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# Stochastic generalized Kolmogorov systems with small diffusion: I. Explicit approximations for invariant probability density function

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**Abstract**

This paper presents Part I of a two-part series on studying the long-term coexistence states of stochastic generalized Kolmogorov systems with small diffusion. Part I establishes a mathematical framework for approximating the invariant probability measures (IPMs) and density functions (IPDFs) of these systems, while Part II will focus on analyzing their non-autonomous periodic counterparts. Compared with the existing approximation methods available only for systems with non-degenerate linear diffusion, this paper introduces two new and easily implementable approximation methods, the log-normal approximation (LNA) and updated normal approximation (uNA), which can be used for systems with not only non-degenerate but also degenerate diffusion. Moreover, we utilize the Kolmogorov-Fokker-Planck (KFP) operator and matrix algebra to develop algorithms for calculating the associated covariance matrix and verifying its positive definiteness. Our new approximation methods exhibit good accuracy in approximating the IPM and IPDF at both local and global levels, and significantly relaxes the minimal criteria for positive definiteness of the solution of the continuous-type Lyapunov equation. We demonstrate the utility of our methods in several application examples from biology and ecology.



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## 1. Introduction

Kolmogorov systems are a class of deterministic systems to characterize the dynamics of interacting populations and have been widely employed in biological and ecological modeling [1]. However, most real processes are subject to small noise perturbations from environmental factors or intrinsic uncertainties, and it is shown that such small perturbations could have a great impact on the dynamics of these processes [13–16,56]. Thus, investigating stochastic Kolmogorov systems with small perturbations is significant to capture the asymptotic behavior of the underlying process. A general form of an $n$-dimensional stochastic Kolmogorov system of Itô type can be expressed as follows:

$$dx_i(t) = x_i(t)b_i(x_1(t), ..., x_n(t))dt + x_i(t)\sigma_i(x_1(t), ..., x_n(t))dW_i(t), \quad i = 1, ..., n, \tag{1.1}$$

where $W_1(\cdot), ..., W_n(\cdot)$ is $n$ independent real-valued Brownian motions. The formulation in (1.1) covers some common dynamical systems in the literature, such as phytoplankton-zooplankton models, Lotka-Volterra population models, and vegetation models [23,60], etc.

A long-standing central issue pertaining to Kolmogorov system is: Under what conditions can interacting populations coexist stably [2]? For deterministic models, it can be well addressed through asymptotic stability of the positive equilibrium state. However, such positive equilibrium may not exist in stochastic settings [11]. Instead, analyzing *asymptotic stability in distribution* (ASD) [3] is relatively applicable in stochastic systems, including the existence and uniqueness of an *invariant probability measure* (IPM) [12]. Two common approaches to investigate ASD are: (i) Lyapunov functional method [4–7] and (ii) a combination of Lyapunov exponents and the tightness of random occupation measures [8–10]. In recent decades, variants of Kolmogorov systems with regard to ASD have been also studied, such as chemostat models [2,61], nutrient-plankton systems [74], and epidemic models [64–67].

In this paper, we consider a generalized stochastic Kolmogorov system with small noise perturbations as follows:

$$\begin{cases} dX_{\epsilon,i}(t) = f_i(\mathbf{X}_\epsilon(t))dt + \sqrt{\epsilon}X_{\epsilon,i}(t)\sum\limits_{j=1}^{N} g_{ij}(\mathbf{X}_\epsilon(t))dW_j(t), \quad i = 1, ..., n, \\ \mathbf{X}_\epsilon(0) = \mathbf{x}_0 \end{cases} \tag{1.2}$$

taking values in $(0, \infty)^n := \mathbb{R}^n_+$, where $\mathbf{X}_\epsilon(t) = (X_{\epsilon,1}(t), ..., X_{\epsilon,n}(t))^\top_{t\geq 0}$ (the superscript "$\top$" stands for the transpose), $\epsilon > 0$ is a small parameter, and $(W_1(t), ..., W_N(t))^\top := \mathbf{W}(t)$ is an $N$-dimensional standard Brownian motions. $f = (f_i)$ is a vector field on $\mathbb{R}^n_+$, called the *drift field*; $G_c = (x_i g_{ij})$ is an $n \times N$ matrix-valued function on $\mathbb{R}^n_+$, called the *Kolmogorov noise matrix*.

In addition to the existence of IPMs, a complete characterization of the coexistence state in practice also necessitates its *invariant probability density function* (IPDF), which is the fundamental solution of the stationary Kolmogorov–Fokker–Planck (KFP) equation. In particular, the KFP equation of the IPDF $\Psi_\epsilon(\cdot)$ associated with system (1.2) is

$$\begin{cases} 0 = -\sum_{i=1}^{n} \partial_i \big( f_i(\mathbf{x}) \Psi_\epsilon(\mathbf{x}) \big) + \frac{\epsilon}{2} \sum_{i,j=1}^{n} \partial_{ij}^2 \big( g_{ij}^c(\mathbf{x}) \Psi_\epsilon(\mathbf{x}) \big) \\ := \mathscr{L}_\epsilon \Psi_\epsilon(\mathbf{x}), \quad \forall\, \mathbf{x} \in \mathbb{R}_+^n, \\ \Psi_\epsilon(\mathbf{x}) \geq 0, \quad \int_{\mathbb{R}_+^n} \Psi_\epsilon(\mathbf{x}) d\mathbf{x} = 1, \end{cases} \tag{1.3}$$

where $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j}$, $(g_{ij}^c)_{n\times n} = G_c G_c^\top$ is the diffusion matrix, and $\mathscr{L}_\epsilon$ is the KFP operator.

To understand quantitatively the impact of noises on coexistence behavior of stochastic models, the IPMs and KFP equations, especially with small diffusion, have attracted much attention in the past decades. The associated analysis is carried out through several aspects including existence [21], noise-vanishing behavior (e.g., [17,18]) and quantification of concentration (e.g., [19,20]), etc. However, there have been virtually no investigations concerning the explicit expression of IPDF in the literature yet. This is likely due to the lack of available techniques and systematic treatments for solving KFP equations. There are a few exceptions, such as the study of one-dimensional ecological models [22–24] and Gibbs density of gradient system [25,26], but these models are relatively simple. In fact, most stochastic systems (in particular, Kolmogorov type) modeled by the continuous-time processes are complex and highly nonlinear [27], making the solving of KFP equations challenging. Therefore, numerical schemes or approximation techniques are often used as viable alternatives.

A few numerical schemes for IPMs have been proposed such as the truncated Euler–Maruyama (EM) method [28,29] for diffusion systems, Wong–Zakai approximation [30] for dissipative periodic equations, and the newly modified EM scheme [31,32] for switching diffusion systems. Although these numerical schemes have shown their abilities to approximate the IPMs, one limitation is that the approximate expressions of their IPDFs cannot be obtained. Recently, resurgent effort has been devoted to solving KFP equations by Monte Carlo simulation and numerical PDE method [33–35,40,50]. However, such techniques are almost unavailable for (1.2) due to two reasons. First, (1.2) is defined on $\mathbb{R}_+^n$ (i.e., an unbounded domain) and does not satisfy the usual "dissipation" conditions [44]; thus, the boundary condition of a discretized KFP equation is not determined. Second, computing the eigenfunction of the KFP operator is quite challenging. Unlike the aforementioned numerical approaches, an interesting new routine is to consider special continuous probability distributions to approximate IPMs [15], and the following expressions of IPDFs can then be explicitly approximated. Our study in this paper adopts this idea.

Inspired by the analysis of gradient systems, a classical assumption is that IPM of (1.2) can be approximated by a Gibbs measure, i.e., the IPDF $\Psi_\epsilon(\mathbf{x})$ satisfies

$$\Psi_\epsilon(\mathbf{x}) \approx \frac{1}{K} e^{-\frac{Q(\mathbf{x})}{\epsilon}}, \quad \mathbf{x} \in \mathbb{R}_+^n,$$

where $Q(\mathbf{x})$ represents the quasi-potential function [26,36,37], and $K = \int_{\mathbb{R}_+^n} e^{-\frac{Q(\mathbf{x})}{\epsilon}} d\mathbf{x}$. However, this assumption is difficult to verify due to the high regularity requirements of $Q(\mathbf{x})$, even in some simple cases [38,39]. Li et al. [56] reinitiated the study by adding small perturbations to a biochemical system with a unique stable equilibrium $\mathbf{x}^*$. They suggested that under non-degenerate diffusion (i.e., the diffusion matrix $\aleph$ is positive definite), the IPM near $\mathbf{x}^*$ may be approximately normally distributed, and the covariance matrix $\Sigma$ satisfies the continuous-type Lyapunov equation $\Sigma \boldsymbol{\varpi}^\top + \boldsymbol{\varpi} \Sigma + \aleph = \mathbb{O}$ ($\Im_c(\Sigma, \boldsymbol{\varpi}, \aleph) = \mathbb{O}$ for short), where $\mathbb{O}$ denotes zero matrix. Zhou et al. [41] further developed their work and proposed a newly developed normal approximation method. Then approximate expressions of the local IPDFs for some low dimensions ($\leq 5$) stochastic epidemic models have been obtained [42,45–48]. These studies rely on the analysis of the "standard $L_0$-algebraic equation", see [41, Lemma 3] and [46, Lemma 3.1]. Then by superposition principle, the explicit form of $\Sigma$ and its positive definiteness can be determined.

It should be noted that the associated study from special epidemic models to that of system (1.2) requires a big leap. Specifically,

(a) By Routh–Hurwitz criterion, a well-known result for Lyaunov equation $\Im_c(\Sigma, \boldsymbol{\varpi}, \aleph) = \mathbb{O}$ is that $\Sigma$ is positive definite if all eigenvalues of $\boldsymbol{\varpi}$ are of negative real part and $\aleph$ is positive definite [49]. However, such property of $\Sigma$ is unknown under degenerate diffusion (i.e., $\aleph$ is positive semi-definite). Moreover, degenerate diffusion is ubiquitous in stochastic modeling such as distributed delays [5,51–53], dependent Brownian motions [57–59] and Ornstein–Uhlenbeck processes [54,55,62]. Thus in this case, when using the existing normal approximation method, $\Sigma$ should be required to be positive definite to obtain an explicit approximate form for the joint marginal density of some subpopulations. In fact, existing works on normal approximations are only established under non-degenerate diffusion.
(b) Although the newly developed normal approximation method has some potential to verify the positive definiteness of $\Sigma$ under degenerate diffusion, there are two main restrictions to its application. Firstly, the analysis of standard $L_0$-algebraic equation is limited to the positive definiteness of solutions, which is computed directly (e.g., [46, Appendix B]) and can become increasingly challenging as the equation's dimension increases. The complete characterization of the solution of $n$-dimensional standard $L_0$-algebraic equation remains a challenging issue. Secondly, this normal approximation method requires that at least one Lyapunov sub-equation can be transformed into standard $L_0$-algebraic equation. Otherwise, additional conditions are needed to verify the positive definiteness of $\Sigma$ (see [45,47,48]).
(c) The current normal approximation methods provide only local approximations for IPDF around a quasi-positive equilibrium, and the approximation accuracy is unknown. A natural extension is to investigate whether it has a good global fitting ability for IPDF. Furthermore, the corresponding error estimates and approximate level for KFP equation have not been explored yet.
(d) In some stochastic risk-adjusted volatility models, IPM exhibits long right tails, as shown in [28, Fig. 4]. According to the Freidlin-Wentzell theory [40], the tails of the probability distribution are non-negligible, which implies that normal approximation methods are not applicable. Therefore, new approximation methods and techniques are required to analyze skewed IPMs.

These challenges motivate our current work. Our aim is to provide unified approximation algorithms for the IPM of stochastic generalized Kolmogorov system (i.e., (1.2)) and to derive an approximate expression for the IPDF. To address the issue of degenerate diffusion, we need to focus on the fundamental properties of the general standard $L_0$-algebraic equation, including the structure of the eigenpolynomial and the positive definiteness of the solution. The relevant analysis is highly challenging as the eigenvalues have to be treated in a complex field. We try to solve it with the help of both the residue theorem and ideas from linear control theory. Although the generalized Kolmogorov-type equation (1.2) is more general and realistic, the study of large-scale approximations for IPDF is much more challenging. To obtain the positive definiteness of the covariance matrix in approximation algorithms under degenerate diffusion, we need to transform each sub-equation into an equivalent form with a higher-dimensional standard $L_0$-algebraic equation structure.

Our main contributions are as follows:

- We develop two easily implementable explicit approaches for approximating the IPM and IPDF of system (1.2) in both local and global horizons: (i) *log-normal approximation* (LNA) for right-skewed measure, and (ii) *updated normal approximation* (uNA) for roughly symmetrical measure. Combining Lyapunov functional method, convergence results in the mean sense are established, and the asymptotic behavior of (1.2) around the positive equilibrium of $f$ is studied.
- New theoretical algorithms for calculating the expression of the covariance matrix of LNA (or uNA) are derived. A novelty of these algorithms is that its positive definiteness can be verified simultaneously. Moreover, two modified approximation algorithms are provided under slightly complex diffusions. It should be mentioned that the matrix transformations we adopted are different from those of the classical command "lyap(·, ·)" in MATLAB software [43], and have some excellent features such as wider applicability and cheaper computational cost; see Remark 3.
- A complete characterization for general standard $L_0$-algebraic equation is presented. The maximal error bounds of the difference between the IPDF of (1.2) and the approximate form obtained from our algorithms in the sense of the KFP equation are calculated.
- As corollaries of our main theorems, the classical conditions for ensuring positive definiteness of the solution of general Lyapunov equation $\Im_c(\Sigma, \boldsymbol{\varpi}, \aleph) = \mathbb{O}$ by using Routh–Hurwitz criterion are substantially relaxed. Furthermore, we demonstrate the utility in several application examples arising in biology and ecology.

The rest of the paper is organized as follows. Section 2 is a preliminary section, where we present mathematical definitions, notations as well as important properties for standard $L_0$-algebraic equations. Section 3 gives a complete framework of LNA approach for the IPM and IPDF of (1.2), including basic formulation, explicit theoretical algorithms, and the approximate effect in local and large-scale ranges. The corresponding framework of uNA approach is developed in Section 4. Section 5 provides several applications of our main results. Finally, we present the proofs of some key auxiliary lemmas and propositions in the Appendix.

## 2. Preliminaries

Throughout this paper, let $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}\}$ be a complete filtered probability space with $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets) [63], and $\mathbb{E}$ denotes the expectation corresponding to $\mathbb{P}$; $\mathbf{W}(t)$ is adapted to $\{\mathscr{F}_t\}_{t\geq 0}$. A glossary of notations used in this paper is shown in Table 1.

Below we introduce the Routh–Hurwitz criterion (Lemma 2.1) and a simple conclusion for the Lyapunov equation (Lemma 2.2).

**Lemma 2.1.** *([49]) Let $\psi_A(\lambda) = \lambda^l + \sum_{i=1}^{l} a_i \lambda^{l-i}$ (with $\lambda$ defined in complex field), then $A \in \overline{\mathbf{RH}}(l)$ if and only if $|\mathscr{H}_{l,A}^{(k)}| > 0$ for any $k \in \mathbb{S}_l^0$, where $\mathscr{H}_{l,A}$ is the $l$-dimensional Hurwitz matrix defined by*

Table 1
A glossary of symbols and notations.

| | |
|---|---|
| $\mathcal{C}^2(\mathbb{R}^n_+; \overline{\mathbb{R}}_+)$[b] | set of all $\mathbb{R}^n_+$-valued nonnegative functions $V(x)$ which are continuously twice differentiable in $x$ |
| $\lvert \cdot \rvert$ | the Euclidean norm (or determinant of a matrix) |
| $\mathcal{L}$ | the classical Itô's operator |
| $\boldsymbol{\delta}^*(\cdot)$ | the Dirac measure |
| $\ln \mathbf{X}$ | $:= (\ln X_1, ..., \ln X_l)^\top$, where $\mathbf{X} = (X_1, ..., X_l)^\top$ |
| $\boldsymbol{a}^{\langle l \rangle}$ | subvector formed by the first $l$ part of vector $\boldsymbol{a}$ |
| $A^{-1}$ | inverse matrix of $A$ |
| $A^\top$ | transpose of $A$ |
| $\mathbf{I}_l$ | $l$-dimensional identity matrix |
| $A^{(k)}$ (or $\boldsymbol{a}^{(k)}$) | submatrix formed by the first $k \times k$ part of matrix $A$ (or the $k$th element of vector $\boldsymbol{a}$) |
| $A^{[k]}$ | submatirx of the first $k$ row of matrix $A$ |
| $\mathbb{O}_{l,q}$ (or $\mathbb{O}_l$)[a] | $l \times q$ (or $l \times l$)-dimensional zero matrix |
| $\mathbb{S}^k_l$[b] | index set $\{k+1, k+2, ..., l\}$ $(-1 \le k \le l)$ |
| $A \succ B$ | $A - B$ is a positive definite matrix |
| $A \succeq B$ | $A - B$ is positive definite or positive semidefinite matrix |
| $\overline{\mathbf{RH}}(l)$ | $:= \{A \in \mathbb{R}^{l \times l} \mid A$ has eigenvalues with all negative real components$\}$ |
| $\mathscr{S}(l)$ | set of all $l$-dimensional standard $\mathbb{k}_l$ *matrices* in Definition 2.1 |
| $\mathcal{U}(l)$ | set of all $l$-dimensional nonsingular upper triangular matrices |
| $\mathcal{U}_q(l)$ | set of all $l$-dimensional upper Hurwitz–Hessenberg matrices in Definition 2.2 |
| $\psi_A(\cdot)$ | the eigenpolynomial of matrix $A$ |
| $\mathscr{H}_{l,A}$ | $l$-dimensional Hurwitz matrix (see Lemma 2.1) of $\psi_A(\cdot)$ |
| $\Im_c(\Sigma, \boldsymbol{\varpi}, \aleph) = \mathbb{O}$ | the continuous-type Lyapunov equation $\Sigma \boldsymbol{\varpi}^\top + \boldsymbol{\varpi} \Sigma + \aleph = \mathbb{O}$, where $\Sigma$ is real symmetric |
| $\mathbf{e}_l$ | $:= (1, 0, 0, ..., 0)^\top \in \mathbb{R}^l$[b] |
| $\boldsymbol{\beta}_l$ | $:= (0, 0, ..., 0, 1) \in \mathbb{R}^{1 \times l}$ |
| $\amalg_{l,k}$ | $:= \mathrm{diag}\{\underbrace{0, 0, ..., 0}_{k-1 \text{ term}}, 1, 0, ..., 0\} \in \mathbb{R}^{l \times l}$ |

[a] If there is no ambiguity in theoretical derivation, $\mathbb{O}_{l,q}$ (or $\mathbb{O}_l$) can be simplified as $\mathbb{O}$.
[b] $\mathbb{R}^l_+ = (0, \infty)^l := \{(x_1, ..., x_l)^\top \in \mathbb{R}^l \mid x_i > 0,\ \forall\, i \in \mathbb{S}^0_l\}$; $\overline{\mathbb{R}}_+ = [0, \infty)$; $\mathbb{S}^l_l = \emptyset$; $\mathbb{R}^l := \mathbb{R}^{l \times 1}$.

$$\mathscr{H}_{l,A} = \begin{pmatrix} a_1 & a_3 & a_5 & \cdots & a_{2l-3} & a_{2l-1} \\ 1 & a_2 & a_4 & \cdots & a_{2l-4} & a_{2l-2} \\ 0 & a_1 & a_3 & \cdots & a_{2l-5} & a_{2l-3} \\ 0 & 1 & a_2 & \cdots & a_{2l-6} & a_{2l-4} \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{l-2} & a_l \end{pmatrix}$$

*with $a_j = 0$ if $j > l$.*

**Lemma 2.2.** *([49]) For any given matrix $A \in \mathbb{R}^{l \times l}$, if all the eigenvalues $s_i$ of $A$ satisfy $s_i + s_j \neq 0$ $(\forall\, i, j \in \mathbb{S}^0_l)$, then for any $D \in \mathbb{R}^{l \times l}$, there is a unique matrix $B$ satisfying the equation $\Im_c(B, A, D) = \mathbb{O}$.*

To obtain the desired results, two necessary mathematical definitions are shown.

