Solutions for Math 300 2009 Midterm

PROBLEM 1: Use the *Method of Characteristics* to solve

$$u_t + \frac{2t}{1+t^2} u_x = -3t^2, \ -\infty < x < \infty, \ t > 0,$$

$$u(x,0) = \tanh(x), \ -\infty < x < \infty.$$

Solution. Observe that the initial data is specified along the x-axis. The characteristic equations are given by

$$\frac{dx}{dt} = \frac{2t}{1+t^2} \text{ subject to } x|_{t=0} = x_0 \in \mathbb{R},$$

which implies

$$\frac{du}{dt} = -3t^2 \text{ subject to } u|_{t=0} = \tanh(x_0).$$

The solution to the characteristic equations is given by

$$dx = \frac{2t}{1+t^2}\, dt \Longrightarrow \int_{x_0}^{x} d\xi = 2\int_0^t \frac{\xi}{1+\xi^2} d\xi$$

$$\Longrightarrow x - x_0 = \ln\left(1+t^2\right) \Longrightarrow x = x_0 + \ln\left(1+t^2\right)$$

$$\Longrightarrow x_0 = x - \ln\left(1+t^2\right),$$

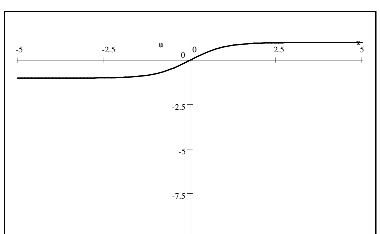
and

$$du = -3t^2\, dt \ \Longrightarrow \int_{\tanh(x_0)}^{u} d\xi = -3\int_0^t \xi^2\, d\xi$$

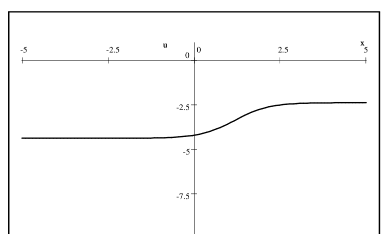
$$\Longrightarrow u = -t^3 + \tanh(x_0)$$

$$\Longrightarrow u(x,t) = -t^3 + \tanh\left[x - \ln\left(1+t^2\right)\right].$$

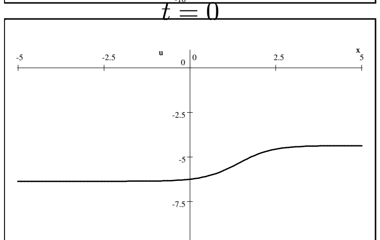
Below, $u(x,t)$ is graphed for $t = 0$, 1.5, 1.75 and 2, respectively.



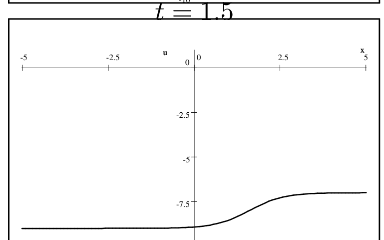
$t = 0$



$t = 1.5$



$t = 1.75$



$t = 2$

PROBLEM 2: Compute the *Fourier Series* (FS) for

$$f(x) = 1 - |x|, \text{ for } -1 \le x \le 1.$$

Solution. Observe that $f(x)$ is an *even* function for $x \in [-1, 1]$, i.e., $f(-x) = f(x)$. Thus, the FS contains only the "cos" terms. The FS for an arbitrary function $f(x)$ on the interval $[-L, L]$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L),$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx,$$

and for $n = 1, 2, 3, ...$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n\pi x/L)\, dx, \;\; b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(n\pi x/L)\, dx.$$

