas for Fourier sine coefficients twice, we obtain

$$A_{nm} = \frac{4}{LH}\int_0^L\int_0^H f(x,y)\sin\frac{m\pi y}{H}\sin\frac{n\pi x}{L}\,dy\,dx.$$

Ex 7.3.4 (a).

$$u(x,y,t) = h(t)\phi(x,y)$$

$$\phi(x,y) = f(x)g(y)$$

$$\Rightarrow \quad \frac{d^2h}{dt^2} = -\lambda c^2 h,$$

$$\begin{cases} \frac{d^2f}{dx^2} = -\mu f \\ f(0) = 0 \\ f(L) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \frac{d^2g}{dy^2} = -(\lambda-\mu)g \\ \frac{\partial g}{\partial y}(0) = 0 \\ \frac{\partial g}{\partial y}(H) = 0 \end{cases}$$

$$\mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots\ldots$$

$$f_n(x) = \sin\frac{n\pi x}{L}$$

$$\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad m = 0, 1, 2, 3, \cdots$$

$$g_{nm}(y) = \cos\frac{m\pi y}{H}$$

Hence, $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$ and $\phi_{nm}(x,y) = \sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}$

$$\frac{d^2h}{dt^2} = -\lambda_{nm} c^2 h \Rightarrow h_{nm}(t) = A_{nm}\cos c\sqrt{\lambda_{nm}}\,t + B_{nm}\sin c\sqrt{\lambda_{nm}}\,t$$

Hence,

$$u = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{nm}\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}\cos c\sqrt{\lambda_{nm}}\,t + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{nm}\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}\sin c\sqrt{\lambda_{nm}}\,t$$

I.C.'s: $u(x,y,0)=0$, $\frac{\partial u}{\partial t}(x,y,0) = f(x,y)$

$\Rightarrow$ $A_{nm} = 0$ for all m and n.

and $$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{nm} c\sqrt{\lambda_{nm}}\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}$$

Fourier cosine coefficients:

$$\sum_{n=1}^{\infty} B_{n0} c\sqrt{\lambda_{n0}}\sin\frac{n\pi x}{L} = \frac{1}{H}\int_0^H f(x,y)\,dy \qquad (m=0)$$

$$\sum_{n=1}^{\infty} B_{nm} c\sqrt{\lambda_{nm}}\sin\frac{n\pi x}{L} = \frac{2}{H}\int_0^H f(x,y)\cos\frac{m\pi y}{H}\,dy \qquad (m\geq 1)$$

Fourier sine coefficients:

$$B_{n0} c\sqrt{\lambda_{n0}} = \frac{2}{L}\int_0^L \left[\frac{1}{H}\int_0^H f(x,y)\,dy\right]\sin\frac{n\pi x}{L}\,dx \qquad (m=0)$$

$$B_{nm} c\sqrt{\lambda_{nm}} = \frac{2}{L}\int_0^L \left[\frac{2}{H}\int_0^H f(x,y)\cos\frac{m\pi y}{H}\,dy\right]\sin\frac{n\pi x}{L}\,dx \qquad (m\geq 1)$$

Hence,

$$B_{n0} = \frac{2}{LHc\sqrt{\lambda_{n0}}}\int_0^L\int_0^H f(x,y)\sin\frac{n\pi x}{L}\,dy\,dx \qquad (m=0)$$

$$B_{nm} = \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H f(x,y)\cos\frac{m\pi y}{H}\sin\frac{n\pi x}{L}\,dy\,dx \qquad (m\geq 1).$$

Ex. ($A_{mn}, B_{mn}, C_{mn}, D_{mn}$ in Section 7.7)

For $m=0$

$$A_{0n} = \frac{\int_0^a \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \alpha(r,\theta)\,d\theta\right] J_0\left(z_{0n}\frac{r}{a}\right) r\,dr}{\int_0^a J_0^2\left(z_{0n}\frac{r}{a}\right) r\,dr}$$

For $m \geqslant 1$,

$$A_{mn} = \frac{\int_0^a \left[\frac{1}{\pi}\int_{-\pi}^{\pi} \alpha(r,\theta)\cos m\theta\,d\theta\right] J_m\left(z_{mn}\frac{r}{a}\right) r\,dr}{\int_0^a J_m^2\left(z_{mn}\frac{r}{a}\right) r\,dr}$$

$$B_{mn} = \frac{\int_0^a \left[\frac{1}{\pi}\int_{-\pi}^{\pi} \alpha(r,\theta)\sin m\theta\,d\theta\right] J_m\left(z_{mn}\frac{r}{a}\right) r\,dr}{\int_0^a J_m^2\left(z_{mn}\frac{r}{a}\right) r\,dr}$$