**Definition 2.1.** $D$ is called an $l$-dimensional standard $\mathbb{k}_l$ matrix if $D \in \overline{\mathbf{RH}}(l)$ and it takes the form

$$D = \begin{pmatrix} -\mathbf{d}^{\langle l-1\rangle} & -d_l \\ \mathbf{I}_{l-1} & \mathbb{O} \end{pmatrix},$$

where $\mathbf{d} = (d_1, d_2, ..., d_l)$.

**Definition 2.2.** $C = (c_{ij})_{l\times l}$ is called an $l$-dimensional upper Hurwitz–Hessenberg matrix if $C \in \overline{\mathbf{RH}}(l)$ and it satisfies $c_{i+1,i} \neq 0$ and $c_{ji} = 0$ for any $i \in \mathbb{S}^0_{l-1}$; $j \in \mathbb{S}^{i+1}_l$.

**Remark 1.** By Lemma 2.1 and Definition 2.1, a consequence of $D \in \mathscr{S}(l)$ is that $\mathbf{d}^\top \in \mathbb{R}^l_+$. If $l = 1$, then $D \in \mathscr{S}(1)$ if and only if $D = (-d_1)$ with $d_1 > 0$. Thus, $\mathcal{U}_q(1) = \mathscr{S}(1)$.

In fact, standard $L_0$-algebraic equation is a special class of Lyapunov equation $\Im_c(\Sigma, \boldsymbol{\alpha}, \aleph) = \mathbb{O}$ with $\boldsymbol{\alpha} \in \mathscr{S}(k)$ and $\aleph = \amalg_{k,1}$ $(k = 1, 2, ...)$.

**Proposition 2.1.** *Consider the following $l$-dimensional standard $L_0$-algebraic equation*

$$\Im_c(\Xi_l, A, \amalg_{l,1}) = \mathbb{O}, \tag{2.1}$$

*where $A \in \mathscr{S}(l)$. Then*

(i) (Positive definiteness) $\Xi_l$ *is unique and* $\Xi_l \succ \mathbb{O}$.
(ii) (Expression) $\Xi_l$ *is of the following form:*

$$\Xi_l = \begin{pmatrix} \theta_1 & 0 & -\theta_2 & 0 & \theta_3 & \cdots \\ 0 & \theta_2 & 0 & -\theta_3 & \cdots & \iddots \\ -\theta_2 & 0 & \theta_3 & \cdots & \iddots & 0 \\ 0 & -\theta_3 & \cdots & \iddots & 0 & -\theta_{l-1} \\ \theta_3 & \cdots & \iddots & 0 & \theta_{l-1} & 0 \\ \vdots & \iddots & 0 & -\theta_{l-1} & 0 & \theta_l \end{pmatrix}, \tag{2.2}$$

*where $(\theta_1, -\theta_2, ..., (-1)^{l-1}\theta_l)^\top := \boldsymbol{\theta}$ is determined by equation $\mathscr{H}_{l,A}\boldsymbol{\theta} = \frac{1}{2}\mathbf{e}_l$, and $\boldsymbol{\theta} \in \mathbb{R}^l_+$. In particular, if all the roots $\lambda_j$ $(j \in \mathbb{S}^0_l)$ of equation $\psi_A(\lambda) = 0$ are simple, then*

$$\theta_k = (-1)^{l-k} \sum_{i=1}^{l} \frac{\lambda_i^{2(l-k)}}{\psi'_A(\lambda_i)\psi_A(-\lambda_i)}, \quad \forall\, k \in \mathbb{S}^0_l. \tag{2.3}$$

(iii) (Determinant)

$$|\Xi_l| = \begin{cases} \dfrac{\varphi_{ol}^2}{2a_{l-1}^2 a_l}, & \text{for odd } l, \\[2ex] \dfrac{\varphi_{el}^2}{4a_{l-1}^2 a_l}, & \text{for even } l, \end{cases}$$

*where* $a_l = \psi_A(0),\ a_{l-1} = \psi'_A(0)$, *and*

$$\varphi_{ol} = \begin{vmatrix} \theta_1 & \cdots & (-1)^{[\frac{l+1}{2}]^\star}\theta_{[\frac{l-1}{2}]^\star} \\ \vdots & \ddots & \vdots \\ (-1)^{[\frac{l+1}{2}]^\star}\theta_{[\frac{l-1}{2}]^\star} & \cdots & (-1)^{l-1}\theta_{l-2} \end{vmatrix},$$

$$\varphi_{el} = \begin{vmatrix} -\theta_2 & \cdots & (-1)^{[\frac{l}{2}]^\star+1}\theta_{[\frac{l}{2}]^\star} \\ \vdots & \ddots & \vdots \\ (-1)^{[\frac{l}{2}]^\star+1}\theta_{[\frac{l}{2}]^\star} & \cdots & (-1)^{l-1}\theta_{l-2} \end{vmatrix},$$

*with* $[\cdot]^\star$ *denoting the bracket function.*

**Proposition 2.2.** *For any matrix* $C \in \mathcal{U}_q(l)$, *let* $M^\top = ((\boldsymbol{\beta}_l C^{l-1})^\top, (\boldsymbol{\beta}_l C^{l-2})^\top, ..., \boldsymbol{\beta}_l^\top)$, *then* $M \in \mathcal{U}(l)$ *and* $MCM^{-1} \in \mathscr{S}(l)$.

The proofs of the above propositions are collected in Appendices A and B.
We impose the following assumptions for system (1.2).

**Assumption 2.1.** The following conditions hold:

(1) For any $\mathbf{x}_0 \in \mathbb{R}^n_+$ and $\epsilon > 0$, system (1.2) has a unique solution $\mathbf{X}_\epsilon(t)$ on $t \geq 0$ and the solution will remain in $\mathbb{R}^n_+$ with probability 1 (a.s.).
(2) There exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$, system (1.2) has a unique IPM $\mu_\epsilon$ on $\mathbb{R}^n_+$, with its IPDF $\Psi_\epsilon(\cdot)$ determined by (1.3).

**Assumption 2.2.** Some of the following conditions hold:

(a) The matrix $(g_{ij}(\mathbf{x}))_{n\times N}(g_{ij}(\mathbf{x}))^\top_{n\times N} \succeq \mathbb{O}$ for any $\mathbf{x} \in \mathbb{R}^n_+$. Furthermore, there is $\epsilon_1 > 0$ such that for any $\epsilon \in [0, \epsilon_1)$, equations $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ have a unique solution $\overline{\mathbf{X}}^*_\epsilon := (\overline{X}^*_{\epsilon,1}, ..., \overline{X}^*_{\epsilon,n})^\top \in \mathbb{R}^n_+$, and the matrix $(\frac{\partial F_i(\mathbf{x})}{\partial(\ln x_j)})_{n\times n}$ at $\mathbf{x} = \overline{\mathbf{X}}^*_\epsilon$ belongs to $\overline{\mathbf{RH}}(n)$, where $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), ..., F_n(\mathbf{x}))^\top$ with

$$F_i(\mathbf{x}) = \frac{f_i(\mathbf{x})}{x_i} - \frac{\epsilon}{2}\sum_{j=1}^N g_{ij}^2(\mathbf{x}).$$

(b) There is an $a > 0$ and a function $V(\cdot) \in \mathcal{C}^2(\mathbb{R}^n_+; \overline{\mathbb{R}}_+)$ satisfying

$$\mathcal{L}V(\mathbf{X}_\epsilon(t)) \leq -a\left|\mathbf{X}_\epsilon(t) - \mathbf{X}^*\right|^2 + \kappa(\epsilon), \tag{2.4}$$

where $\mathbf{X}^*$ is the unique root of $f(\mathbf{x}) = 0$ on $\mathbb{R}^n_+$, $\kappa(\epsilon) > 0$ is a continuous function satisfying $\lim\limits_{\epsilon\to 0}\kappa(\epsilon) = 0$.

(c) The diffusion matrix $G_c G_c^\top \succeq \mathbb{O}$ for any $\mathbf{x} \in \mathbb{R}_+^n$. In addition, the matrix $(\frac{\partial f_i(\mathbf{x})}{\partial(x_j)})_{n\times n}$ at $\mathbf{x} = \mathbf{X}^*$ belongs to $\overline{\mathbf{RH}}(n)$.

**Remark 2.** Assumption 2.2(b) ensures that the solution of (1.2) will oscillate around $\mathbf{X}^*$ under some small diffusions. In particular, if further the function $V$ is positive-definite decrescent radially unbounded, then $\mathbf{X}^*$ is stochastically asymptotically stable in the large; see [63]. We need Assumption 2.2(a) (resp., (c)) to guarantee that our LNA (resp., uNA) method can be carried out for approximating the IPM $\mu_\epsilon$ and its IPDF $\Psi_\epsilon(\cdot)$. Both Assumptions 2.2(a) and (c) imply that our approximation methods can work in non-degenerate diffusion.

## 3. Log-normal approximation (LNA) for IPDF

This section is devoted to providing a LNA method to approximate $\mu_\epsilon$ and $\Psi_\epsilon(\cdot)$.

### 3.1. Main transformations of (1.2)

We first consider the logarithmic transformation of (1.2), i.e.,

$$\begin{cases} d(\ln X_{\epsilon,i}(t)) = F_i(\mathbf{X}_\epsilon(t))dt + \sqrt{\epsilon}\sum\limits_{j=1}^{N} g_{ij}(\mathbf{X}_\epsilon(t))dW_j(t), \quad i \in \mathbb{S}_n^0, \\ \mathbf{X}_\epsilon(0) = \mathbf{x} \in \mathbb{R}_+^n. \end{cases} \tag{3.1}$$

Under Assumption 2.2(a), we can define a quasi-positive equilibrium $\overline{\mathbf{X}}_\epsilon^*$, which uniquely satisfies the equation $\mathbf{F}(\mathbf{x}) = \mathbf{0}$. By calculation,

$$\lim_{\epsilon\to 0^+} \overline{\mathbf{X}}_\epsilon^* = \mathbf{X}^*. \tag{3.2}$$

This implies that $\overline{\mathbf{X}}_\epsilon^*$ is biologically reasonable assumptions involved in the stochasticity. Then the linearized equation of the solution $\ln \mathbf{X}_\epsilon(t)$ of (3.1) around $\ln \overline{\mathbf{X}}_\epsilon^*$ is as follows:

$$\begin{aligned} d\mathbf{Y}_\epsilon(t) =& \Big(\frac{\partial F_i(\mathbf{x})}{\partial(\ln x_j)}\Big)_{n\times n}\Big|_{\mathbf{x}=\overline{\mathbf{X}}_\epsilon^*}\mathbf{Y}_\epsilon(t)dt + \sqrt{\epsilon}(g_{ij}(\overline{\mathbf{X}}_\epsilon^*))_{n\times N}d\mathbf{W}(t) \\ :=& B_\epsilon \mathbf{Y}_\epsilon(t)dt + \sqrt{\epsilon}\Theta_\epsilon d\mathbf{W}(t), \end{aligned} \tag{3.3}$$

where the initial value $\mathbf{Y}_\epsilon(0) = \ln \mathbf{x}_0 - \ln \overline{\mathbf{X}}_\epsilon^*$.

Let $\lambda_k^+$ ($k \in \mathbb{S}_\xi^0$) be all the nonzero eigenvalues of $\Theta_\epsilon\Theta_\epsilon^\top$. As in Assumption 2.2(a), we have $\lambda_k^+ > 0$, $\forall\, k \in \mathbb{S}_\xi^0$. Thus, there exists an orthogonal matrix $\mathcal{G}_\epsilon$ such that

$$\mathcal{G}_\epsilon(\Theta_\epsilon\Theta_\epsilon^\top)\mathcal{G}_\epsilon^\top = \sum_{k=1}^{\xi} \lambda_k^+ \amalg_{n,\phi_k}, \tag{3.4}$$

where $1 \le \phi_i < \phi_j \le n$, $\forall\, i < j$. Obviously, $\xi = \mathrm{rank}(\Theta_\epsilon\Theta_\epsilon^\top)$.

### 3.2. Local approximation-I

Let $\boldsymbol{\phi} = \{\phi_1, ..., \phi_\xi\}$ and $A_\epsilon = \mathcal{G}_\epsilon B_\epsilon \mathcal{G}_\epsilon^{-1}$.

**Theorem 3.1.** *Under Assumptions 2.1 and 2.2(a), the IPM $\mu_\epsilon$ around $\overline{\mathbf{X}}_\epsilon^*$ is approximated by a log-normal distribution $\mathbb{LN}_n(\ln \overline{\mathbf{X}}_\epsilon^*, \Sigma_\epsilon)$ (with $\Phi_\epsilon(\cdot)$ denoting its density), where*

$$\Sigma_\epsilon = \epsilon \mathcal{G}_\epsilon^\top \Big( \sum_{k=1}^{\xi} \lambda_k^+ \Sigma_{\phi_k,\epsilon} \Big) \mathcal{G}_\epsilon, \tag{3.5}$$

*with $\Sigma_{\phi_k,\epsilon}$ obtained by Algorithm 1. Moreover, for any constant vector $\mathbf{X} = (X_1, ..., X_n)^\top \in \mathbb{R}^n$, the following assertion holds:*

$$\mathbf{X}^\top \Sigma_\epsilon \mathbf{X} \ge \rho_\epsilon \sum_{k=1}^{\xi} \Big( Y_{\phi_k}^2 + \sum_{j=2}^{\eta_k} (\mathbf{H}_{\phi_k,j}^{(j)})^2 \Big), \tag{3.6}$$

*where $\mathbf{Y} = \mathcal{G}_\epsilon \mathbf{X} := (Y_1, ..., Y_n)^\top$, $\rho_\epsilon > 0$ is defined in (3.35), and $\mathbf{H}_{\phi_k,j} = \big[\prod_{i=0}^{j-1} (Q_{\phi_k,i}^{-1})^\top P_{\phi_k,i}\big] J_{\phi_k} \mathbf{Y}$, with $\eta_k$, $J_{\phi_k}$, $P_{\phi_k,i}$ and $Q_{\phi_k,i}$ shown in Algorithm 1.*

**Proof.** It is readily seen that system (3.3) has a unique explicit solution

$$\mathbf{Y}_\epsilon(t) = e^{B_\epsilon t}\mathbf{Y}_\epsilon(0) + \sqrt{\epsilon} \int_0^t e^{B_\epsilon(t-\tau)} \Theta_\epsilon d\mathbf{W}(\tau).$$

Note that $\Theta_\epsilon$ is a constant matrix, by a standard argument [63,68], $\sqrt{\epsilon}\int_0^t e^{B_\epsilon(t-\tau)}\Theta_\epsilon d\mathbf{W}(\tau)$ follows a Gaussian distribution $\mathbb{N}_n(\mathbf{0}, \Sigma(t))$ with

$$\Sigma(t) = \epsilon \int_0^t e^{B_\epsilon(t-\tau)} \Theta_\epsilon \Theta_\epsilon^\top e^{B_\epsilon^\top (t-\tau)} d\tau.$$

That is, system (3.3) admits a transient distribution $\mathbb{N}_n(e^{B_\epsilon t}\mathbf{Y}_\epsilon(0), \Sigma(t))$ at time $t$. Under Assumption 2.2(a), we have $B_\epsilon \in \overline{\mathbf{RH}}(n)$, and

$$\lim_{t\to\infty} e^{B_\epsilon t}\mathbf{Y}_\epsilon(0) = \mathbf{0}, \quad \lim_{t\to\infty} \Sigma(t) = \epsilon \int_0^\infty e^{B_\epsilon t} \Theta_\epsilon \Theta_\epsilon^\top e^{B_\epsilon^\top t} dt \triangleq \Sigma_\epsilon. \tag{3.7}$$

Hence, the process $\{\mathbf{Y}_\epsilon(t)\}_{t\ge 0}$ has a unique IPM $\mathbb{N}_n(\mathbf{0}, \Sigma_\epsilon)$.

Based on the relationship between (3.3) around $\ln \overline{\mathbf{X}}_\epsilon^*$ and (1.2), we determine that $\ln \frac{\mathbf{X}_\epsilon(t)}{\overline{\mathbf{X}}_\epsilon^*}$ $(:= \ln \mathbf{X}_\epsilon(t) - \ln \overline{\mathbf{X}}_\epsilon^*)$ around $\mathbf{X}_\epsilon(t) = \overline{\mathbf{X}}_\epsilon^*$ can be approximated by $\mathbf{Y}_\epsilon(t)$, i.e., the probability measure $\mu_\epsilon$ of (1.2) near $\overline{\mathbf{X}}_\epsilon^*$ is approximately a log-normal distribution $\mathbb{LN}_n(\ln \overline{\mathbf{X}}_\epsilon^*, \Sigma_\epsilon)$.

**Algorithm 1:** Algorithm for obtaining $\Sigma_{\phi_k,\epsilon}$.

**Input:** $A_\epsilon$, $\phi_k$ (or $\boldsymbol{\phi}$).
**Output:** $\eta_k$, $\Sigma_{\phi_k,\epsilon} = (\prod_{i=0}^{\eta_k-1}\overline{a}^{\lfloor\phi_k,i\rfloor}_{\nu_{k(i)},i})^2[M_{\phi_k,\eta_k}(\prod_{i=0}^{\eta_k-1}Q_{\phi_k,i}P_{\phi_k,i})J_{\phi_k}]^{-1}\Delta_{\phi_k,\eta_k}\{[M_{\phi_k,\eta_k}(\prod_{i=0}^{\eta_k-1}Q_{\phi_k,i}P_{\phi_k,i})J_{\phi_k}]^{-1}\}^{\top\mathrm{a}}$.

```
(Initialization): η_k = 1;
(Order transformation): Ā_{φ_k,1} = J_{φ_k} A_ε J_{φ_k}^{-1};
for i = 1 : n − 1 do
    if Σ_{j=i+1}^{n} (ā_{ji}^{⌊φ_k,i⌋})^2 = 0 then
        η_k = i;
        break;
    else
        Choose a "suitable^a "ν_{k(i)} ∈ S_n^i such that ā^{⌊φ_k,i⌋}_{ν_{k(i)},i} ≠ 0;
        (Rotation transformation): Â_{φ_k,i} = P_{φ_k,i} Ā_{φ_k,i} P_{φ_k,i}^{-1 c};
        (Elimination transformation): Ā_{φ_k,i+1} := Q_{φ_k,i} Â_{φ_k,i} Q_{φ_k,i}^{-1} tsupc;
    end
    η_k++;
end
(Standardized transformation): A_{s,φ_k} = M_{φ_k,η_k} Ā_{φ_k,η_k} M_{φ_k,η_k}^{-1};
Consider a standard L_0-algebraic equation ℑ_c(Ξ_{φ_k,η_k}, A_{s,φ_k}^{(η_k)}, Ш_{η_k,1}) = 𝕆^c;
return η_k, Ξ_{φ_k,η_k}, Σ_{φ_k,ε}.
```

[a] $\overline{a}^{\lfloor\cdot\rfloor}_{ji}$ (or $\overline{a}^{\lfloor\cdot\rfloor}_{j,i}$) denotes the $i$th element of the $j$th row of $\overline{A}_{(\cdot)}$. $J_{\phi_k}$ and $M_{\phi_k,\eta_k}$ are called the order and standardized $\phi_k$-$A_\epsilon$ matrices, respectively. $P_{\phi_k,i}$ (resp., $Q_{\phi_k,i}$) is called the $i$th rotation (resp., elimination) $\phi_k$-$A_\epsilon$ matrix. Specifically, $\overline{a}^{\lfloor\phi_k,0\rfloor}_{\nu_{k(0)},0}=1$ and $P_{\phi_k,l}=Q_{\phi_k,l}=\mathbf{I}_n$, where $l\in\{0,n-1\}$. Furthermore,

$$M_{\phi_k,\eta_k}=\begin{pmatrix}\mathcal{M}_{\eta_k} & \mathbb{O}\\ \mathbb{O} & \mathbf{I}_{n-\eta_k}\end{pmatrix},\quad \mathcal{M}_{\eta_k}=\begin{pmatrix}\boldsymbol{\beta}_{\eta_k}(\overline{A}^{(\eta_k)}_{\phi_k,\eta_k})^{\eta_k-1}\\ \boldsymbol{\beta}_{\eta_k}(\overline{A}^{(\eta_k)}_{\phi_k,\eta_k})^{\eta_k-2}\\ \cdots\\ \boldsymbol{\beta}_{\eta_k}\end{pmatrix},\quad J_{\phi_k}=\begin{pmatrix}\mathbb{O} & 1 & \mathbb{O}\\ \mathbf{I}_{\phi_k-1} & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & \mathbb{O} & \mathbf{I}_{n-\phi_k}\end{pmatrix},$$

$$\Delta_{\phi_k,\eta_k}=\begin{pmatrix}\Xi_{\phi_k,\eta_k} & \mathbb{O}\\ \mathbb{O} & \mathbb{O}\end{pmatrix},\quad P_{\phi_k,i}=\begin{pmatrix}\mathbf{I}_i & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & \mathbb{O} & \mathbf{I}_{n+1-\nu_{k(i)}}\\ \mathbb{O} & \mathbf{I}_{\nu_{k(i)}-1-i} & \mathbb{O}\end{pmatrix},\quad Q_{\phi_k,i}=\begin{pmatrix}\mathbf{I}_i & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & 1 & \mathbb{O}\\ \mathbb{O} & \boldsymbol{\ell}_{k,n-1-i} & \mathbf{I}_{n-1-i}\end{pmatrix},$$

where $\boldsymbol{\ell}_{k,n-1-i}=\frac{-1}{\widehat{a}^{\lfloor\phi_k,i\rfloor}_{i+1,i}}(\widehat{a}^{\lfloor\phi_k,i\rfloor}_{i+2,i},\ldots,\widehat{a}^{\lfloor\phi_k,i\rfloor}_{n,i})^\top$. [The paraphrase of $\widehat{a}^{\lfloor\cdot\rfloor}_{j,i}$ is the same as $\overline{a}^{\lfloor\cdot\rfloor}_{j,i}$.]