Here $L = 1$, so that the FS is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x),$$

where

$$b_n = \int_{-1}^{1} f(x) \sin(n\pi x)\, dx = 0,$$

since $f(x)$ is an *even* function, and

$$a_0 = \frac{1}{2} \int_{-1}^{1} f(x)\, dx = \int_{0}^{1} 1 - x\, dx = x - \frac{x^2}{2} \bigg|_0^1 = \frac{1}{2},$$

and for $n = 1, 2, 3, ...$

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx = 2 \int_{0}^{1} (1 - x) \cos(n\pi x)\, dx$$

$$= \frac{2}{n\pi} \int_0^1 (1 - x) \frac{d \sin(n\pi x)}{dx}\, dx = \frac{2}{n\pi} \left[(1 - x) \sin(n\pi x)|_0^1 + \int_0^1 \sin(n\pi x)\, dx \right]$$

$$= -\frac{2}{(n\pi)^2} \cos(nx)|_0^{\pi} = \frac{2\left[1 - (-1)^n\right]}{(n\pi)^2} = \frac{4}{(n\pi)^2} \text{ if } n \text{ is odd and } 0 \text{ otherwise.}$$

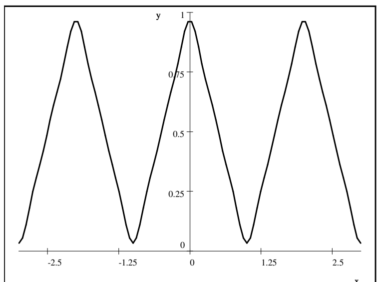
So the FS is

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos\left[(2n+1)\pi x\right]}{(2n+1)^2}.$$

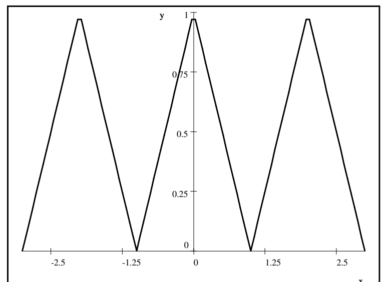
Let us define the *nth* partial sum as

$$F_N(x) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=0}^{N} \frac{\cos[(2n+1)\pi x]}{(2n+1)^2}.$$

Below, we show $F_2(x)$ and $F_{50}(x)$.



$F_2(x)$



$F_{50}(x)$

PROBLEM 3: Using *Separation of Variables*, solve the initial-boundary-value problem

$$u_{tt} - u_{xx} = 0,\ 0 < x < 1,\ t > 0,$$

$$u(0,t) = u_x(1,t) = 0,\ t > 0,$$

$$u(x,0) = \sin(\pi x/2) \text{ and } u_t(x,0) = \sin(3\pi x/2),\ 0 < x < 1.$$

Solution. Introduce the decomposition $u(x,t) = G(t)\Phi(x)$. Substitution into the PDE leads to

$$\frac{G''}{G} = \frac{\Phi''}{\Phi} = -\lambda^2,$$

so that

$$\Phi'' + \lambda^2 \Phi = 0,\ \Phi(0) = \Phi'(1) = 0.$$

The general solution is given by

$$\Phi(x) = D\sin(\lambda x) + C\cos(\lambda x),$$

where A and B are free amplitude constants. Application of the boundary conditions lead to

$$\Phi(0) = C = 0,$$

and

$$\Phi'(1) = 0 = D\lambda\cos(\lambda) \Longrightarrow \lambda = \lambda_n \equiv (2n+1)\pi/2 \text{ for } n = 0, 1, 2, ...$$

Hence

$$\Phi_n(x) = D_n \sin\left[\frac{(2n+1)\pi x}{2}\right], \quad \text{for } n = 0, 1, 2, ...,$$

where D_n is a free amplitude constant. The orthogonality condition associated with the eigenfunctions is given by

$$\int_0^1 \sin\left[\frac{(2n+1)\,\pi x}{2}\right]\sin\left[\frac{(2m+1)\,\pi x}{2}\right]\,dx = \frac{\delta_{nm}}{2}.$$