For $m=0$,

$$C_{0n} = \frac{a}{c z_{0n}} \cdot \frac{\int_0^a \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \beta(r,\theta)\,d\theta\right] J_0\left(z_{0n}\frac{r}{a}\right) r\,dr}{\int_0^a J_0^2\left(z_{0n}\frac{r}{a}\right) r\,dr}$$

For $m \geqslant 1$,

$$C_{mn} = \frac{a}{c z_{mn}} \cdot \frac{\int_0^a \left[\frac{1}{\pi}\int_{-\pi}^{\pi} \beta(r,\theta)\cos m\theta\,d\theta\right] J_m\left(z_{mn}\frac{r}{a}\right) r\,dr}{\int_0^a J_m^2\left(z_{mn}\frac{r}{a}\right) r\,dr}$$

$$D_{mn} = \frac{a}{c z_{mn}} \cdot \frac{\int_0^a \left[\frac{1}{\pi}\int_{-\pi}^{\pi} \beta(r,\theta)\sin m\theta\,d\theta\right] J_m\left(z_{mn}\frac{r}{a}\right) r\,dr}{\int_0^a J_m^2\left(z_{mn}\frac{r}{a}\right) r\,dr}$$

Ex (E_{mn} , F_{mn} in Section 7.9)

$$r(\theta, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} E_{mn} I_m\left(\frac{n\pi}{H}a\right)\sin\frac{n\pi z}{H}\cos m\theta$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} I_m\left(\frac{n\pi}{H}a\right)\sin\frac{n\pi z}{H}\sin m\theta$$

Fourier coefficients:

$$\sum_{n=1}^{\infty} E_{0n} I_0\left(\frac{n\pi}{H}a\right)\sin\frac{n\pi z}{H} = \frac{1}{2\pi}\int_{-\pi}^{\pi} r(\theta, z)\,d\theta \qquad (m=0)$$
$$\sum_{n=1}^{\infty} E_{mn} I_m\left(\frac{n\pi}{H}a\right)\sin\frac{n\pi z}{H} = \frac{1}{\pi}\int_{-\pi}^{\pi} r(\theta, z)\cos m\theta\,d\theta \qquad (m\geq 1)$$
$$\sum_{n=1}^{\infty} F_{mn} I_m\left(\frac{n\pi}{H}a\right)\sin\frac{n\pi z}{H} = \frac{1}{\pi}\int_{-\pi}^{\pi} r(\theta, z)\sin m\theta\,d\theta \qquad (m\geq 1)$$

Fourier sine coefficients:

$$E_{0n} I_0\left(\frac{n\pi}{H}a\right) = \frac{2}{H}\int_0^H\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} r(\theta, z)\,d\theta\right]\sin\frac{n\pi z}{H}\,dz \qquad (m=0)$$
$$E_{mn} I_m\left(\frac{n\pi}{H}a\right) = \frac{2}{H}\int_0^H\left[\frac{1}{\pi}\int_{-\pi}^{\pi} r(\theta, z)\cos m\theta\,d\theta\right]\sin\frac{n\pi z}{H}\,dz \qquad (m\geq 1)$$
$$F_{mn} I_m\left(\frac{n\pi}{H}a\right) = \frac{2}{H}\int_0^H\left[\frac{1}{\pi}\int_{-\pi}^{\pi} r(\theta, z)\sin m\theta\,d\theta\right]\sin\frac{n\pi z}{H}\,dz \qquad (m\geq 1)$$

Hence, For $m=0$,
$$E_{0n} = \frac{1}{\pi H I_0\left(\frac{n\pi}{H}a\right)}\int_0^H\int_{-\pi}^{\pi} r(\theta, z)\sin\frac{n\pi z}{H}\,d\theta\,dz$$

For $m\geq 1$,
$$E_{mn} = \frac{2}{\pi H I_m\left(\frac{n\pi}{H}a\right)}\int_0^H\int_{-\pi}^{\pi} r(\theta, z)\cos m\theta\sin\frac{n\pi z}{H}\,d\theta\,dz$$
$$F_{mn} = \frac{2}{\pi H I_m\left(\frac{n\pi}{H}a\right)}\int_0^H\int_{-\pi}^{\pi} r(\theta, z)\sin m\theta\sin\frac{n\pi z}{H}\,d\theta\,dz$$

Ex 8.2.1.

(a)

$$\begin{cases} \dfrac{d^2u_E}{dx^2} = 0 \\ u_E(0) = A \\ \dfrac{du_E}{dx}(L) = B \end{cases} \Rightarrow \quad u_E(x) = A + Bx$$

Let $v(x,t) = u(x,t) - u_E(x)$, then

$$\begin{cases} \dfrac{\partial v}{\partial t} = k\dfrac{\partial^2 v}{\partial x^2} \\ v(0) = 0 \\ \dfrac{\partial v}{\partial x}(L) = 0 \end{cases}$$

Separation of variables $\Rightarrow$ $v = \sum_{n=1}^{\infty} a_n \sin(n-\frac{1}{2})\frac{\pi x}{L} e^{-[(n-\frac{1}{2})\frac{\pi}{L}]^2 kt}$,

where $a_n = \frac{2}{L}\int_0^L g(x)\sin(n-\frac{1}{2})\frac{\pi x}{L}dx$.