[b] The aim of "suitable" is that the choice of $\nu_{k(i)}$ is helpful to prove $\Sigma_\epsilon\succ\mathbb{O}$. More details refer to Sections 5.1-5.4.

[c] $P_{\phi_k,i}$ and $Q_{\phi_k,i}$ are determined by $\nu_{k(i)}$. $\Xi_{\phi_k,\eta_k}$ is shown in (3.20).

Below we combine the superposition principle to derive the specific form of $\Sigma_\epsilon$. As similarly in (A.2), $\Sigma_\epsilon$ can be determined by the Lyapunov equation

$$\Im_c\big(\Sigma_\epsilon, B_\epsilon, \epsilon\Theta_\epsilon\Theta_\epsilon^\top\big)=\mathbb{O}. \tag{3.8}$$

In view of (3.4) and $A_\epsilon$, Eq. (3.8) is equivalently transformed into

$$\Im_c\Big(\frac{1}{\epsilon}\mathcal{G}_\epsilon\Sigma_\epsilon\mathcal{G}_\epsilon^\top, A_\epsilon, \sum_{k=1}^{\xi}\lambda_k^+ \amalg_{n,\phi_k}\Big) = \mathbb{O}. \tag{3.9}$$

Consider the following algebraic equations

$$\Im_c(\Sigma_{\phi_k,\epsilon}, A_\epsilon, \amalg_{n,\phi_k}) = \mathbb{O}, \quad \forall\, k \in \mathbb{S}_\xi^0, \tag{3.10}$$

where $\Sigma_{\phi_k,\epsilon}$ is the same as in Algorithm 1.
Combining (3.9) and (3.10) yields

$$\mathcal{G}_\epsilon\Sigma_\epsilon\mathcal{G}_\epsilon^\top = \epsilon\sum_{k=1}^{\xi}\lambda_k^+\Sigma_{\phi_k,\epsilon}.$$

Then (3.5) is obtained by the orthogonality of $\mathcal{G}_\epsilon$.

Inspired by the Gaussian elimination method, we divide the remaining proof of Theorem 3.1 into three steps. The first is to solve Eq. (3.10) (i.e., $\Sigma_{\phi_k,\epsilon}$) by Algorithm 1. The second is to obtain an important property of $\mathbf{H}_{\phi_k,j}$, $\forall\, j \in \mathbb{S}_{\eta_k}^0$, and the third is to verify (3.6).

**Step 1.** For any $k \in \mathbb{S}_\xi^0$, let $\overline{A}_{\phi_k,1} = J_{\phi_k} A_\epsilon J_{\phi_k}^{-1}$. It is clear that $J_{\phi_k} \amalg_{n,\phi_k} J_{\phi_k}^\top = \amalg_{n,1}$, and (3.10) can then be equivalently transformed into

$$\Im_c\big((Q_{\phi_k,0}P_{\phi_k,0}J_{\phi_k})\Sigma_{\phi_k,\epsilon}(Q_{\phi_k,0}P_{\phi_k,0}J_{\phi_k})^\top, \overline{A}_{\phi_k,1}, \amalg_{n,1}\big) = \mathbb{O}. \tag{3.11}$$

Based on the definition of $\eta_k$ in Algorithm 1, we have $\eta_k \ge 1$, and

$$\text{(1-i)}\ \sum_{j=i+1}^{n}(\overline{a}_{ji}^{\lfloor\phi_k,i\rfloor})^2 \neq 0, \ \ \forall\, i \in \mathbb{S}_{\eta_k-1}^0, \qquad \text{(1-ii)}\ \ \overline{a}_{j,\eta_k}^{\lfloor\phi_k,\eta_k\rfloor} = 0, \ \ \forall\, j \in \mathbb{S}_n^{\eta_k+1}, \tag{3.12}$$

where each $\overline{a}_{ji}^{\lfloor\phi_k,i\rfloor}$ is obtained by the following iterative scheme:

$$\begin{cases} \widehat{A}_{\phi_k,i} = P_{\phi_k,i}\overline{A}_{\phi_k,i}P_{\phi_k,i}^{-1}, \\ \overline{A}_{\phi_k,i+1} := Q_{\phi_k,i}\widehat{A}_{\phi_k,i}Q_{\phi_k,i}^{-1}, \quad \forall\, i \in \mathbb{S}_{\eta_k}^0. \end{cases} \tag{3.13}$$

We first illustrate (3.12) for two special cases

$$(\mathscr{A}_1)\ \ \eta_k = 1, \quad (\mathscr{A}_2)\ \ \eta_k = n.$$

Under $(\mathscr{A}_1)$, (3.12) is simplified as condition (1-ii), whereas it is simplified as (1-i) if $(\mathscr{A}_2)$ holds. Combining (3.12), (3.13), and the forms of $P_{\phi_k,i}$ and $Q_{\phi_k,i}$ yields that

$$\overline{a}_{i+1,i}^{\lfloor\phi_k,\eta_k\rfloor} = \overline{a}_{\nu_{k(i)},i}^{\lfloor\phi_k,i\rfloor}(\neq 0), \ \ \text{and}\ \ \overline{a}_{j,i}^{\lfloor\phi_k,\eta_k\rfloor} = 0, \ \ \forall\, i \in \mathbb{S}_{\eta_k-1}^{-1};\ j \in \mathbb{S}_n^{i}. \tag{3.14}$$

Moreover, Eq. (3.11) is equivalent to

$$\Im_c\Big(\Big(\Big(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i}P_{\phi_k,i}\Big)J_{\phi_k}\Big)\Sigma_{\phi_k,\epsilon}\Big(\Big(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i}P_{\phi_k,i}\Big)J_{\phi_k}\Big)^\top, \overline{A}_{\phi_k,\eta_k}, Ш_{n,1}\Big) = \mathbb{O}. \tag{3.15}$$

In the display above, we have used

$$\Big[\Big(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i}P_{\phi_k,i}\Big)J_{\phi_k}\Big] Ш_{n,1} \Big[\Big(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i}P_{\phi_k,i}\Big)J_{\phi_k}\Big]^\top = Ш_{n,1}.$$

Using (3.14) and Definition 2.2, we get $\overline{A}^{(\eta_k)}_{\phi_k,\eta_k} \in \mathcal{U}_q(\eta_k)$. Combined with Proposition 2.2, we can determine a matrix $\mathcal{M}_{\eta_k} \in \mathcal{U}(\eta_k)$:

$$\mathcal{M}_{\eta_k} = \begin{pmatrix} \boldsymbol{\beta}_{\eta_k}(\overline{A}^{(\eta_k)}_{\phi_k,\eta_k})^{\eta_k-1} \\ \boldsymbol{\beta}_{\eta_k}(\overline{A}^{(\eta_k)}_{\phi_k,\eta_k})^{\eta_k-2} \\ \cdots \\ \boldsymbol{\beta}_{\eta_k} \end{pmatrix},$$

which forces $\mathcal{M}_{\eta_k}\overline{A}^{(\eta_k)}_{\phi_k,\eta_k}\mathcal{M}^{-1}_{\eta_k} \in \mathscr{S}(\eta_k)$. A similar argument in (B.2), (B.3) coupled with (3.14) leads to

$$\big(\boldsymbol{\beta}_{\eta_k}(\overline{A}^{(\eta_k)}_{\phi_k,\eta_k})^{\eta_k-1}\big)^{(1)} = \prod_{i=0}^{\eta_k-1} \overline{a}^{\lfloor \phi_k,\eta_k \rfloor}_{i+1,i} \neq 0. \tag{3.16}$$

By Algorithm 1, let

$$M_{\phi_k,\eta_k} = \begin{pmatrix} \mathcal{M}_{\eta_k} & \mathbb{O} \\ \mathbb{O} & \mathbf{I}_{n-\eta_k} \end{pmatrix}, \quad A_{s,\phi_k} = M_{\phi_k,\eta_k}\overline{A}_{\phi_k,\eta_k}M^{-1}_{\phi_k,\eta_k}.$$

Combining (3.16) and Definition 2.1, one has

$$M_{\phi_k,\eta_k}\, Ш_{n,1}\, M^\top_{\phi_k,\eta_k} = \Big(\prod_{j=0}^{\eta_k-1} \overline{a}^{\lfloor \phi_k,\eta_k \rfloor}_{j+1,j}\Big)^2 Ш_{n,1}, \tag{3.17}$$

and there is a vector $\boldsymbol{\alpha}^\top_{\phi_k} = (\alpha_{\phi_k,1}, ..., \alpha_{\phi_k,\eta_k})^\top \in \mathbb{R}^{\eta_k}_+$ satisfying

$$A^{(\eta_k)}_{s,\phi_k} = \begin{pmatrix} -\boldsymbol{\alpha}^{\langle \eta_k-1 \rangle}_{\phi_k} & -\alpha_{\phi_k,\eta_k} \\ \mathbf{I}_{\eta_k-1} & \mathbb{O} \end{pmatrix} \in \mathscr{S}(\eta_k). \tag{3.18}$$

Consider an $\eta_k$-dimensional algebraic equation

$$\Im_c\big(\Xi_{\phi_k,\eta_k}, A^{(\eta_k)}_{s,\phi_k}, Ш_{\eta_k,1}\big) = \mathbb{O}. \tag{3.19}$$

By (3.18) and Proposition 2.1, we determine that $\Xi_{\phi_k,\eta_k} \succ \mathbb{O}$ and

$$\Xi_{\phi_k,\eta_k} = \begin{pmatrix} \zeta_1 & 0 & -\zeta_2 & 0 & \zeta_3 & \cdots \\ 0 & \zeta_2 & 0 & -\zeta_3 & \cdots & \iddots \\ -\zeta_2 & 0 & \zeta_3 & \cdots & \iddots & 0 \\ 0 & -\zeta_3 & \cdots & \iddots & 0 & -\zeta_{\eta_k-1} \\ \zeta_3 & \cdots & \iddots & 0 & \zeta_{\eta_k-1} & 0 \\ \vdots & \iddots & 0 & -\zeta_{\eta_k-1} & 0 & \zeta_{\eta_k} \end{pmatrix}, \tag{3.20}$$

where $(\zeta_1, -\zeta_2, ..., (-1)^{\eta_k-1}\zeta_{\eta_k})^\top = \frac{1}{2}\mathscr{H}^{-1}_{\eta_k, A^{(\eta_k)}_{s,\phi_k}} \mathbf{e}_{\eta_k}$.

To proceed, we define

$$\widetilde{\Sigma}_{\phi_k,\epsilon} = \Big[\Big(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i} P_{\phi_k,i}\Big) J_{\phi_k}\Big] \Sigma_{\phi_k,\epsilon} \Big[\Big(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i} P_{\phi_k,i}\Big) J_{\phi_k}\Big]^\top.$$

Using (3.17), Eq. (3.15) is equivalent to

$$\Im_c\Bigg(M_{\phi_k,\eta_k} \widetilde{\Sigma}_{\phi_k,\epsilon} M^\top_{\phi_k,\eta_k}, A_{s,\phi_k}, \Big(\prod_{j=0}^{\eta_k-1} \overline{a}^{\lfloor \phi_k,\eta_k \rfloor}_{j+1,j}\Big)^2 \amalg_{n,1}\Bigg) = \mathbb{O}, \tag{3.21}$$

after which the related analysis can be divided into the following two conditions:

$$(\mathscr{A}_1') \;\; \eta_k \in \mathbb{S}^0_{n-1}, \quad (\mathscr{A}_2') \;\; \eta_k = n.$$

Case 1. Under $(\mathscr{A}_1')$, it follows from (3.14) that $\overline{A}_{\phi_k,\eta_k}$ takes the form

$$\overline{A}_{\phi_k,\eta_k} = \begin{pmatrix} \overline{A}^{(\eta_k)}_{\phi_k,\eta_k} & \underline{A}_{1,\eta_k} \\ \mathbb{O} & \underline{A}_{2,\eta_k} \end{pmatrix} \tag{3.22}$$

with $\underline{A}_{1,\eta_k} \in \mathbb{R}^{\eta_k \times (n-\eta_k)}$. For simplicity, we assume that

$$\widetilde{\Sigma}_{\phi_k,\epsilon} := \begin{pmatrix} \widetilde{\Sigma}^{(\eta_k)}_{\phi_k,\epsilon} & \pounds_1 \\ \pounds_1^\top & \pounds_2 \end{pmatrix}, \tag{3.23}$$

where $\pounds_2$ is real symmetric.

Below we need to prove $\pounds_i = \mathbb{O}$ for any $i = 1, 2$. Applying (3.22) and (3.23) to Eq. (3.21) yields

$$\begin{cases} \Im_c\Bigg(\mathcal{M}_{\eta_k} \widetilde{\Sigma}^{(\eta_k)}_{\phi_k,\epsilon} \mathcal{M}^\top_{\eta_k}, A^{(\eta_k)}_{s,\phi_k}, \Big(\prod_{j=0}^{\eta_k-1} \overline{a}^{\lfloor \phi_k,\eta_k \rfloor}_{j+1,j}\Big)^2 \amalg_{\eta_k,1} \\ \qquad\qquad + \mathcal{M}_{\eta_k} \underline{A}_{1,\eta_k} (\mathcal{M}_{\eta_k}\pounds_1)^\top + \mathcal{M}_{\eta_k}\pounds_1(\mathcal{M}_{\eta_k}\underline{A}_{1,\eta_k})^\top\Bigg) = \mathbb{O}, \\ A^{(\eta_k)}_{s,\phi_k}\mathcal{M}_{\eta_k}\pounds_1 + \mathcal{M}_{\eta_k}\pounds_1\underline{A}^\top_{2,\eta_k} + \mathcal{M}_{\eta_k}\underline{A}_{1,\eta_k}\pounds_2 = \mathbb{O}, \\ \Im_c\big(\pounds_2, \underline{A}_{2,\eta_k}, \mathbb{O}\big) = \mathbb{O}. \end{cases} \tag{3.24}$$

In view of $\overline{A}^{(\eta_k)}_{\phi_k,\eta_k} \in \overline{\mathbf{RH}}(\eta_k)$, we get $\underline{A}_{2,\eta_k} \in \overline{\mathbf{RH}}(n-\eta_k)$. Then by (A.2) and the third equality of (3.24), one has $\pounds_2 = \mathbb{O}$. Thus,

$$\begin{cases} \Im_c\Big(\mathcal{M}_{\eta_k}\widetilde{\Sigma}^{(\eta_k)}_{\phi_k,\epsilon}\mathcal{M}^\top_{\eta_k},\, A^{(\eta_k)}_{s,\phi_k},\, \Big(\prod_{j=0}^{\eta_k-1}\overline{a}^{\lfloor\phi_k,\eta_k\rfloor}_{j+1,j}\Big)^2 \text{Ш}_{\eta_k,1} \\ \qquad\qquad +\mathcal{M}_{\eta_k}\underline{A}_{1,\eta_k}(\mathcal{M}_{\eta_k}\pounds_1)^\top + \mathcal{M}_{\eta_k}\pounds_1(\mathcal{M}_{\eta_k}\underline{A}_{1,\eta_k})^\top\Big) = \mathbb{O}, \\ A^{(\eta_k)}_{s,\phi_k}\mathcal{M}_{\eta_k}\pounds_1 + \mathcal{M}_{\eta_k}\pounds_1\underline{A}^\top_{2,\eta_k} = \mathbb{O}. \end{cases} \tag{3.25}$$

Since $\widetilde{\Sigma}_{\phi_k,\epsilon}$ is unique, the solution $(\widetilde{\Sigma}^{(\eta_k)}_{\phi_k,\epsilon}, \pounds_1, \mathbb{O})$ of (3.24) is unique. Let $\mathscr{F} := A^{(\eta_k)}_{s,\phi_k}\mathcal{M}_{\eta_k}\pounds_1$, we obtain from (3.25) that $\mathcal{M}_{\eta_k}\pounds_1\underline{A}^\top_{2,\eta_k} = -\mathscr{F}$, i.e., $\mathcal{M}_{\eta_k}\pounds_1 = (A^{(\eta_k)}_{s,\phi_k})^{-1}\mathscr{F} = -\mathscr{F}(\underline{A}^\top_{2,\eta_k})^{-1}$. Then,

$$A^{(\eta_k)}_{s,\phi_k}\mathscr{F} + \mathscr{F}\underline{A}^\top_{2,\eta_k} = \mathbb{O}.$$

Below we prove $\mathscr{F} = \mathcal{M}_{\eta_k}\pounds_1$ by reductio ad absurdum. If $\mathcal{M}^{-1}_{\eta_k}\mathscr{F} \neq \pounds_1$, we consider an $\eta_k$-dimensional algebraic equation of $\widehat{\Sigma}^{\lceil \mathscr{F}\rceil}_{\phi_k,\epsilon}$:

$$\Im_c\Big(\mathcal{M}_{\eta_k}\widehat{\Sigma}^{\lceil \mathscr{F}\rceil}_{\phi_k,\epsilon}\mathcal{M}^\top_{\eta_k},\, A^{(\eta_k)}_{s,\phi_k},\, \Big(\prod_{j=0}^{\eta_k-1}\overline{a}^{\lfloor\phi_k,\eta_k\rfloor}_{j+1,j}\Big)^2 \text{Ш}_{\eta_k,1} + \mathcal{M}_{\eta_k}\underline{A}_{1,\eta_k}\mathscr{F}^\top + \mathscr{F}(\mathcal{M}_{\eta_k}\underline{A}_{1,\eta_k})^\top\Big) = \mathbb{O}.$$