The associated time problem is

$$G'' + \left[\frac{(2n+1)\,\pi}{2}\right]^2 G = 0,$$

$$\Longrightarrow G_n(t) = A_n \sin\left[\frac{(2n+1)\,\pi t}{2}\right] + B_n \cos\left[\frac{(2n+1)\,\pi t}{2}\right],$$

where A_n and B_n are free amplitude constants. Thus, the general solution for $u(x,t)$ that satisfies the boundary conditions can be written in the form

$$u(x,t) = \sum_{n=0}^{\infty}\left\{A_n \sin\left[\frac{(2n+1)\,\pi t}{2}\right] + B_n \cos\left[\frac{(2n+1)\,\pi t}{2}\right]\right\}\sin\left[\frac{(2n+1)\,\pi x}{2}\right],$$

where the D_n coefficient has been "absorbed" into the A_n and B_n coefficients. Application of the initial condition $u(x,0) = \sin(\pi x/2)$ implies

$$\sin(\pi x/2) = \sum_{n=0}^{\infty} B_n \sin\left[\frac{(2n+1)\,\pi x}{2}\right],$$

which implies that

$$B_0 = 1 \text{ and } B_n = 0 \text{ for } n = 1, 2, 3, ...$$

Application of the initial condition $u_t(x,0) = \sin(3\pi x/2)$ implies

$$\sin(3\pi x/2) = \sum_{n=0}^{\infty} A_n \frac{(2n+1)\,\pi}{2} \sin\left[\frac{(2n+1)\,\pi x}{2}\right],$$

which implies that

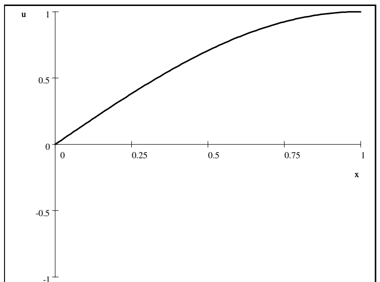
$$A_1 = \frac{2}{3\pi} \text{ and } A_n = 0 \text{ for } n = 0, 2, 3, ...$$

Hence the solution is given by simply

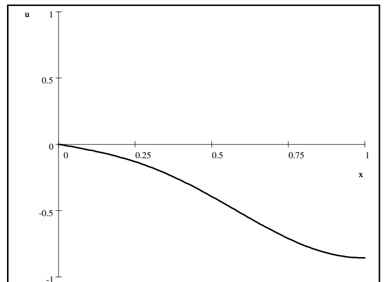
$$u(x,t) = \cos\left(\frac{\pi t}{2}\right)\sin\left(\frac{\pi x}{2}\right) + \frac{2}{3\pi}\sin\left(\frac{3\pi t}{2}\right)\sin\left(\frac{3\pi x}{2}\right)$$

$$= \frac{1}{2}\left\{\sin\left[\frac{\pi(x-t)}{2}\right] + \sin\left[\frac{\pi(x+t)}{2}\right]\right\} + \frac{1}{3\pi}\left\{\cos\left[\frac{3\pi(x-t)}{2}\right] - \cos\left[\frac{3\pi(x+t)}{2}\right]\right\}.$$

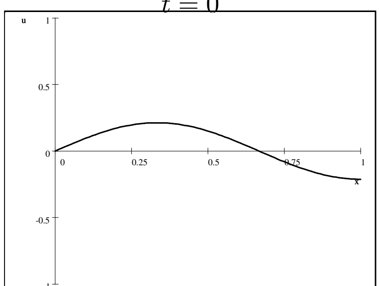
The solution is periodic in time with period 4 time units. The solution is shown below for $t = 0$, 1.5, 3 and 4, respectively.



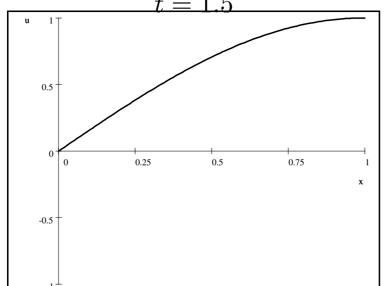
$t = 0$



$t = 1.5$



$t = 3$



$t = 4$