Hence

$$u(x,t) = v(x,t) + u_E(x)$$
$$= A + Bx + \sum_{n=1}^{\infty} a_n \sin(n-\tfrac{1}{2})\frac{\pi x}{L} e^{-[(n-\frac{1}{2})\frac{\pi}{L}]^2 kt}.$$

(b)

$$\left.\begin{cases} \dfrac{d^2u_E}{dx^2} = 0 \\ \dfrac{du_E}{dx}(0) = 0 \end{cases}\right\} \Rightarrow \quad u_E(x) = \alpha + \beta x \text{ and } \beta = 0$$

$$\dfrac{du_E}{dx}(L) = B \neq 0 \Rightarrow \quad \beta = B \neq 0$$

Contradiction

Thus, no equilibrium exists.

Find a reference function $r(x,t)$ to satisfy

$$\begin{cases} \frac{\partial r}{\partial x}(0,t)=0 \\ \frac{\partial r}{\partial x}(L,t)=B\neq 0 \end{cases}$$

Choose $r(x,t)=\frac{Bx^2}{2L}$

Let $v(x,t)=u(x,t)-r(x,t)$, then

$$\begin{cases} \frac{\partial v}{\partial t}=k\frac{\partial^2 v}{\partial x^2}+\frac{kB}{L} \\ \frac{\partial v}{\partial x}(0,t)=0 \\ \frac{\partial v}{\partial x}(L,t)=0 \\ v(x,0)=f(x)-\frac{Bx^2}{2L} \end{cases}$$

has homogeneous boundary conditions.

(f)
$$\begin{cases} k\frac{d^2u_E}{dx^2}+\sin\frac{2\pi x}{L}=0 \\ \frac{du_E}{dx}(0)=0 \\ \frac{du_E}{dx}(L)=0 \end{cases}$$

$$\Rightarrow u_E(x)=u_E(0)+\frac{1}{k}\left(\frac{L}{2\pi}\right)^2\sin\frac{2\pi x}{L}-\frac{1}{k}\cdot\frac{L}{2\pi}x$$

Here, $u_E(0)$ can be any value, thus there are infinitely many equilibria with ~~the~~ same shape.

For example, choose $u_E(x)=\frac{1}{k}\left(\frac{L}{2\pi}\right)^2\sin\frac{2\pi x}{L}-\frac{1}{k}\cdot\frac{L}{2\pi}x$ (i.e. choose $u_E(0)=0$)

Let $v(x,t)=u(x,t)-u_E(x)$, then

$$\begin{cases} \dfrac{\partial v}{\partial t} = k\dfrac{\partial^2 v}{\partial x^2} \\ \dfrac{\partial v}{\partial x}(0,t) = 0 \\ \dfrac{\partial v}{\partial x}(L,t) = 0 \\ v(x,0) = f(x) - u_E(x) \end{cases}$$

Separation of variables can be applied. We obtain

$$v(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}, \quad \text{where}$$

$$A_0 = \frac{1}{L}\int_0^L [f(x) - u_E(x)]\,dx$$

$$(n \geq 1) \quad A_n = \frac{2}{L}\int_0^L [f(x) - u_E(x)] \cos\frac{n\pi x}{L}\,dx.$$

One solution to the original nonhomogeneous problem is

$$u(x,t) = v(x,t) + u_E(x).$$

(Note: We don't know whether there is only one solution. You can use the general expression with $u_E(0)$ for all equilibria to obtain all solutions. It's possible that different $u_E(0)$ give different solutions of the nonhomogeneous problem.)

Ex 8.2.2 (a).

Choose a reference function

$$r(x,t) = A(t)\,x + \frac{x^2}{2L}\left[B(t) - A(t)\right]$$

$\left(\text{integration of } A(t) + \frac{x}{L}\left[B(t) - A(t)\right]\right)$

Satisfying $\frac{\partial r}{\partial x}(0,t) = A(t)$ and $\frac{\partial r}{\partial x}(L,t) = B(t)$

Let $v(x,t) = u(x,t) - r(x,t)$, then

$$\begin{cases} \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \left[Q(x,t) - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2}\right] \\ \frac{\partial v}{\partial x}(0,t) = 0 \\ \frac{\partial v}{\partial x}(L,t) = 0 \\ v(x,0) = f(x) - \left[A(0)x + \frac{x^2}{2L}\left(B(0) - A(0)\right)\right] \end{cases}$$

has homogeneous boundary conditions.