Using Lemma 2.2, $\widehat{\Sigma}^{\lceil \mathscr{F}\rceil}_{\phi_k,\epsilon}$ exists and is unique. This leads to a contradiction that there are two different solutions $(\widetilde{\Sigma}^{(\eta_k)}_{\phi_k,\epsilon}, \pounds_1, \mathbb{O})$ and $(\widehat{\Sigma}^{\lceil \mathscr{F}\rceil}_{\phi_k,\epsilon}, \mathcal{M}^{-1}_{\eta_k}\mathscr{F}, \mathbb{O})$ satisfying Eq. (3.24). Hence, $\pounds_1 = \mathcal{M}^{-1}_{\eta_k}\mathscr{F}$, which means

$$(A^{(\eta_k)}_{s,\phi_k} - \mathbf{I}_{\eta_k})\mathscr{F} = \mathbb{O}. \tag{3.26}$$

As in (3.18), we calculate

$$\big|A^{(\eta_k)}_{s,\phi_k} - \mathbf{I}_{\eta_k}\big| = (-1)^{\eta_k}\Big(1 + \sum_{i=1}^{\eta_k}\alpha_{\phi_k,i}\Big) \neq 0.$$

According to (3.26), one has $\mathscr{F} = \mathbb{O}$ and $\pounds_1 = \mathbb{O}$. Then (3.21) can be equivalently transformed into

$$\Im_c\bigg(\Big(\prod_{j=0}^{\eta_k-1}\overline{a}^{\lfloor\phi_k,\eta_k\rfloor}_{j+1,j}\Big)^{-2}\mathcal{M}_{\eta_k}\widetilde{\Sigma}^{(\eta_k)}_{\phi_k,\epsilon}\mathcal{M}^\top_{\eta_k},\, A^{(\eta_k)}_{s,\phi_k},\, \text{Ш}_{\eta_k,1}\bigg) = \mathbb{O}.$$

This together with Proposition 2.1 and Eq. (3.19) implies that

$$\Big(\prod_{j=0}^{\eta_k-1}\overline{a}^{\lfloor\phi_k,\eta_k\rfloor}_{j+1,j}\Big)^{-2}\mathcal{M}_{\eta_k}\widetilde{\Sigma}^{(\eta_k)}_{\phi_k,\epsilon}\mathcal{M}^\top_{\eta_k} = \Xi_{\phi_k,\eta_k} \succ \mathbb{O}.$$

Thus,

$$\begin{aligned}\Sigma_{\phi_k,\epsilon} =&\Big(\Big(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i} P_{\phi_k,i}\Big) J_{\phi_k}\Big)^{-1} \begin{pmatrix} \widetilde{\Sigma}_{\phi_k,\epsilon}^{(\eta_k)} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \Big[\Big(\Big(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i} P_{\phi_k,i}\Big) J_{\phi_k}\Big)^{-1}\Big]^\top \\ =&\Big(\prod_{j=0}^{\eta_k-1} \overline{a}_{j+1,j}^{\lfloor\phi_k,\eta_k\rfloor}\Big)^2 \Big(M_{\phi_k,\eta_k}\Big(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i} P_{\phi_k,i}\Big) J_{\phi_k}\Big)^{-1} \\ &\Delta_{\phi_k,\eta_k}\Big[\Big(M_{\phi_k,\eta_k}\Big(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i} P_{\phi_k,i}\Big) J_{\phi_k}\Big)^{-1}\Big]^\top, \end{aligned} \tag{3.27}$$

where $\Delta_{\phi_k,\eta_k}$ is the same as in Algorithm 1.

Case 2. Under $(\mathscr{A}_2')$, by $M_{\phi_k,n} = \mathcal{M}_n$ and $A_{s,\phi_k} \in \mathscr{S}(n)$, we can equivalently transform Eq. (3.21) into

$$\begin{aligned}\Im_c\Big(&\Big(\prod_{j=0}^{n-1} \overline{a}_{j+1,j}^{\lfloor\phi_k,n\rfloor}\Big)^{-2}\Big(M_{\phi_k,n}\Big(\prod_{i=0}^{n-1} Q_{\phi_k,i} P_{\phi_k,i}\Big) J_{\phi_k}\Big)\Sigma_{\phi_k,\epsilon} \\ &\times\Big(M_{\phi_k,n}\Big(\prod_{i=0}^{n-1} Q_{\phi_k,i} P_{\phi_k,i}\Big) J_{\phi_k}\Big)^\top, A_{s,\phi_k}, \amalg_{n,1}\Big) = \mathbb{O}.\end{aligned}$$

Using Lemma 2.2 and (3.19), we have

$$\Sigma_{\phi_k,\epsilon} = \Big(\prod_{j=0}^{n-1} \overline{a}_{j+1,j}^{\lfloor\phi_k,n\rfloor}\Big)^2 \Big(M_{\phi_k,n}\Big(\prod_{i=0}^{n-1} Q_{\phi_k,i} P_{\phi_k,i}\Big) J_{\phi_k}\Big)^{-1} \Xi_{\phi_k,n}\Big[\Big(M_{\phi_k,n}\Big(\prod_{i=0}^{n-1} Q_{\phi_k,i} P_{\phi_k,i}\Big) J_{\phi_k}\Big)^{-1}\Big]^\top. \tag{3.28}$$

Combining (3.27) and (3.28), the expression of $\Sigma_{\phi_k,\epsilon}$ in Algorithm 1 is derived, and $\Sigma_{\phi_k,\epsilon} \succeq \mathbb{O}$, $\forall\, k \in \mathbb{S}_\xi^0$. In this sense, $\Sigma_\epsilon$ can be determined by (3.5).

**Step 2.** To verify (3.6), for any $\eta_k \geq 2$, we show an important assertion of $\mathbf{H}_{\phi_k,j}$ below.

$$\mathbf{H}_{\phi_k,m}^{\langle j\rangle} = \mathbf{H}_{\phi_k,j}^{\langle j\rangle}, \quad \forall\, m \in \mathbb{S}_{\eta_k}^j. \tag{3.29}$$

By Algorithm 1, it is clear that

$$(Q_{\phi_k,i}^{-1})^\top = \begin{pmatrix} \mathbf{I}_i & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & 1 & -\boldsymbol{\ell}_{k,n-1-i}^\top \\ \mathbb{O} & \mathbb{O} & \mathbf{I}_{n-1-i} \end{pmatrix}, \quad ((Q_{\phi_k,i}^{-1})^\top P_{\phi_k,i})^{[i]} = \begin{pmatrix} \mathbf{I}_i & \mathbb{O} \end{pmatrix},$$

which implies

$$\begin{aligned}\mathbf{H}^{\langle j\rangle}_{\phi_k,j+1} =&\big[\big((Q^{-1}_{\phi_k,j})^\top P_{\phi_k,j}\big)\mathbf{H}_{\phi_k,j}\big]^{\langle j\rangle}\\ =&\Big[\big((Q^{-1}_{\phi_k,j})^\top P_{\phi_k,j}\big)^{[j]}\mathbf{H}_{\phi_k,j}\Big]^{\langle j\rangle} = \mathbf{H}^{\langle j\rangle}_{\phi_k,j}, \quad \forall\, j \in \mathbb{S}^0_{\eta_k-1}.\end{aligned}$$

Then by a natural result $(\mathbf{H}^{\langle c-1\rangle}_{\phi_k,c-1})^{\langle j\rangle} = (\mathbf{H}^{\langle c-2\rangle}_{\phi_k,c-1})^{\langle j\rangle}$, $\forall\, c \in \mathbb{S}^{j+1}_{\eta_k}$,

$$\big(\mathbf{H}^{\langle c-1\rangle}_{\phi_k,c}\big)^{\langle j\rangle} = \big(\mathbf{H}^{\langle c-2\rangle}_{\phi_k,c-1}\big)^{\langle j\rangle}. \tag{3.30}$$

Using (3.30) recursively, we obtain

$$\mathbf{H}^{\langle j\rangle}_{\phi_k,m} = \big(\mathbf{H}^{\langle m-1\rangle}_{\phi_k,m}\big)^{\langle j\rangle} = \big(\mathbf{H}^{\langle j\rangle}_{\phi_k,j+1}\big)^{\langle j\rangle} = \mathbf{H}^{\langle j\rangle}_{\phi_k,j}.$$

Therefore, the desired assertion follows.

**Step 3.** In view of (3.5), (3.14), (3.27)-(3.29) and the definition of $\mathbf{Y}$, we obtain

$$\begin{aligned}\mathbf{X}^\top\Sigma_\epsilon\mathbf{X} =&\mathbf{X}^\top\Big[\epsilon\mathcal{G}_\epsilon^\top\Big(\sum_{k=1}^{\xi}\lambda_k^+\Sigma_{\phi_k,\epsilon}\Big)\mathcal{G}_\epsilon\Big]\mathbf{X}\\ \geq&\epsilon\min_{k\in\mathbb{S}^0_\xi}\{\lambda_k^+\}\Big[\sum_{k=1}^{\xi}(\mathcal{G}_\epsilon\mathbf{X})^\top\Sigma_{\phi_k,\epsilon}(\mathcal{G}_\epsilon\mathbf{X})\Big]\\ \geq&\epsilon\min_{k\in\mathbb{S}^0_\xi}\Big\{\lambda_k^+\Big(\prod_{i=0}^{\eta_k-1}\overline{a}^{\lfloor\phi_k,i\rfloor}_{\nu_{k(i)},i}\Big)^2\Big\}\Bigg\{\sum_{k=1}^{\xi}\mathbf{Y}^\top\Big(M_{\phi_k,\eta_k}\Big(\prod_{i=0}^{\eta_k-1}Q_{\phi_k,i}P_{\phi_k,i}\Big)J_{\phi_k}\Big)^{-1}\\ &\times\Delta_{\phi_k,\eta_k}\Big[\Big(M_{\phi_k,\eta_k}\Big(\prod_{i=0}^{\eta_k-1}Q_{\phi_k,i}P_{\phi_k,i}\Big)J_{\phi_k}\Big)^{-1}\Big]^\top\mathbf{Y}\Bigg\}\\ =&\epsilon\min_{k\in\mathbb{S}^0_\xi}\Big\{\lambda_k^+\Big(\prod_{i=0}^{\eta_k-1}\overline{a}^{\lfloor\phi_k,i\rfloor}_{\nu_{k(i)},i}\Big)^2\Big\}\sum_{k=1}^{\xi}\Bigg\{\Big[\Big(\Big(\Big(\prod_{i=0}^{\eta_k-1}Q_{\phi_k,i}P_{\phi_k,i}\Big)J_{\phi_k}\Big)^{-1}\Big)^\top\mathbf{Y}\Big]^\top\\ &\times\Big(M^{-1}_{\phi_k,\eta_k}\Delta_{\phi_k,\eta_k}(M^{-1}_{\phi_k,\eta_k})^\top\Big)\Big[\Big(\Big(\Big(\prod_{i=0}^{\eta_k-1}Q_{\phi_k,i}P_{\phi_k,i}\Big)J_{\phi_k}\Big)^{-1}\Big)^\top\mathbf{Y}\Big]\Bigg\}.\end{aligned} \tag{3.31}$$

Intuitively,

$$M^{-1}_{\phi_k,\eta_k}\Delta_{\phi_k,\eta_k}(M^{-1}_{\phi_k,\eta_k})^\top = \begin{pmatrix}\mathcal{M}^{-1}_{\eta_k}\Xi_{\phi_k,\eta_k}(\mathcal{M}^{-1}_{\eta_k})^\top & \mathbb{O}\\ \mathbb{O} & \mathbb{O}\end{pmatrix}. \tag{3.32}$$

Let $\widetilde{\lambda}_k$ be the minimal eigenvalue of $\mathcal{M}^{-1}_{\eta_k}\Xi_{\phi_k,\eta_k}(\mathcal{M}^{-1}_{\eta_k})^\top$, an application of $\Xi_{\phi_k,\eta_k}\succ\mathbb{O}$ for (3.32) implies that $\widetilde{\lambda}_k > 0$ and

$$M^{-1}_{\phi_k,\eta_k}\Delta_{\phi_k,\eta_k}(M^{-1}_{\phi_k,\eta_k})^\top \succeq \widetilde{\lambda}_k\begin{pmatrix}\mathbf{I}_{\eta_k} & \mathbb{O}\\ \mathbb{O} & \mathbb{O}\end{pmatrix}. \tag{3.33}$$

Note that $J_{\phi_k}^{-1} = J_{\phi_k}^{\top}$ and $P_{\phi_k,i}^{-1} = P_{\phi_k,i}^{\top}$, we determine that

$$\left(\left(\left(\prod_{i=0}^{\eta_k-1} Q_{\phi_k,i} P_{\phi_k,i}\right) J_{\phi_k}\right)^{-1}\right)^{\top} \mathbf{Y} = \mathbf{H}_{\phi_k,\eta_k}. \tag{3.34}$$

Applying (3.29), (3.33) and (3.34) to (3.31) leads to

$$\begin{aligned}
\mathbf{X}^{\top}\Sigma_\epsilon \mathbf{X} &\geq \epsilon \min_{k\in\mathbb{S}_\xi^0}\left\{\lambda_k^+\left(\prod_{i=0}^{\eta_k-1} \overline{a}_{\nu_{k(i)},i}^{\lfloor\phi_k,i\rfloor}\right)^2\right\}\sum_{k=1}^{\xi}\widetilde{\lambda}_k(\mathbf{H}_{\phi_k,\eta_k}^{\langle\eta_k\rangle})^{\top}\mathbf{H}_{\phi_k,\eta_k}^{\langle\eta_k\rangle} \\
&\geq \rho_\epsilon \sum_{k=1}^{\xi}\sum_{j=1}^{\eta_k}(\mathbf{H}_{\phi_k,\eta_k}^{(j)})^2 \\
&= \rho_\epsilon \sum_{k=1}^{\xi}\sum_{j=1}^{\eta_k}(\mathbf{H}_{\phi_k,j}^{(j)})^2,
\end{aligned} \tag{3.35}$$

where

$$\rho_\epsilon = \epsilon \min_{k\in\mathbb{S}_\xi^0}\left\{\widetilde{\lambda}_k\lambda_k^+\left(\prod_{i=0}^{\eta_k-1} \overline{a}_{\nu_{k(i)},i}^{\lfloor\phi_k,i\rfloor}\right)^2\right\} > 0.$$

The assertion (3.6) easily follows from (3.35) and $\mathbf{H}_{\phi_k,1}^{(1)} = Y_{\phi_k}$. This completes the proof. □

**Remark 3.** It should be mentioned that throughout the above proof, only the existence of the IPM $\mu_\epsilon$ is required for Theorem 3.1, not its uniqueness. In other words, the conditions of Theorem 3.1 can be reduced to Assumptions 2.1(1) and 2.2(a) as well as the existence of stationary distribution of (1.2). However, if the IPM $\mu_\epsilon$ is not unique, the local approximation accuracy of the distribution $\mathbb{LN}_n(\ln\overline{\mathbf{X}}_\epsilon^*, \Sigma_\epsilon)$ (resp., $\Phi_\epsilon(\cdot)$) on $\mu_\epsilon$ (resp., $\Psi_\epsilon(\cdot)$) around $\overline{\mathbf{X}}$ cannot be verified by computer simulations; see Remark 8 and Example 5.5 for details.

By Theorem 3.1, we provide a local approximation $\mathbb{LN}_n(\ln\overline{\mathbf{X}}_\epsilon^*, \Sigma_\epsilon)$ for the IPM $\mu_\epsilon$ around $\overline{\mathbf{X}}_\epsilon^*$ and also obtain the expression and positive definiteness of $\Sigma_\epsilon$. As in (3.7), $\Sigma_\epsilon$ has an explicit form $\int_0^\infty e^{B_\epsilon t}\Theta_\epsilon\Theta_\epsilon^{\top}e^{B_\epsilon^{\top}t}dt$. However, this method still has some limitations: (i) It is difficult to compute matrix integral as it requires an accurate result of $e^{B_\epsilon t}$ for any $t \geq 0$ (ii) Only $\Sigma_\epsilon \succeq \mathbb{O}$ can be derived under Assumption 2.2(a) (i.e., $\Theta_\epsilon\Theta_\epsilon^{\top} \succeq \mathbb{O}$), but $\Sigma_\epsilon \succ \mathbb{O}$ is unknown. Hence, we consider studying $\Sigma_\epsilon$ from a matrix equation perspective; see Eq. (3.8). An effective approach for solving Lyapunov equation $\Im_c(\Sigma, \boldsymbol{\varpi}, \aleph)$ is to obtain a simple canonical form of $\boldsymbol{\varpi}$ (called $\mathscr{C}_f(\boldsymbol{\varpi})$) by matrix transformations, thereby simplifying calculations and deriving implementable iteration schemes.

Bartels–Stewart method [43] is currently the most popular algorithm along this line and has been widely adopted, the associated command "lyap(·, ·)" in MATLAB is thus developed [82]. This numerical method is based on Schur factorization, i.e., find an orthogonal matrix $Q$ satisfying $Q^{-1}\boldsymbol{\varpi}Q = \mathscr{C}_f(\boldsymbol{\varpi}) := R \in \mathcal{U}(\cdot)$, then $\Im_c(\Sigma, \boldsymbol{\varpi}, \aleph)$ is equivalently transformed into $\Im_c(Q^{\top}\Sigma Q, R, Q^{\top}\aleph Q)$, which is easily treated. By now, using different matrix factorizations, several modified Bartels–Stewart methods have been developed including (i) Hessenberg–Schur method [69], where $\mathscr{C}_f(\boldsymbol{\varpi})$ is an upper Hessenberg matrix, and (ii) Hammarling method [70–72], where the condition $\aleph \succ \mathbb{O}$ is required.

In the study of LNA method for locally approximating the IPM $\mu_\epsilon$, we develop a novel numerical framework for solving the Lyapunov equation, as shown in Algorithm 1. A key idea is to introduce a new canonical form $\mathscr{S}(\cdot)$ and combine similarity transformations such that

$$\mathscr{C}_f(\boldsymbol{\varpi}^{(k)}) \in \mathscr{S}(k), \quad \forall\, \boldsymbol{\varpi} \in \overline{\mathbf{RH}}(l),$$

where $k \in \mathbb{S}_l^0$.

The aim of Algorithm 1 is to obtain the expression of $\Sigma_\epsilon$ in (3.8). To highlight, our main advantages here are as follows:

- (Simpler Computational Routine): Based on the superposition principle and (3.4), our first step is to transform Eq. (3.8) into the equations $\Im_c(\Sigma_{\phi_k,\epsilon}, A_\epsilon, \amalg_{n,\phi_k}) = \mathbb{O}$ for all $k \in \mathbb{S}_\xi^0$, i.e., Eq. (3.10). Unlike the aforementioned algorithms that focus mainly on the factorization of $B_\epsilon$, our method only requires the computation of large $\eta_k \in \mathbb{S}_n^0$ that satisfy $\mathscr{C}_f(A_\epsilon^{(\eta_k)}) \in \mathscr{S}(\eta_k)$. This is achieved using a Gaussian-like elimination method (see (3.11)-(3.13)), which is simpler and more effective than Householder transformation method. The computational routine of our method is thus easier to understand and implement. Moreover, for each $k$, Eq. (3.10) can be equivalent to an $\eta_k$-dimensional standard $L_0$ algebraic equation, except for the zero matrix equations. Then by Proposition 2.1, the expression of $\Sigma_\epsilon$ is easily derived. Compared with existing numerical methods that use complex iterative schemes for $\Theta_\epsilon\Theta_\epsilon^\top$ and factorization of $B_\epsilon$, our Algorithm 1 has a lower computational cost.
- (Characterization of Positive Definiteness): Although Bartels–Stewart method and its variants can be used to solve (3.8), the positive definiteness of $\Sigma_\epsilon$ cannot be verified. Instead, this question is positively addressed by Algorithm 1. Specifically, combining Gaussian-like elimination method and Proposition 2.2, for any $k \in \mathbb{S}_\xi^0$, we determine a sequence $\{\mathbf{H}_{\phi_k,i}\}_{i=1}^{\eta_k}$, which records the form and minimal rank (i.e., $\eta_k$) of all column components of $\Sigma_{\phi_k,\epsilon}$. Then (3.6), a criterion for analyzing $\Sigma_\epsilon \succ \mathbb{O}$, is derived. To be specific, for each $j \in \mathbb{S}_{\eta_k}^0$, there is a vector $\boldsymbol{h}_{\phi_k,j}$ such that $\mathbf{H}_{\phi_k,j}^{(j)} = \boldsymbol{h}_{\phi_k,j}\mathbf{Y}$, where $\mathcal{G}_\epsilon^\top\mathbf{Y} = \mathbf{X}$. To prove $\Sigma_\epsilon \succ \mathbb{O}$, it is sufficient to verify that $\mathbf{X}^\top\Sigma_\epsilon\mathbf{X} = 0$ holds if and only if $\mathbf{Y} = \mathbf{0}$.
- (General Applicability) The existing theories approximating IPDFs can be at most applied to five-dimensional stochastic models established in non-degenerate diffusion; see Zhou et al. [73]. Our results (i.e., (3.6) and Algorithm 1) will cover, improve and generalize the relevant theories, and can work on stochastic systems with arbitrary dimension setting and degenerate diffusion.

As was mentioned, we obtain a criterion (3.6) for verifying $\Sigma_\epsilon \succ \mathbb{O}$. However, under a special case, where the diffusion matrix $\Theta_\epsilon\Theta_\epsilon^\top$ of (3.3) is complicated and its drift term $B_\epsilon$ is "simple" in the sense that approaching the canonical form $\mathscr{S}(\cdot)$, we can further simplify the analysis of $\Sigma_\epsilon \succ \mathbb{O}$ by introducing a modified criterion as follows.

Using Assumption 2.2(a), there are a constant $\lambda_\theta^+ > 0$ and a set $\boldsymbol{\phi}^\circ := \{\phi_1^\circ, ..., \phi_{\xi^\circ}^\circ\} \subseteq \mathbb{S}_n^0$ such that

$$\Theta_\epsilon\Theta_\epsilon^\top \succeq \lambda_\theta^+ \sum_{k=1}^{\xi^\circ} \amalg_{n,\phi_k^\circ}, \tag{3.36}$$

where $\phi_i < \phi_j$, $\forall\, i < j$. In particular, we stipulate $\lambda_\theta^+ = 1$ if $\boldsymbol{\phi}^\circ = \emptyset$. In view of (3.4), one has $\xi^\circ \leq \xi$.

**Theorem 3.2.** *Under Assumptions 2.1 and 2.2(a), the following result is true:*

$$\mathbf{X}^\top \Sigma_\epsilon \mathbf{X} \geq \rho_\epsilon^\circ \sum_{k=1}^{\xi^\circ} \Big( X_{\phi_k^\circ}^2 + \sum_{j=2}^{\eta_k^\circ} (\underline{\mathbf{H}}_{\phi_k^\circ, j}^{(j)})^2 \Big), \tag{3.37}$$

*where* $\mathbf{X}$ *is the same as in Theorem 3.1,* $\rho_\epsilon^\circ > 0$ *is shown in (3.41), and* $\underline{\mathbf{H}}_{\phi_k^\circ, j} = [\prod_{i=0}^{j-1} (\underline{Q}_{\phi_k^\circ, i}^{-1})^\top \underline{P}_{\phi_k^\circ, i}] \underline{J}_{\phi_k^\circ} \mathbf{X}$ $(j \in \mathbb{S}_{\eta_k^\circ}^0)$*, with* $\eta_k^\circ$, $\underline{J}_{\phi_k^\circ}$, $\underline{P}_{\phi_k^\circ, i}$ *and* $\underline{Q}_{\phi_k^\circ, i}$ $(i \in \mathbb{S}_{j-1}^0)$ *determined in Algorithm 2.*

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**Algorithm 2:** Algorithm for obtaining $\{\underline{\mathbf{H}}_{\phi_k^\circ, j}\}_{j=1}^{\eta_k^\circ}$.

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**Input:** $B_\epsilon$, $\phi_k^\circ$ (or $\boldsymbol{\phi}^\circ$).

**Output:** $\underline{\mathbf{H}}_{\phi_k^\circ, j} = [\sum_{i=0}^{j-1} (\underline{Q}_{\phi_k^\circ, i}^{-1})^\top \underline{P}_{\phi_k^\circ, i}] \underline{J}_{\phi_k^\circ} \mathbf{X}$[a].

1 **(Initialization):** $\eta_k^\circ = 1$, $\overline{B}_{\phi_k^\circ, 1} = \underline{J}_{\phi_k^\circ} B_\epsilon \underline{J}_{\phi_k^\circ}^{-1}$;

2 **(Technical framework):** By a FOR loop similar to Algorithm 1, we can determine the values of $\eta_k^\circ$ and some suitable $\nu_{k(i)}^\circ \in \mathbb{S}_n^i$, which yield

3 $\quad$ (2-i) $\overline{b}_{\nu_{k(i)}^\circ, i}^{\lfloor \phi_k^\circ, i \rfloor} \neq 0$[a], $\forall\, i \in \mathbb{S}_{\eta_k^\circ - 1}^0$, $\quad$ (2-ii) $\overline{b}_{j, \eta_k^\circ}^{\lfloor \phi_k^\circ, \eta_k^\circ \rfloor} = 0$, $\forall\, j \in \mathbb{S}_n^{\eta_k^\circ + 1}$,

4 where each $\overline{b}_{ji}^{\lfloor \phi_k^\circ, i \rfloor}$ is obtained by the iterative scheme:

5 $\quad \widehat{B}_{\phi_k^\circ, i} = \underline{P}_{\phi_k^\circ, i} \overline{B}_{\phi_k^\circ, i} \underline{P}_{\phi_k^\circ, i}^{-1}$[a], $\overline{B}_{\phi_k^\circ, i+1} := \underline{Q}_{\phi_k^\circ, i} \widehat{B}_{\phi_k^\circ, i} \underline{Q}_{\phi_k^\circ, i}^{-1}$;

6 **return** $\eta_k^\circ$, $\underline{J}_{\phi_k^\circ}$, $\underline{P}_{\phi_k^\circ, i}$, $\underline{Q}_{\phi_k^\circ, i}$ $(i \in \mathbb{S}_{\eta_k^\circ - 1}^0)$.

[a] $\underline{J}_{\phi_k^\circ}$, $\underline{P}_{\phi_k^\circ, i}$ and $\underline{Q}_{\phi_k^\circ, i}$ have the same form as $J_{\phi_k}$, $P_{\phi_k, i}$ and $Q_{\phi_k, i}$ (see Algorithm 1) by replacing $(\phi_k, \nu_{k(i)}, \boldsymbol{\ell}_{k, n-1-i})$ with $(\phi_k^\circ, \nu_{k(i)}^\circ, \underline{\boldsymbol{\ell}}_{k, n-1-i})$, where $\underline{\boldsymbol{\ell}}_{k, n-1-i} = \frac{-1}{\widehat{b}_{i+1, i}^{\lfloor \phi_k^\circ, i \rfloor}} (\widehat{b}_{i+2, i}^{\lfloor \phi_k^\circ, i \rfloor}, \ldots, \widehat{b}_{n, i}^{\lfloor \phi_k^\circ, i \rfloor})^\top$. [The paraphrase of $\overline{b}_{j,i}^{\lfloor \cdot \rfloor}$ (resp., $\widehat{b}_{j,i}^{\lfloor \cdot \rfloor}$) is the same as $\overline{a}_{j,i}^{\lfloor \cdot \rfloor}$ (resp., $\widehat{a}_{j,i}^{\lfloor \cdot \rfloor}$).] In addition, we stipulate that $\overline{b}_{\nu_{k(0)}^\circ, 0}^{\lfloor \phi_k^\circ, 0 \rfloor} = 1$ and $\underline{P}_{\phi_k, l} = \underline{Q}_{\phi_k, l} = \mathbf{I}_n$, $\forall\, k \in \mathbb{S}_{\xi^\circ}^0$; $l \in \{0, n-1\}$.

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**Proof.** Consider an $n$-dimensional algebraic equation

$$\Im_c \bigg( \Sigma_\theta^+, B_\epsilon, \epsilon \Big( \Theta_\epsilon \Theta_\epsilon^\top - \lambda_\theta^+ \sum_{k=1}^{\xi^\circ} \amalg_{n, \phi_k^\circ} \Big) \bigg) = \mathbb{O}. \tag{3.38}$$

Similar to (A.2) and (A.3), it follows from Lemma 2.2 and (3.36) that

$$\Sigma_\theta^+ = \epsilon \int_0^\infty e^{B_\epsilon t} \Big( \Theta_\epsilon \Theta_\epsilon^\top - \lambda_\theta^+ \sum_{k=1}^{\xi^\circ} \amalg_{n, \phi_k^\circ} \Big) e^{B_\epsilon^\top t} dt \succeq \mathbb{O}.$$

We construct the following auxiliary Lyapunov equations:

$$\Im_c\left(\underline{\Sigma}_{\phi_k^\circ,\epsilon}, B_\epsilon, \amalg_{n,\phi_k^\circ}\right) = \mathbb{O}, \quad \forall\, k \in \mathbb{S}^0_{\xi^\circ}. \tag{3.39}$$

By a slight modification of (3.11)-(3.35), an application of (3.10) and Algorithm 2 for (3.39) yields that there exists a $\widetilde{\lambda}_\theta^+ > 0$ such that

$$\mathbf{X}^\top\Big(\lambda_\theta^+ \sum_{k=1}^{\xi^\circ} \underline{\Sigma}_{\phi_k^\circ,\epsilon}\Big)\mathbf{X} \geq \lambda_\theta^+ \widetilde{\lambda}_\theta^+ \min_{k\in\mathbb{S}^0_{\xi^\circ}} \Big\{\Big(\prod_{i=0}^{\eta_k^\circ-1} \overline{b}^{\lfloor \phi_k^\circ, i\rfloor}_{\nu^\circ_{k(i)},i}\Big)^2\Big\} \sum_{k=1}^{\xi^\circ}\sum_{j=1}^{\eta_k^\circ} (\underline{\mathbf{H}}^{(j)}_{\phi_k^\circ,j})^2. \tag{3.40}$$

Let

$$\rho_\epsilon^\circ = \epsilon \lambda_\theta^+ \widetilde{\lambda}_\theta^+ \min_{k\in\mathbb{S}^0_{\xi^\circ}} \Big\{\Big(\prod_{i=0}^{\eta_k^\circ-1} \overline{b}^{\lfloor \phi_k^\circ, i\rfloor}_{\nu^\circ_{k(i)},i}\Big)^2\Big\}. \tag{3.41}$$

Combining $\underline{\mathbf{H}}^{(1)}_{\phi_k^\circ,1} = X_{\phi_k^\circ}$, $\forall\, k \in \mathbb{S}^0_{\xi^\circ}$, (3.40) is then simplified as

$$\mathbf{X}^\top\Big(\lambda_\theta^+ \sum_{k=1}^{\xi^\circ} \underline{\Sigma}_{\phi_k^\circ,\epsilon}\Big)\mathbf{X} \geq \frac{\rho_\epsilon^\circ}{\epsilon} \sum_{k=1}^{\xi^\circ}\Big(X^2_{\phi_k^\circ} + \sum_{j=2}^{\eta_k^\circ} (\underline{\mathbf{H}}^{(j)}_{\phi_k^\circ,j})^2\Big). \tag{3.42}$$

In view of (3.8), (3.38) and (3.39), we have

$$\Sigma_\epsilon = \epsilon \lambda_\theta^+ \sum_{k=1}^{\xi^\circ} \underline{\Sigma}_{\phi_k^\circ,\epsilon} + \Sigma_\theta^+. \tag{3.43}$$

This coupled with (3.42) completes the proof. $\square$

Applying Theorems 3.1 and 3.2, we present some special cases ensuring $\Sigma_\epsilon \succ \mathbb{O}$, which are stated as in the following Corollary.

**Corollary 3.1.** *Under Assumptions 2.1 and 2.2(a), if one of the following four conditions holds:*

$$\text{(1-a)}\ \ \xi = n, \quad \text{(1-b)}\ \ \eta_{k_1} = n, \ \ \exists\, k_1 \in \mathbb{S}^0_\xi, \quad \text{(1-c)}\ \ \xi^\circ = n, \quad \text{(1-d)}\ \ \eta^\circ_{k_2} = n, \ \ \exists\, k_2 \in \mathbb{S}^0_{\xi^\circ}.$$

*Then* $\Sigma_\epsilon \succ \mathbb{O}$.

**Proof.** If case (1-a) is satisfied, it is easily seen that $\phi_j = j$, $\forall\, j \in \mathbb{S}^0_n$. Using (3.6),

$$\mathbf{X}^\top \Sigma_\epsilon \mathbf{X} \geq \rho_\epsilon \sum_{k=1}^n Y_k^2 = \rho_\epsilon |\mathbf{Y}|^2,$$

implying that $\mathbf{X}^\top \Sigma_\epsilon \mathbf{X} = 0$ if and only if $\mathbf{Y} = \mathbf{0}$. Then by Remark 3, $\Sigma_\epsilon \succ \mathbb{O}$.
If case (1-b) holds, by (3.6),

$$\begin{aligned}\mathbf{X}^\top \Sigma_\epsilon \mathbf{X} \geq &\rho_\epsilon \left( Y^2_{\phi_{k_1}} + \sum_{j=2}^{\eta_{k_1}} (\mathbf{H}^{(j)}_{\phi_{k_1},j})^2 \right) \\ =&\rho_\epsilon \sum_{j=1}^{n} (\mathbf{H}^{(j)}_{\phi_{k_1},n})^2 = \rho_\epsilon |\mathbf{H}_{\phi_{k_1},n}|^2,\end{aligned}$$

where in the first equality, we have used (3.29). In view of $\mathbf{H}_{\phi_{k_1},n} = \left[\prod_{i=0}^{n-1} (Q^{-1}_{\phi_{k_1},i})^\top P_{\phi_{k_1},i}\right] J_{\phi_{k_1}} \mathbf{Y}$, we deduce that $\mathbf{X}^\top \Sigma_\epsilon \mathbf{X} = 0$ if and only if $\mathbf{Y} = \mathbf{0}$. Using case (1-a), then $\Sigma_\epsilon \succ \mathbb{O}$.
As for case (1-c) or (1-d), the desired results can be similarly concluded from (3.37), and are omitted. □

**Remark 4.** To supplement, we first show another proof for $\Sigma_\epsilon \succ \mathbb{O}$ in case (1-b). A consequence of (3.20) and (3.28) under (1-b) is that $\Sigma_{\phi_{k_1},\epsilon} \succ \mathbb{O}$. Combining (3.5) and $\Sigma_{\phi_l,\epsilon} \succeq \mathbb{O}$ ($\forall\ l \in \mathbb{S}^0_\xi \setminus \{k_1\}$) yields $\Sigma_\epsilon \succ \mathbb{O}$. Now consider a special diffusion of (1.2):

$$G_c G_c^\top = \mathrm{diag}\{\sigma_1^2 X^2_{\epsilon,1}, ..., \sigma_n^2 X^2_{\epsilon,n}\}, \tag{3.44}$$

which is known as linear diffusion [75,76], and is a well-established way of introducing stochasticity into biologically realistic dynamic models. In this case, we have $\Theta_\epsilon \Theta_\epsilon^\top = \mathrm{diag}\{\sigma_1^2, ..., \sigma_n^2\}$. If $\sigma_i^2 \neq 0,\ \forall\ i \in \mathbb{S}^0_n$, i.e., $\xi = n$, we obtain $\Sigma_\epsilon \succ \mathbb{O}$, which substantially eases the cumbersome proofs for examining the positive definiteness of some covariance matrices in the literature (e.g., [41,45–48,73,81]). In particular, for a stochastic SICA-type HIV/AIDS infection model [45], the condition $\omega \neq \alpha + d + \frac{\sigma_4^2}{2} - \frac{\sigma_3^2}{2}$ is not required in their Theorem 5.4. To summarize, Corollary 3.1 is a generalization of existing results. Moreover, it should be noted that similar to Remark 3, the uniqueness of the IPM $\mu_\epsilon$ is not required for Theorem 3.2 and Corollary 3.1.

### 3.3. Global approximation-I

By Theorem 3.1, we determine that the density function $\Phi_\epsilon(\cdot)$ of the distribution $\mathbb{LN}_n(\ln \overline{\mathbf{X}}^*_\epsilon, \Sigma_\epsilon)$ is a local approximation for the IPDF $\Psi_\epsilon(\cdot)$ near $\overline{\mathbf{X}}^*_\epsilon$. This section is further devoted to studying the fitting effect of such approximation in global horizon.

**Theorem 3.3.** *Under Assumptions 2.1 and 2.2(a)-(b), for sufficiently small $\epsilon$,*

(i) *The IPM $\mu_\epsilon$ (resp., IPDF $\Psi_\epsilon(\cdot)$) can be globally approximated by the distribution $\mathbb{LN}_n(\ln \overline{\mathbf{X}}^*_\epsilon, \Sigma_\epsilon)$ (resp., $\Phi_\epsilon(\cdot)$). If $\Sigma_\epsilon \succ \mathbb{O}$, then*

$$\Phi_\epsilon(\mathbf{X}_\epsilon(t)) = (2\pi)^{-\frac{n}{2}} |\Sigma_\epsilon|^{-\frac{1}{2}} \left( \prod_{i=1}^{n} X_{\epsilon,i} \right)^{-1} e^{-\frac{1}{2}(\ln \mathbf{X}_\epsilon - \ln \overline{\mathbf{X}}^*_\epsilon)^\top \Sigma_\epsilon^{-1} (\ln \mathbf{X}_\epsilon - \ln \overline{\mathbf{X}}^*_\epsilon)}. \tag{3.45}$$

(ii) *For any $\gamma \in (0, 2]$, one has*

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n_+} \left|\mathbf{y} - \mathbf{X}^*\right|^\gamma \left|\Psi_\epsilon(\mathbf{y}) - \Phi_\epsilon(\mathbf{y})\right| d\mathbf{y} = 0.$$

(iii) *Under linear diffusion (3.44) and $\Sigma_\epsilon \succ \mathbb{O}$, then $\mathscr{L}_\epsilon \Phi_\epsilon(\overline{\mathbf{X}}_\epsilon^*) = 0$. If there further exist constants $\widetilde{K} > 0$, $\alpha \in (0,1)$ and a vector $\boldsymbol{\gamma}_1$ such that*

$$\sum_{i=1}^{n}\Big(\Big|\frac{f_i(\mathbf{x})}{x_i}\Big| + \big|\partial_i f_i(\mathbf{x})\big|\Big) \leq \widetilde{K} e^{\frac{1}{\epsilon^\alpha}|\ln \mathbf{x} - \boldsymbol{\gamma}_1|^2}, \quad \forall\, \mathbf{x} \in \mathbb{R}_+^n. \tag{3.46}$$

*Then*

$$\sup_{\mathbf{x}\in\mathbb{R}_+^n} \big|\mathscr{L}_\epsilon \Phi_\epsilon(\mathbf{x})\big| \leq \wp_1(\epsilon), \tag{3.47}$$

*where* $\lim_{\epsilon\to 0} \frac{\wp_1(\epsilon)}{\epsilon^2} = 0$.

**Proof.** We divide the proof into three steps.

**Step 1.** (Proof of (i)): By Assumption 2.2(b) and the Itô's formula, we obtain

$$\frac{V(\mathbf{X}_\epsilon(t))}{t} - \frac{V(\mathbf{X}_\epsilon(0))}{t} \leq -\frac{a}{t}\int_0^t \big|\mathbf{X}_\epsilon(s) - \mathbf{X}^*\big|^2 ds + \kappa(\epsilon). \tag{3.48}$$

Applying the expectation on (3.48) yields

$$\limsup_{t\to\infty} \frac{\mathbb{E}V(\mathbf{X}_\epsilon(t))}{t} \leq -a \limsup_{t\to\infty} \mathbb{E}\Big(\frac{1}{t}\int_0^t \big|\mathbf{X}_\epsilon(s) - \mathbf{X}^*\big|^2\Big) ds + \kappa(\epsilon),$$

which implies

$$\limsup_{t\to\infty} \mathbb{E}\Big(\frac{1}{t}\int_0^t \big|\mathbf{X}_\epsilon(s) - \mathbf{X}^*\big|^2\Big) ds \leq \frac{\kappa(\epsilon)}{a}. \tag{3.49}$$

Under Assumption 2.1(2), it follows from (3.49) and the dominated convergence theorem that

$$\begin{aligned}\int_{\mathbb{R}_+^n} \big|\mathbf{y} - \mathbf{X}^*\big|^2 \Psi_\epsilon(\mathbf{y}) d\mathbf{y} =& \mathbb{E}\Big(\int_{\mathbb{R}_+^n} \big|\mathbf{y} - \mathbf{X}^*\big|^2 \Psi_\epsilon(\mathbf{y}) d\mathbf{y}\Big) \\ =& \mathbb{E}\Big(\lim_{t\to\infty} \frac{1}{t}\int_0^t \big|\mathbf{X}_\epsilon(s) - \mathbf{X}^*\big|^2\Big) ds \leq \frac{\kappa(\epsilon)}{a}.\end{aligned} \tag{3.50}$$

Let $\mathbf{M}_\epsilon$ be the random variable of $\Psi_\epsilon(\cdot)$, by (3.50), then

$$\begin{aligned}\mu_\epsilon\big(\mathcal{O}(\mathbf{X}^*,\varsigma)\cap\mathbb{R}^n_+\big) =&1-\mathbb{P}\big(\big|\mathbf{M}_\epsilon-\mathbf{X}^*\big|\geq\varsigma\big)\\ \geq&1-\frac{1}{\varsigma^2}\int\limits_{\mathrm{dist}(\mathbf{y},\mathbf{X}^*)\geq\varsigma}\big|\mathbf{y}-\mathbf{X}^*\big|^2\mu_\epsilon(d\mathbf{y})\\ \geq&1-\frac{1}{\varsigma^2}\int\limits_{\mathbb{R}^n_+}\big|\mathbf{y}-\mathbf{X}^*\big|^2\Psi_\epsilon(\mathbf{y})d\mathbf{y}\\ \geq&1-\frac{\kappa(\epsilon)}{a\varsigma^2},\quad \text{for all } \varsigma>0,\end{aligned}$$

where $\mathcal{O}(\mathbf{X}^*,\varsigma):=\{z\in\mathbb{R}^n:|z-\mathbf{X}^*|\leq\varsigma\}$. Thus, $\mu_\epsilon(\mathbf{x})\xrightarrow{w}\boldsymbol{\delta}^*(\mathbf{x}-\mathbf{X}^*)$ as $\epsilon\to0$. In addition, let $\varsigma=\sqrt[3]{\kappa(\epsilon)}$. By Fatou's lemma, we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t\mathbb{P}\big(\big|\mathbf{X}_\epsilon(s)-\mathbf{X}^*\big|\leq\sqrt[3]{\kappa(\epsilon)}\big)ds\geq1-\frac{\sqrt[3]{\kappa(\epsilon)}}{a}. \tag{3.51}$$

This means that, the solution $\mathbf{X}_\epsilon(t)$ is distributed in $\mathcal{O}(\mathbf{X}^*,\sqrt[3]{\kappa(\epsilon)})$ in a large probability for some small $\epsilon$.

In view of (3.3) and (3.7), we determine that the process $\{e^{\mathbf{Y}_\epsilon(t)+\ln\overline{\mathbf{X}}^*_\epsilon}\}_{t\geq0}$ has a unique IPM $\mathbb{LN}(\overline{\mathbf{X}}^*_\epsilon,\Sigma_\epsilon)$, where $e^{\mathbf{Y}_\epsilon(t)+\ln\overline{\mathbf{X}}^*_\epsilon}:=(e^{Y_{\epsilon,1}(t)+\ln\overline{X}^*_{\epsilon,1}},...,e^{Y_{\epsilon,n}(t)+\ln\overline{X}^*_{\epsilon,n}})^\top$. Then

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t\big|e^{\mathbf{Y}_\epsilon(s)+\ln\overline{\mathbf{X}}^*_\epsilon}-\overline{\mathbf{X}}^*_\epsilon\big|^2ds=\int\limits_{\mathbb{R}^n_+}\big|\mathbf{y}-\overline{\mathbf{X}}^*_\epsilon\big|^2\Phi_\epsilon(\mathbf{y})d\mathbf{y}=\sum_{i,j=1}^n\big|\Sigma_\epsilon(i,j)\big|, \tag{3.52}$$

where $\Sigma_\epsilon(i,j)$ is the $i$th element of the $j$th row of $\Sigma_\epsilon$. Using (3.2) and (3.5), we have $\mathbb{LN}(\overline{\mathbf{X}}^*_\epsilon,\Sigma_\epsilon)\xrightarrow{w}\boldsymbol{\delta}^*(\mathbf{x}-\mathbf{X}^*)$ as $\epsilon\to0$. Thus,

$$\mu_\epsilon(\cdot)\xrightarrow{w}\mathbb{LN}(\overline{\mathbf{X}}^*_\epsilon,\Sigma_\epsilon),\quad \text{as } \epsilon\to0. \tag{3.53}$$

Intuitively, such weak convergence is established in the sense of the Dirac measure. To this end, we should further focus on the approximate degree of $\mu_\epsilon(\cdot)$ and $\mathbb{LN}(\overline{\mathbf{X}}^*_\epsilon,\Sigma_\epsilon)$ in terms of curve shape when $\epsilon\to0$.

By (3.50) and (3.52), there holds

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\int\limits_0^t\big|\mathbf{X}_\epsilon(s)-e^{\mathbf{Y}_\epsilon(s)+\ln\overline{\mathbf{X}}^*_\epsilon}\big|^2ds\leq&3\limsup_{t\to\infty}\frac{1}{t}\int\limits_0^t\big|\mathbf{X}_\epsilon(s)-\mathbf{X}^*_\epsilon\big|^2ds\\ &+3\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t\big|\mathbf{X}^*_\epsilon-\overline{\mathbf{X}}^*_\epsilon\big|^2ds\end{aligned}$$

$$
\begin{aligned}
&+3\lim_{t\to\infty}\frac{1}{t}\int_0^t \left|e^{\mathbf{Y}_\epsilon(s)+\ln\overline{\mathbf{X}}_\epsilon^*}-\overline{\mathbf{X}}_\epsilon^*\right|^2 ds\\
\leq&3\Big(\frac{\kappa(\epsilon)}{a}+\left|\mathbf{X}_\epsilon^*-\overline{\mathbf{X}}_\epsilon^*\right|^2+\sum_{i,j=1}^n\left|\Sigma_\epsilon(i,j)\right|\Big)\\
:=&\widehat{\kappa}(\epsilon).
\end{aligned}
\tag{3.54}
$$

Clearly, $\lim_{\epsilon\to 0}\widehat{\kappa}(\epsilon)=0$. Hence, the difference of two solutions in the mean sense is small for sufficiently small $\epsilon$. Combined with (3.51), (3.53) and (3.54), the desired result (i) is obtained.

**Step 2.** (Proof of (ii)): For any $\gamma\in(0,2]$, an application of Hölder inequality for (3.50) and (3.52) yields

$$
\begin{aligned}
&\int_{\mathbb{R}_+^n}\left|\mathbf{y}-\mathbf{X}^*\right|^\gamma\left|\Psi_\epsilon(\mathbf{y})-\Phi_\epsilon(\mathbf{y})\right|d\mathbf{y}\\
&\leq\int_{\mathbb{R}_+^n}\left|\mathbf{y}-\mathbf{X}^*\right|^\gamma\Psi_\epsilon(\mathbf{y})d\mathbf{y}+\int_{\mathbb{R}_+^n}\left|\mathbf{y}-\overline{\mathbf{X}}_\epsilon^*\right|^2\Phi_\epsilon(\mathbf{y})d\mathbf{y}\\
&\leq\Big(\int_{\mathbb{R}_+^n}\left|\mathbf{y}-\mathbf{X}^*\right|^2\Psi_\epsilon(\mathbf{y})d\mathbf{y}\Big)^{\frac{\gamma}{2}}+\Big(\int_{\mathbb{R}_+^n}\left|\mathbf{y}-\overline{\mathbf{X}}_\epsilon^*\right|^2\Phi_\epsilon(\mathbf{y})d\mathbf{y}\Big)^{\frac{\gamma}{2}}\\
&\leq\Big(\frac{\kappa(\epsilon)}{a}\Big)^{\frac{\gamma}{2}}+\Big(\sum_{i,j=1}^n\left|\Sigma_\epsilon(i,j)\right|\Big)^{\frac{\gamma}{2}}.
\end{aligned}
$$

Then by (3.5) and Assumption 2.2(b), we obtain the assertion (ii).

**Step 3.** (Proof of (iii)): Denote

$$
\mathcal{T}_p=\sum_{i=1}^n\partial_i\big(f_i(\mathbf{x})\Phi_\epsilon(\mathbf{x})\big),\quad \mathcal{T}_q=\frac{\epsilon}{2}\sum_{i,j=1}^n\partial_{ij}^2\big(g_{ij}^c(\mathbf{x})\Phi_\epsilon(\mathbf{x})\big),\quad \forall\,\mathbf{x}\in\mathbb{R}_+^n.
$$

Under (3.44), by (1.3) and (3.45), one has

$$
\begin{cases}
\mathcal{T}_q=\frac{\epsilon}{2}\sum\limits_{i=1}^n\sigma_i^2\big(2\Phi_\epsilon(\mathbf{x})+4x_i\partial_i\Phi_\epsilon(\mathbf{x})+x_i^2\partial_{ii}^2\Phi_\epsilon(\mathbf{x})\big),\\
\mathcal{T}_p=\sum\limits_{i=1}^n\big(\Phi_\epsilon(\mathbf{x})\partial_i f_i(\mathbf{x})+f_i(\mathbf{x})\partial_i\Phi_\epsilon(\mathbf{x})\big).
\end{cases}
$$

This yields

$$
\begin{aligned}
\left|\mathscr{L}_\epsilon\Phi_\epsilon(\mathbf{x})\right|=&\left|\mathcal{T}_q-\mathcal{T}_p\right|\\
=&\left|\sum_{i=1}^n\Big(\big(\epsilon\sigma_i^2-\partial_i f_i(\mathbf{x})\big)\Phi_\epsilon(\mathbf{x})+\big(2\epsilon\sigma_i^2x_i-f_i(\mathbf{x})\big)\partial_i\Phi_\epsilon(\mathbf{x})+\frac{\epsilon\sigma_i^2}{2}x_i^2\partial_{ii}^2\Phi_\epsilon(\mathbf{x})\Big)\right|.
\end{aligned}
\tag{3.55}
$$

To proceed, let $\Sigma_\epsilon^{-1} := (m_{ij})_{n\times n}$. By direct calculation,

$$\begin{aligned}\partial_i \Phi_\epsilon(\mathbf{x}) =&(2\pi)^{-\frac{n}{2}}|\Sigma_\epsilon|^{-\frac{1}{2}}\Big[-x_i^{-1}\Big(\prod_{i=1}^n x_i\Big)^{-1} e^{-\frac{1}{2}(\ln\mathbf{x}-\ln\overline{\mathbf{X}}_\epsilon^*)^\top\Sigma_\epsilon^{-1}(\ln\mathbf{x}-\ln\overline{\mathbf{X}}_\epsilon^*)} \\ &- e^{-\frac{1}{2}(\ln\mathbf{x}-\ln\overline{\mathbf{X}}_\epsilon^*)^\top\Sigma_\epsilon^{-1}(\ln\mathbf{x}_\epsilon-\ln\overline{\mathbf{X}}_\epsilon^*)}\Big(\prod_{i=1}^n x_i\Big)^{-1}\sum_{k=1}^n \frac{m_{ki}(\ln x_k-\ln\overline{X}_{\epsilon,k}^*)}{x_i}\Big] \\ =&-\frac{\Phi_\epsilon(\mathbf{x})}{x_i}\Big(1+\sum_{k=1}^n m_{ki}\big(\ln x_k-\ln\overline{X}_{\epsilon,k}^*\big)\Big).\end{aligned} \tag{3.56}$$

Then by (3.56),

$$\begin{aligned}\partial_{ii}^2\Phi_\epsilon(\mathbf{x}) =& -\partial_i\bigg(\frac{\Phi_\epsilon(\mathbf{x})}{x_i}\Big(1+\sum_{k=1}^n m_{ki}\big(\ln x_k-\ln\overline{X}_{\epsilon,k}^*\big)\Big)\bigg) \\ =&\frac{\Phi_\epsilon(\mathbf{x})}{x_i^2}\bigg[\Big(1+\sum_{k=1}^n m_{ki}\big(\ln x_k-\ln\overline{X}_{\epsilon,k}^*\big)\Big)^2+\Big(1+\sum_{k=1}^n m_{ki}\big(\ln x_k-\ln\overline{X}_{\epsilon,k}^*\big)\Big)-m_{ii}\bigg] \\ =&\frac{\Phi_\epsilon(\mathbf{x})}{x_i^2}\bigg[(2-m_{ii})+3\sum_{k=1}^n m_{ki}\big(\ln x_k-\ln\overline{X}_{\epsilon,k}^*\big)+\Big(\sum_{k=1}^n m_{ki}\big(\ln x_k-\ln\overline{X}_{\epsilon,k}^*\big)\Big)^2\bigg].\end{aligned} \tag{3.57}$$

For simplicity, let

$$d_{\mathbb{L},i}(\mathbf{x},\overline{\mathbf{X}}_\epsilon^*) = \sum_{k=1}^n m_{ki}\big(\ln x_k-\ln\overline{X}_{\epsilon,k}^*\big).$$

Inserting (3.56) and (3.57) into (3.55) leads to

$$\begin{aligned}\big|\mathscr{L}_\epsilon\Phi_\epsilon(\mathbf{x})\big| =&\Phi_\epsilon(\mathbf{x})\bigg|\sum_{i=1}^n\Big[\big(\epsilon\sigma_i^2-\partial_i f_i(\mathbf{x})\big)-\frac{2\epsilon\sigma_i^2 x_i-f_i(\mathbf{x})}{x_i}\big(1+d_{\mathbb{L},i}(\mathbf{x},\overline{\mathbf{X}}_\epsilon^*)\big) \\ &+\frac{\epsilon\sigma_i^2}{2}\Big((2-m_{ii})+3d_{\mathbb{L},i}(\mathbf{x},\overline{\mathbf{X}}_\epsilon^*)+d_{\mathbb{L},i}^2(\mathbf{x},\overline{\mathbf{X}}_\epsilon^*)\Big)\Big]\bigg| \\ =&\Phi_\epsilon(\mathbf{x})\bigg|\sum_{i=1}^n\Big(\frac{f_i(\mathbf{x})}{x_i}-\frac{\epsilon\sigma_i^2 m_{ii}}{2}-\partial_i f_i(\mathbf{x})\Big) \\ &+\sum_{i=1}^n d_{\mathbb{L},i}(\mathbf{x},\overline{\mathbf{X}}_\epsilon^*)\Big(\frac{f_i(\mathbf{x})}{x_i}+\frac{\epsilon\sigma_i^2}{2}\big(d_{\mathbb{L},i}(\mathbf{x},\overline{\mathbf{X}}_\epsilon^*)-1\big)\Big)\bigg|.\end{aligned} \tag{3.58}$$

Intuitively, (3.8) under $\Sigma_\epsilon \succ \mathbb{O}$ is equivalent to

$$B_\epsilon^\top + \Sigma_\epsilon^{-1}B_\epsilon\Sigma_\epsilon = -\epsilon\Sigma_\epsilon^{-1}\Theta_\epsilon\Theta_\epsilon^\top,$$

and hence

$$-\mathrm{trace}(\Sigma_\epsilon^{-1}\Theta_\epsilon\Theta_\epsilon^\top)=\frac{1}{\epsilon}\big(\mathrm{trace}(B_\epsilon^\top)+\mathrm{trace}(\Sigma_\epsilon^{-1}B_\epsilon\Sigma_\epsilon)\big)=\frac{2}{\epsilon}\mathrm{trace}(B_\epsilon).$$

As in (3.44), we have $\Theta_\epsilon\Theta_\epsilon^\top=\mathrm{diag}\{\sigma_1^2,...,\sigma_n^2\}$, i.e., $\mathrm{trace}(\Sigma_\epsilon^{-1}\Theta_\epsilon\Theta_\epsilon^\top)=\sum_{i=1}^n\sigma_i^2 m_{ii}$. Using Assumption 2.2(a) and $\partial_j f_j(\mathbf{x})=x_j^{-1}\frac{\partial f_j(\mathbf{x})}{\partial(\ln x_j)}$, $\forall\, j\in\mathbb{S}_n^0$, then

$$\mathrm{trace}(B_\epsilon)=-\sum_{i=1}^n\Big(\frac{f_i(\mathbf{x})}{x_i}-\partial_i f_i(\mathbf{x})\Big)\Big|_{\mathbf{x}=\overline{\mathbf{X}}_\epsilon^*},$$

which implies

$$\sum_{i=1}^n\Big(\frac{f_i(\mathbf{x})}{x_i}-\frac{\epsilon\sigma_i^2 m_{ii}}{2}-\partial_i f_i(\mathbf{x})\Big)\Big|_{\mathbf{x}=\overline{\mathbf{X}}_\epsilon^*}=0. \tag{3.59}$$

Thus, $\mathscr{L}_\epsilon\Phi_\epsilon(\overline{\mathbf{X}}_\epsilon^*)=0$. By (3.2), (3.3) and Assumption 2.2(a), there holds

$$\begin{cases}\lim\limits_{\epsilon\to0}\Theta_\epsilon\Theta_\epsilon^\top=(g_{ij}(\mathbf{X}^*))_{n\times N}(g_{ij}(\mathbf{X}^*))_{n\times N}^\top\succeq\mathbb{O},\\ \lim\limits_{\epsilon\to0}B_\epsilon=\Big(\dfrac{\partial F_i(\mathbf{X}^*)}{\partial(\ln x_j)}\Big)_{n\times n}\in\overline{\mathbf{RH}}(n).\quad\text{(constant matrix)}\end{cases}$$

Combining (3.5) and (3.8) yields that, for any $i,j\in\mathbb{S}_n^0$ and $u>0$,

$$\lim_{\epsilon\to0}\epsilon^{-1}|\Sigma_\epsilon(i,j)|=\iota_{ij}<\infty,\quad\text{and}\quad\lim_{\epsilon\to0}\epsilon^{-(1+u)}|\Sigma_\epsilon(i,j)|=\infty\ (\text{or }0). \tag{3.60}$$

By virtue of (3.46) and (3.58)-(3.60), we determine that there are constants $\varsigma_0\in(0,\min_{k\in\mathbb{S}_n^0}\overline{X}_{\epsilon,k}^*)$ and $m_0>0$ such that

$$\lim_{\epsilon\to0}\Big(\frac{1}{\epsilon^2}\sup_{\mathbf{x}\in\mathcal{O}(\overline{\mathbf{X}}_\epsilon^*,\varsigma_0)}\big|\mathscr{L}_\epsilon\Phi_\epsilon(\mathbf{x})\big|\Big)=0, \tag{3.61}$$

and

$$\begin{aligned}\big|\mathscr{L}_\epsilon\Phi_\epsilon(\mathbf{x})\big|\le&\Phi_\epsilon(\mathbf{x})\Big|m_0\Big(1+\sum_{i=1}^n d_{\mathbb{L},i}^2(\mathbf{x},\overline{\mathbf{X}}_\epsilon^*)\Big)\widetilde{K}e^{\frac{1}{\epsilon^\alpha}|\ln\mathbf{x}-\boldsymbol{\gamma}_1|^2}\Big|\\ =&m_0\widetilde{K}(2\pi)^{-\frac{n}{2}}\frac{(1+\sum_{i=1}^n d_{\mathbb{L},i}^2(\mathbf{x},\overline{\mathbf{X}}_\epsilon^*))e^{\frac{1}{\epsilon^\alpha}|\ln\mathbf{x}-\boldsymbol{\gamma}_1|^2}}{|\Sigma_\epsilon|^{\frac12}e^{\frac12(\ln\mathbf{x}-\ln\overline{\mathbf{X}}_\epsilon^*)^\top\Sigma_\epsilon^{-1}(\ln\mathbf{x}-\ln\overline{\mathbf{X}}_\epsilon^*)}\prod_{j=1}^n x_j},\quad\forall\,\mathbf{x}\in\mathbb{R}_+^n\setminus\mathcal{O}(\overline{\mathbf{X}}_\epsilon^*,\varsigma_0).\end{aligned} \tag{3.62}$$

Using (3.60), we obtain

$$\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^2}\sup_{\mathbf{x}\in\mathbb{R}_+^n\setminus\mathcal{O}(\overline{\mathbf{X}}_\epsilon^*,\varsigma_0)}\left\{\frac{(1+\sum_{i=1}^n d_{\mathbb{L},i}^2(\mathbf{x},\overline{\mathbf{X}}_\epsilon^*))e^{\frac{1}{\epsilon^\alpha}|\ln\mathbf{x}-\boldsymbol{\gamma}_1|^2}}{|\Sigma_\epsilon|^{\frac{1}{2}}e^{\frac{1}{2}(\ln\mathbf{x}-\ln\overline{\mathbf{X}}_\epsilon^*)^\top\Sigma_\epsilon^{-1}(\ln\mathbf{x}-\ln\overline{\mathbf{X}}_\epsilon^*)}\prod_{j=1}^n x_j}\right\}\right)=0. \tag{3.63}$$

Then the desired result (iii) follows from (3.61)-(3.63). □

**Remark 5.** Two remarks on Theorem 3.3 are shown as follows.

- Throughout the rest of the paper, let $\mu_\epsilon^\partial(X_{\epsilon,k_1},...,X_{\epsilon,k_l})$ be the marginal measure of $\mu_\epsilon$ with respect to the variables $X_{\epsilon,k_1},...,X_{\epsilon,k_l}$, and its probability density is denoted by $\Psi_\epsilon^\partial(X_{\epsilon,k_1},...,X_{\epsilon,k_l})$. Similarly, we define $\Phi_\epsilon^\partial(X_{k_1},...,X_{k_l})$ as the marginal density of $\mathbb{LN}_n(\ln\overline{\mathbf{X}}_\epsilon^*,\Sigma_\epsilon)$ involving the components $(X_{k_1},...,X_{k_l})$. In this sense, we combine Theorem 3.3 to obtain that, for sufficiently small $\epsilon>0$, the measure $\mu_\epsilon^\partial(X_{\epsilon,i})$ approximately has a log-normal density function $\Phi_\epsilon^\partial(X_{\epsilon,i})$, which takes the form

$$\Phi_\epsilon^\partial(X_{\epsilon,i})=\frac{1}{X_{\epsilon,i}\sqrt{2\pi\,\Sigma_\epsilon(i,i)}}e^{-\frac{(\ln X_{\epsilon,i}-\ln\overline{X}_{\epsilon,i}^*)^2}{2\Sigma_\epsilon(i,i)}},\quad \forall\, i\in\mathbb{S}_n^0,$$

  where $\Sigma_\epsilon\succ\mathbb{O}$ is required.
- Under linear diffusion, (3.47) is equivalent to

$$\begin{cases}\left|\mathscr{L}_\epsilon\Phi_\epsilon(\mathbf{x})\right|\le\wp_1(\epsilon),\quad \forall\,\mathbf{x}\in\mathbb{R}_+^n,\\ \Phi_\epsilon(\mathbf{x})\ge 0,\quad \int_{\mathbb{R}_+^n}\Phi_\epsilon(\mathbf{x})d\mathbf{x}=1.\end{cases}\tag{3.64}$$

  Then in practical terms, $\Phi_\epsilon(\cdot)$ can be regarded as the numerical solution of (1.3) if $\wp(\epsilon)$ is in the allowed error range. That is, $\Phi_\epsilon(\cdot)$ is relatively close to $\Psi_\epsilon(\cdot)$ in the sense of KFP equation. Clearly, such property can be generalized to slightly complex diffusion settings (e.g., $G_cG_c^\top=(\sigma_{ij}X_{\epsilon,i}^2)_{n\times n}$).

## 4. Updated normal approximation (uNA) for IPDF

Section 3 provides a LNA method for the IPM $\mu_\epsilon$ (or IPDF $\Psi_\epsilon(\cdot)$) of (1.2) in local and global horizons. Inspired by the idea, an updated version of the existing normal approximation for $\Psi_\epsilon(\cdot)$ is presented in this section.

### 4.1. Local approximation-II

By Taylor expansion, the linearized equation of (1.2) near $\mathbf{X}^*$ is

$$\begin{cases}d\mathbf{Z}_\epsilon(t)=\left(\frac{\partial f_i(\boldsymbol{x})}{\partial x_j}\right)_{n\times n}\Big|_{\boldsymbol{x}=\mathbf{X}^*}\mathbf{Z}_\epsilon(t)dt+\sqrt{\epsilon}G_c|_{\boldsymbol{x}=\mathbf{X}^*}d\mathbf{W}(t)\\ \qquad\quad :=C_{[o]}\mathbf{Z}_\epsilon(t)dt+\sqrt{\epsilon}\varGamma d\mathbf{W}(t),\\ \mathbf{Z}_\epsilon(0)=\mathbf{x}_0-\mathbf{X}^*.\end{cases}\tag{4.1}$$

Under Assumption 2.2(b), by a similar argument in (3.4), we can find an orthogonal matrix $\mathcal{H}_{[o]}$ and a set $\overline{\boldsymbol{\phi}}:=\{\overline{\phi}_1,...,\overline{\phi}_{\overline{\xi}}\}\subseteq\mathbb{S}_n^0$ such that

$$\mathcal{H}_{[o]}(\Gamma\Gamma^\top)\mathcal{H}_{[o]}^\top = \sum_{k=1}^{\overline{\xi}} \overline{\lambda}_k^+ \amalg_{n,\overline{\phi}_k}, \tag{4.2}$$

where $\overline{\lambda}_k^+$ ($k \in \mathbb{S}_{\overline{\xi}}^0$) are all the positive eigenvalues of $\Gamma\Gamma^\top$, and $\overline{\phi}_j > \overline{\phi}_i$, $\forall\ j > i$. Clearly, $\overline{\xi} = \mathrm{rank}(\Gamma\Gamma^\top)$.
To proceed, let

$$A_{[o]} = \mathcal{H}_{[o]} C_{[o]} \mathcal{H}_{[o]}^{-1}.$$

Then we can mimic the proof of Theorem 3.1 to obtain the uNA method for $\mu_\epsilon(\cdot)$ in local horizon, which is stated as in the following Theorem.

**Theorem 4.1.** *Under Assumptions 2.1 and 2.2(c), the IPM $\mu_\epsilon$ near $\mathbf{X}^*$ can be approximately by a normal distribution $\mathbb{N}_n(\mathbf{X}^*, \Sigma_{[o]\epsilon})$ (with $\Phi_{[o]\epsilon}(\cdot)$ denoting its density), where*

$$\Sigma_{[o]\epsilon} = \epsilon \mathcal{H}_{[o]}^\top \Big( \sum_{k=1}^{\overline{\xi}} \overline{\lambda}_k^+ \Sigma_{[o]\overline{\phi}_k,\epsilon} \Big) \mathcal{H}_{[o]}, \tag{4.3}$$

*with $\Sigma_{[o]\overline{\phi}_k,\epsilon}$ shown in Algorithm 3. In addition, let $\mathbf{Z} = \mathcal{H}_0\mathbf{X}$ and $\mathbf{G}_{\overline{\phi}_k,j} = [\sum_{i=0}^{j-1} (Q_{[o]\overline{\phi}_k,i}^{-1})^\top P_{[o]\overline{\phi}_k,i}] J_{[o]\overline{\phi}_k} \mathbf{Z}$, then*

$$\mathbf{X}^\top \Sigma_{[o]\epsilon} \mathbf{X} \geq \varrho_\epsilon \sum_{k=1}^{\overline{\xi}} \Big( Z_{\overline{\phi}_k}^2 + \sum_{j=2}^{\overline{\eta}_k} (\mathbf{G}_{\overline{\phi}_k,j}^{(j)})^2 \Big), \tag{4.4}$$

*where $\mathbf{X}$ is the same as in (3.6), $\varrho_\epsilon > 0$ is determined later, and $\overline{\eta}_k$, $J_{[o]\overline{\phi}_k}$, $P_{[o]\overline{\phi}_k,i}$ and $Q_{[o]\overline{\phi}_k,i}$ ($\forall\ i \in \mathbb{S}_{\overline{\eta}_k}^0$) are defined in Algorithm 3.*

**Proof.** System (4.1) has an explicit solution

$$\mathbf{Z}_\epsilon(t) = e^{C_{[o]}t}\mathbf{Z}_\epsilon(0) + \sqrt{\epsilon} \int_0^t e^{C_{[o]}(t-\tau)} \Gamma d\mathbf{W}(\tau).$$

Based on Assumption 2.2(c), we have $C_{[o]} \in \overline{\mathbf{RH}}(n)$. Using a similar argument in (3.7) yields that, the solution process $\{\mathbf{Z}_\epsilon(t)\}_{t\geq 0}$ of (4.1) has a unique stationary distribution $\mathbb{N}_n(\mathbf{0}, \Sigma_{[o]\epsilon})$, where

$$\Sigma_{[o]\epsilon} = \epsilon \int_0^\infty e^{C_{[o]}t} \Gamma\Gamma^\top e^{C_{[o]}^\top t} dt. \tag{4.5}$$

As a consequence of (4.2), (4.5), (A.2) and (A.3), $\Sigma_{[o]\epsilon} \succeq \mathbb{O}$ and it satisfies

**Algorithm 3:** Algorithm for obtaining $\Sigma_{[o]\overline{\phi}_k,\epsilon}$.

**Input:** $A_{[o]}$, $\overline{\phi}_k$ (or $\overline{\boldsymbol{\phi}}$).

**Output:** $\overline{\eta}_k$, $\Sigma_{[o]\overline{\phi}_k,\epsilon} = (\prod_{i=0}^{\overline{\eta}_k-1} \overline{a}_{\overline{v}_{k(i)},i}^{\lfloor [o]\overline{\phi}_k,i \rfloor})^2 \times$
$[M_{[o]\overline{\phi}_k,\overline{\eta}_k}(\prod_{i=0}^{\overline{\eta}_k-1} Q_{[o]\overline{\phi}_k,i} P_{[o]\overline{\phi}_k,i}) J_{[o]\overline{\phi}_k}]^{-1} \Delta_{[o]\overline{\phi}_k,\overline{\eta}_k} \{[M_{[o]\overline{\phi}_k,\overline{\eta}_k}(\prod_{i=0}^{\overline{\eta}_k-1} Q_{[o]\overline{\phi}_k,i} P_{[o]\overline{\phi}_k,i}) J_{[o]\overline{\phi}_k}]^{-1}\}^{\top}$[a].

**(Initialization):** $\overline{\eta}_k = 1$, $\overline{A}_{[o]\overline{\phi}_k,1} = J_{[o]\overline{\phi}_k} A_{[o]} J_{[o]\overline{\phi}_k}^{-1}$;
**for** $i = 1 : n-1$ **do**
  **if** $\sum_{j=i+1}^{n} (\overline{a}_{ji}^{\lfloor [o]\overline{\phi}_k,i \rfloor})^2 = 0$ **then**
    $\overline{\eta}_k = i$;
    **break**;
  **else**
    Choose a "**<u>suitable</u>**[b]" $\overline{v}_{k(i)} \in \mathbb{S}_n^i$ such that $\overline{a}_{\overline{v}_{k(i)},i}^{\lfloor [o]\overline{\phi}_k,i \rfloor} \neq 0$;
    Let $\widehat{A}_{[o]\overline{\phi}_k,i} = P_{[o]\overline{\phi}_k,i} \overline{A}_{[o]\overline{\phi}_k,i} P_{[o]\overline{\phi}_k,i}^{-1}$ and $\overline{A}_{[o]\overline{\phi}_k,i+1} := Q_{[o]\overline{\phi}_k,i} \widehat{A}_{[o]\overline{\phi}_k,i} Q_{[o]\overline{\phi}_k,i}^{-1}$;
  **end**
  $\overline{\eta}_k$++;
**end**
Let $A_{[o]s,\overline{\phi}_k} = M_{[o]\overline{\phi}_k,\overline{\eta}_k} \overline{A}_{[o]\overline{\phi}_k,\overline{\eta}_k} M_{[o]\overline{\phi}_k,\overline{\eta}_k}^{-1}$;
Obtain a standard $L_0$-algebraic equation $\Im_c\big(\Xi_{[o]\overline{\phi}_k,\overline{\eta}_k}, A_{[o]s,\overline{\phi}_k}^{(\overline{\eta}_k)}, \amalg_{\overline{\eta}_k,1}\big) = \mathbb{O}$;
**return** $\overline{\eta}_k$, $\Xi_{[o]\overline{\phi}_k,\overline{\eta}_k}$, $\Sigma_{[o]\overline{\phi}_k,\epsilon}$.

[a] $J_{[o]\overline{\phi}_k}$, $M_{\overline{\phi}_k,\overline{\eta}_k}$, $P_{[o]\overline{\phi}_k,i}$ and $Q_{[o]\overline{\phi}_k,i}$ are called the order, standardized, the $i$th rotation and elimination $\overline{\phi}_k$-$A_{[o]}$ matrices, respectively, $\forall\, i \in \mathbb{S}_{\overline{\eta}_k-1}^{-1}$. Similar to Algorithm 1, $\overline{a}_{\overline{v}_{k(0)},0}^{\lfloor [o]\overline{\phi}_k,0 \rfloor} = 1$, $P_{[o]\overline{\phi}_k,l} = Q_{[o]\overline{\phi}_k,l} = \mathbf{I}_n$ ($\forall\, l \in \{0, n-1\}$), and $J_{[o]\overline{\phi}_k}$, $P_{[o]\overline{\phi}_k,i}$ and $Q_{[o]\overline{\phi}_k,i}$ ($i \in \mathbb{S}_{\overline{\eta}_k-1}^{0}$) have the same form as $J_{\phi_k}$, $P_{\phi_k,i}$ and $Q_{\phi_k,i}$ by replacing $(\phi_k, \nu_{k(i)}, \boldsymbol{\ell}_{k,n-1-i})$ with $(\overline{\phi}_k, \overline{v}_{k(i)}, \boldsymbol{\ell}_{[o]k,n-1-i})$, where $\boldsymbol{\ell}_{[o]k,n-1-i} = \frac{-1}{\widehat{a}_{i+1,i}^{\lfloor [o]\overline{\phi}_k,i \rfloor}} (\widehat{a}_{i+2,i}^{\lfloor [o]\overline{\phi}_k,i \rfloor}, \ldots, \widehat{a}_{n,i}^{\lfloor [o]\overline{\phi}_k,i \rfloor})^{\top}$. Moreover, $\Xi_{[o]\overline{\phi}_k,\overline{\eta}_k}$ is shown in (4.10), and

$$M_{[o]\overline{\phi}_k,\overline{\eta}_k} = \begin{pmatrix} \mathcal{M}_{[o]\overline{\eta}_k} & \mathbb{O} \\ \mathbb{O} & \mathbf{I}_{n-\overline{\eta}_k} \end{pmatrix}, \quad \mathcal{M}_{[o]\overline{\eta}_k} = \begin{pmatrix} \boldsymbol{\beta}_{\overline{\eta}_k} (\overline{A}_{[o]\overline{\phi}_k,\overline{\eta}_k}^{(\overline{\eta}_k)})^{\overline{\eta}_k-1} \\ \boldsymbol{\beta}_{\overline{\eta}_k} (\overline{A}_{[o]\overline{\phi}_k,\overline{\eta}_k}^{(\overline{\eta}_k)})^{\overline{\eta}_k-2} \\ \cdots \\ \boldsymbol{\beta}_{\overline{\eta}_k} \end{pmatrix}, \quad \Delta_{[o]\overline{\phi}_k,\overline{\eta}_k} = \begin{pmatrix} \Xi_{[o]\overline{\phi}_k,\overline{\eta}_k} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}.$$

[b] The choice of $\overline{v}_{k(i)}$ is conducive to verifying $\Sigma_{[o]\epsilon} \succ \mathbb{O}$. More details refer to Section 5.5.

$$\Im_c\big(\Sigma_{[o]\epsilon}, C_{[o]}, \epsilon \Gamma_\epsilon \Gamma_\epsilon^{\top}\big) = \mathbb{O}, \tag{4.6}$$

or equivalently,

$$\Im_c\Big(\frac{1}{\epsilon} \mathcal{H}_{[o]} \Sigma_{[o]\epsilon} \mathcal{H}_{[o]}^{\top}, A_{[o]}, \sum_{k=1}^{\overline{\xi}} \overline{\lambda}_k^{+} \amalg_{n,\overline{\phi}_k}\Big) = \mathbb{O}. \tag{4.7}$$

Let $\Sigma_{[o]\overline{\phi}_k,\epsilon}$ ($k \in \mathbb{S}_{\overline{\xi}}$) be the solutions of the following algebraic equations, respectively:

$$\Im_c\left(\Sigma_{[o]\overline{\phi}_k,\epsilon}, A_{[o]}, \amalg_{n,\phi_k}\right) = \mathbb{O}. \tag{4.8}$$

The assertion (4.3) can follows from (4.7), (4.8) and the superposition principle.

In view of (4.6), and the relationship between (4.1) near $\mathbf{X}^*$ and (1.2), we obtain that the solution $\mathbf{X}_\epsilon(t)$ near $\mathbf{X}^*$ is approximated by $\mathbf{Z}_\epsilon(t) + \mathbf{X}^*$, i.e., the distribution $\mathbb{N}_n(\mathbf{X}^*, \Sigma_{[o]\epsilon})$ can be a local approximation for $\mu_\epsilon$ around $\mathbf{X}^*$. Therefore, the first part of Theorem 4.1 is proved.

According to the theoretical schemes in Algorithm 3, for any $k \in \mathbb{S}^0_{\overline{\xi}}$, we construct an $\overline{\eta}_k$-dimensional Lyapunov equation

$$\Im_c\left(\Xi_{[o]\overline{\phi}_k,\overline{\eta}_k}, A^{(\overline{\eta}_k)}_{[o]s,\overline{\phi}_k}, \amalg_{\overline{\eta}_k,1}\right) = \mathbb{O}. \tag{4.9}$$

Then we can mimic the analysis of (3.10)-(3.20) to prove that (4.9) is a standard $L_0$-algebraic equation, i.e., $A^{(\overline{\eta}_k)}_{[o]s,\overline{\phi}_k} \in \mathscr{S}(\overline{\eta}_k)$ and

$$\Xi_{[o]\overline{\phi}_k,\overline{\eta}_k} = \begin{pmatrix} \overline{\zeta}_1 & 0 & -\overline{\zeta}_2 & 0 & \overline{\zeta}_3 & \cdots \\ 0 & \overline{\zeta}_2 & 0 & -\overline{\zeta}_3 & \cdots & \iddots \\ -\overline{\zeta}_2 & 0 & \overline{\zeta}_3 & \cdots & \iddots & 0 \\ 0 & -\overline{\zeta}_3 & \cdots & \iddots & 0 & -\overline{\zeta}_{\overline{\eta}_k-1} \\ \overline{\zeta}_3 & \cdots & \iddots & 0 & \overline{\zeta}_{\overline{\eta}_k-1} & 0 \\ \vdots & \iddots & 0 & -\overline{\zeta}_{\overline{\eta}_k-1} & 0 & \overline{\zeta}_{\overline{\eta}_k} \end{pmatrix} \succ \mathbb{O}, \tag{4.10}$$

where $(\overline{\zeta}_1, -\overline{\zeta}_2, ..., (-1)^{\overline{\eta}_k-1}\overline{\zeta}_{\overline{\eta}_k})^\top = \frac{1}{2}\mathscr{H}^{-1}_{\overline{\eta}_k, A^{(\overline{\eta}_k)}_{[o]s,\overline{\phi}_k}} \mathbf{e}_{\overline{\eta}_k}$.

Also by proceeding a complex procedure similar to (3.20)-(3.30), we derive

$$\mathbf{G}^{\langle j\rangle}_{\overline{\phi}_k,m} = \mathbf{G}^{\langle j\rangle}_{\overline{\phi}_k,j}, \quad \forall\, j \in \mathbb{S}^0_{\overline{\eta}_k};\, m \in \mathbb{S}^j_{\overline{\eta}_k}, \tag{4.11}$$

and

$$\begin{aligned}\Sigma_{[o]\overline{\phi}_k,\epsilon} =& \left(\prod_{i=0}^{\overline{\eta}_k-1} a^{\lfloor [o]\overline{\phi}_k,i\rfloor}_{\overline{v}_{k(i)},i}\right)^2 \left(M_{[o]\overline{\phi}_k,\overline{\eta}_k}\Big(\prod_{i=0}^{\overline{\eta}_k-1} Q_{[o]\overline{\phi}_k,i} P_{[o]\overline{\phi}_k,i}\Big) J_{[o]\overline{\phi}_k}\right)^{-1} \\ &\times \begin{pmatrix} \Xi_{[o]\overline{\phi}_k,\overline{\eta}_k} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \left[\left(M_{[o]\overline{\phi}_k,\overline{\eta}_k}\Big(\prod_{i=0}^{\overline{\eta}_k-1} Q_{[o]\overline{\phi}_k,i} P_{[o]\overline{\phi}_k,i}\Big) J_{[o]\overline{\phi}_k}\right)^{-1}\right]^\top.\end{aligned} \tag{4.12}$$

Thus, $\Sigma_{[o]\overline{\phi}_k,\overline{\eta}_k}$ in Algorithm 3 is verified.

To proceed, let $\widehat{\lambda}_k$ be the minimal eigenvalue of $\mathcal{M}^{-1}_{[o]\overline{\eta}_k}\Xi_{[o]\overline{\phi}_k,\overline{\eta}_k}(\mathcal{M}^{-1}_{[o]\overline{\eta}_k})^\top$. Using (4.10), we have $\widehat{\lambda}_k > 0$ and

$$\mathcal{M}^{-1}_{[o]\overline{\eta}_k}\Xi_{[o]\overline{\phi}_k,\overline{\eta}_k}(\mathcal{M}^{-1}_{[o]\overline{\eta}_k})^\top \succeq \widehat{\lambda}_k \mathbf{I}_{\overline{\eta}_k}. \tag{4.13}$$

Combining (4.3), (4.11)-(4.13) and the definition of $\mathbf{Z}$ yields

$$
\begin{aligned}
\mathbf{X}^\top \Sigma_{[o]\epsilon}\mathbf{X} =& \mathbf{X}^\top \left[ \epsilon \mathcal{H}_{[o]}^\top \Big( \sum_{k=1}^{\overline{\xi}} \overline{\lambda}_k^+ \Sigma_{[o]\overline{\phi}_k,\epsilon} \Big) \mathcal{H}_{[o]} \right] \mathbf{X} \\
\geq & \epsilon \min_{k\in \mathbb{S}^0_{\overline{\xi}}} \left\{ \overline{\lambda}_k^+ \Big( \prod_{i=0}^{\overline{\eta}_k-1} \overline{a}_{\overline{v}_{k(i)},i}^{\lfloor [o]\overline{\phi}_k,i \rfloor} \Big)^2 \right\} \sum_{k=1}^{\overline{\xi}} \left\{ \left[ \left( \left( \Big( \Big( \prod_{i=0}^{\overline{\eta}_k-1} Q_{[o]\overline{\phi}_k,i} P_{[o]\overline{\phi}_k,i} \Big) J_{[o]\overline{\phi}_k} \Big)^{-1} \right)^\top \mathbf{Z} \right]^\top \right. \\
& \left. \times M^{-1}_{[o]\overline{\phi}_k,\overline{\eta}_k} \Delta_{[o]\overline{\phi}_k,\overline{\eta}_k} (M^{-1}_{[o]\overline{\phi}_k,\overline{\eta}_k})^\top \left[ \left( \Big( \Big( \prod_{i=0}^{\overline{\eta}_k-1} Q_{[o]\overline{\phi}_k,i} P_{[o]\overline{\phi}_k,i} \Big) J_{[o]\overline{\phi}_k} \Big)^{-1} \right)^\top \mathbf{Z} \right] \right\} \\
=& \epsilon \min_{k\in \mathbb{S}^0_{\overline{\xi}}} \left\{ \overline{\lambda}_k^+ \Big( \prod_{i=0}^{\overline{\eta}_k-1} \overline{a}_{\overline{v}_{k(i)},i}^{\lfloor [o]\overline{\phi}_k,i \rfloor} \Big)^2 \right\} \\
& \sum_{k=1}^{\overline{\xi}} \left[ \mathbf{G}^\top_{\overline{\phi}_k,\overline{\eta}_k} \begin{pmatrix} \mathcal{M}^{-1}_{[o]\overline{\eta}_k} \Xi_{[o]\overline{\phi}_k,\overline{\eta}_k} (\mathcal{M}^{-1}_{[o]\overline{\eta}_k})^\top & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \mathbf{G}_{\overline{\phi}_k,\overline{\eta}_k} \right] \\
\geq & \epsilon \min_{k\in \mathbb{S}^0_{\overline{\xi}}} \left\{ \overline{\lambda}_k^+ \widehat{\lambda}_k \Big( \prod_{i=0}^{\overline{\eta}_k-1} \overline{a}_{\overline{v}_{k(i)},i}^{\lfloor [o]\overline{\phi}_k,i \rfloor} \Big)^2 \right\} \sum_{k=1}^{\overline{\xi}} \sum_{j=1}^{\overline{\eta}_k} (\mathbf{G}^{(j)}_{\overline{\phi}_k,\overline{\eta}_k})^2 \\
:=& \varrho_\epsilon \sum_{k=1}^{\overline{\xi}} \sum_{j=1}^{\overline{\eta}_k} (\mathbf{G}^{(j)}_{\overline{\phi}_k,j})^2 \\
=& \varrho_\epsilon \sum_{k=1}^{\overline{\xi}} \Big( Z^2_{\overline{\phi}_k} + \sum_{j=2}^{\overline{\eta}_k} (\mathbf{G}^{(j)}_{\overline{\phi}_k,j})^2 \Big),
\end{aligned}
$$

where $\varrho_\epsilon = \epsilon \min_{k\in \mathbb{S}^0_{\overline{\xi}}} \{ \overline{\lambda}_k^+ \widehat{\lambda}_k (\prod_{i=0}^{\overline{\eta}_k-1} \overline{a}_{\overline{v}_{k(i)},i}^{\lfloor [o]\overline{\phi}_k,i \rfloor})^2 \}$. In the display above, we have used the fact

$$
\mathbf{G}^{(1)}_{\overline{\phi}_k,1} = Z_{\overline{\phi}_k}, \quad \text{and} \quad \left( \Big( \Big( \prod_{i=0}^{\overline{\eta}_k-1} Q_{[o]\overline{\phi}_k,i} P_{[o]\overline{\phi}_k,i} \Big) J_{[o]\overline{\phi}_k} \Big)^{-1} \right)^\top \mathbf{Z} = \mathbf{G}_{\overline{\phi}_k,\overline{\eta}_k}, \quad \forall\, k \in \mathbb{S}^0_{\overline{\xi}}.
$$

Thus we obtain (4.4). This completes the proof. □

It should be mentioned that in the case of complex diffusion $\Gamma\Gamma^\top$, if the drift term $C_{[o]}$ of (4.1) is "simple" in the sense that approaching the canonical form $\mathscr{S}(\cdot)$, the criterion (4.4) may be tedious for verifying $\Sigma_{[o]\epsilon} \succ \mathbb{O}$. To this end, another available criterion is supplemented below.

Analogous to (3.36), if Assumption 2.2(c) holds, we can determine a set $\overline{\boldsymbol{\phi}}^\diamond := \{\overline{\phi}^\diamond_1, \dots, \overline{\phi}^\diamond_{\overline{\xi}^\diamond_k}\} \subseteq \mathbb{S}^0_n$ (including $\overline{\boldsymbol{\phi}}^\diamond = \emptyset$) and a constant $\overline{\lambda}^+_\theta > 0$ satisfying

$$
\Gamma\Gamma^\top \succeq \overline{\lambda}^+_\theta \sum_{k=1}^{\overline{\xi}^\diamond} \amalg_{n,\overline{\phi}^\diamond_k}, \tag{4.14}
$$

where $\overline{\phi}^{\diamond}_j > \overline{\phi}^{\diamond}_j$, $\forall\, i < j$. By (4.2), we obtain $\overline{\xi}^{\diamond}_k \leq \overline{\xi}$.

**Theorem 4.2.** *Under Assumptions 2.1 and 2.2(c), the following assertion is true:*

$$\mathbf{X}^{\top} \Sigma_{[o]\epsilon} \mathbf{X} \geq \varrho^{\diamond}_{\epsilon} \sum_{k=1}^{\overline{\xi}^{\diamond}} \Big( X^2_{\overline{\phi}^{\diamond}_k} + \sum_{j=2}^{\overline{\eta}^{\diamond}_k} (\mathbf{G}^{(j)}_{\overline{\phi}^{\diamond}_k, j})^2 \Big), \tag{4.15}$$

*where* $\underline{\mathbf{G}}_{\overline{\phi}^{\diamond}_k, j} = \big[\sum_{i=0}^{j-1} (\underline{Q}^{-1}_{[o]\overline{\phi}^{\diamond}_k, i})^{\top} \underline{P}_{[o]\overline{\phi}^{\diamond}_k, i}\big] \underline{J}_{[o]\overline{\phi}^{\diamond}_k} \mathbf{X}$*, with* $\overline{\eta}^{\diamond}_k$*,* $\underline{J}_{[o]\overline{\phi}^{\diamond}_k}$*,* $\underline{P}_{[o]\overline{\phi}^{\diamond}_k, i}$ *and* $\underline{Q}_{[o]\overline{\phi}^{\diamond}_k, i}$ *obtained in Algorithm 4,* $\forall\, i \in \mathbb{S}^0_{j-1}$*. Furthermore,* $\mathbf{X}$ *is the same as in (3.6), and* $\varrho^{\diamond}_{\epsilon} > 0$ *can be similarly determined like* $\varrho_{\epsilon}$*.*

---

**Algorithm 4:** Algorithm for obtaining $\{\underline{\mathbf{G}}_{\overline{\phi}^{\diamond}_k, j}\}_{j=1}^{\overline{\eta}^{\diamond}_k}$.

---

**Input:** $C_{[o]}$, $\overline{\phi}^{\diamond}_k$.

**Output:** $\underline{\mathbf{G}}_{\overline{\phi}^{\diamond}_k, j} = \big[\sum_{i=0}^{j-1} (\underline{Q}^{-1}_{[o]\overline{\phi}^{\diamond}_k, i})^{\top} \underline{P}_{[o]\overline{\phi}^{\diamond}_k, i}\big] \underline{J}_{[o]\overline{\phi}^{\diamond}_k} \mathbf{X}$[a], $\forall\, j \in \mathbb{S}^0_{\overline{\eta}^{\diamond}_k}$.

**(Initialization):** $\overline{\eta}^{\diamond}_k = 1$, $\overline{C}_{[o]\overline{\phi}^{\diamond}_k, 1} = \underline{J}_{[o]\overline{\phi}^{\diamond}_k} C_{[o]} \underline{J}^{-1}_{[o]\overline{\phi}^{\diamond}_k}$;

**(Technical framework):** By a FOR loop similar to Algorithm 2, we determine the values of $\overline{\eta}^{\diamond}_k$ and some suitable $\overline{\nu}^{\diamond}_{k(i)} \in \mathbb{S}^i_n$, which satisfies

$$\text{(4-i) } \overline{c}^{\lfloor [o]\overline{\phi}^{\diamond}_k, i \rfloor}_{\overline{\nu}^{\diamond}_{k(i)}, i} \neq 0, \;\; \forall\, i \in \mathbb{S}^0_{\overline{\eta}^{\diamond}_k - 1}, \quad \text{(4-ii) } \overline{c}^{\lfloor [o]\overline{\phi}^{\diamond}_k, \overline{\eta}^{\diamond}_k \rfloor}_{\overline{\eta}^{\diamond}_k, j} = 0, \;\; \forall\, j \in \mathbb{S}^{\overline{\eta}^{\diamond}_k + 1}_n \text{[a]},$$

where each $\overline{c}^{\lfloor [o]\overline{\phi}^{\diamond}_k, i \rfloor}_{ji}$ is determined by the iterative scheme:

$$\widehat{C}_{[o]\overline{\phi}^{\diamond}_k, i} = \underline{P}_{[o]\overline{\phi}^{\diamond}_k, i} \overline{C}_{[o]\overline{\phi}^{\diamond}_k, i} \underline{P}^{-1}_{[o]\overline{\phi}^{\diamond}_k, i} \text{[a]}, \; \overline{C}_{[o]\overline{\phi}^{\diamond}_k, i+1} := \underline{Q}_{[o]\overline{\phi}^{\diamond}_k, i} \widehat{C}_{[o]\overline{\phi}^{\diamond}_k, i} \underline{Q}^{-1}_{[o]\overline{\phi}^{\diamond}_k, i};$$

**return** $\overline{\eta}^{\diamond}_k$, $\underline{J}_{[o]\overline{\phi}^{\diamond}_k}$, $\underline{P}_{[o]\overline{\phi}^{\diamond}_k, i}$, $\underline{Q}_{[o]\overline{\phi}^{\diamond}_k, i}$ $(i \in \mathbb{S}^0_{\overline{\eta}^{\diamond}_k - 1})$.

[a] The matrices $\underline{J}_{[o]\overline{\phi}^{\diamond}_k}$, $\underline{P}_{[o]\overline{\phi}^{\diamond}_k, i}$ and $\underline{Q}_{[o]\overline{\phi}^{\diamond}_k, i}$ have the same form as $J_{\phi_k}$, $P_{\phi_k, i}$ and $Q_{\phi_k, i}$ by replacing $(\phi_k, \nu_{k(i)}, \boldsymbol{\ell}_{k, n-1-i})$ with $(\overline{\phi}^{\diamond}_k, \overline{\nu}^{\diamond}_{k(i)}, \underline{\boldsymbol{\ell}}_{[o]k, n-1-i})$, where $\underline{\boldsymbol{\ell}}_{[o]k, n-1-i} = \frac{-1}{\widehat{c}^{\lfloor [o]\overline{\phi}^{\diamond}_k, i \rfloor}_{i+1, i}} (\widehat{c}^{\lfloor [o]\overline{\phi}^{\diamond}_k, i \rfloor}_{i+2, i}, \ldots, \widehat{c}^{\lfloor [o]\overline{\phi}^{\diamond}_k, i \rfloor}_{n, i})^{\top}$. [The paraphrase of $\overline{c}^{\lfloor \cdot \rfloor}_{j,i}$ (resp., $\widehat{c}^{\lfloor \cdot \rfloor}_{j,i}$) is the same as $\overline{a}^{\lfloor \cdot \rfloor}_{j,i}$ (resp., $\widehat{a}^{\lfloor \cdot \rfloor}_{j,i}$).] Analogously, $\overline{c}^{\lfloor [o]\overline{\phi}^{\diamond}_k, 0 \rfloor}_{\overline{\nu}^{\diamond}_{k(0)}, 0} := 1$ and $\underline{P}_{\overline{\phi}^{\diamond}_k, l} = \underline{Q}_{\overline{\phi}^{\diamond}_k, l} := \mathbf{I}_n$, $\forall\, l \in \{0, n-1\}$; $k \in \mathbb{S}^0_{\overline{\xi}^{\diamond}}$.

---

**Proof.** For any $k \in \mathbb{S}^0_{\overline{\xi}^{\diamond}}$, let $\underline{\Sigma}_{[o]\overline{\phi}^{\diamond}_k, \epsilon}$ be the solution of the following algebraic equation:

$$\Im_c\big(\underline{\Sigma}_{[o]\overline{\phi}^{\diamond}_k, \epsilon}, C_{[o]}, \amalg_{n, \overline{\phi}^{\diamond}_k}\big) = \mathbb{O}. \tag{4.16}$$

Using (4.6), (4.14) and (4.16), there holds

$$\Sigma_{[o]\epsilon} \succeq \epsilon \overline{\lambda}^{+}_{\theta} \sum_{k=1}^{\overline{\xi}^{\diamond}} \underline{\Sigma}_{[o]\overline{\phi}^{\diamond}_k, \epsilon}.$$

The remainder of the proof is similar to that of (3.39)-(3.42), and is omitted. □

Below we show some sufficient conditions for $\Sigma_{[o]\epsilon} \succ \mathbb{O}$ (with the motivation originating from the study on the local approximation for the IPDFs of existing ecological and biological models).

**Corollary 4.1.** *Under Assumptions 2.1 and 2.2(c), if one of the following four conditions holds:*

$$\text{(2-a)}\ \ \overline{\xi} = n, \quad \text{(2-b)}\ \ \overline{\eta}_{k_3} = n, \ \ \exists\, k_3 \in \mathbb{S}^0_{\overline{\xi}}, \quad \text{(2-c)}\ \ \overline{\xi}^{\diamond} = n, \quad \text{(2-d)}\ \ \overline{\eta}^{\diamond}_{k_4} = n, \ \ \exists\, k_4 \in \mathbb{S}^0_{\overline{\xi}^{\diamond}}.$$

*Then* $\Sigma_{[o]\epsilon} \succ \mathbb{O}$.

**Proof.** This is a direct corollary of (4.4) and (4.15); see a standard argument in Corollary 3.1. □

**Remark 6.** Most theoretical results regarding the positive definiteness of covariance matrices ($\Sigma_{[o]\epsilon} \succ \mathbb{O}$) are established only under case (2-b) or non-degenerate linear diffusion (a special case of (2-a)), based on existing normal approximation methods; see [41,45–48,79–82]. Then, some models based on the Ornstein–Uhlenbeck process, such as those in Yang et al. [62] and Zhou et al. [90], cannot verify $\Sigma_{[o]\epsilon} \succ \mathbb{O}$ using these methods. However, our method can address this issue. Although these models do not fall into the general setting of Corollary 4.1, the desired result $\Sigma_{[o]\epsilon} \succ \mathbb{O}$ can still be obtained from Theorems 4.1 and 4.2. The relevant analysis is left for the reader. Our uNA method will advance and outperform existing results of normal approximation for IPDFs. Furthermore, it should be mentioned that similar to Remark 3, the uniqueness of the IPM $\mu_\epsilon$ is not required for Theorems 4.1, 4.2 and Corollary 4.1.

### *4.2. Global approximation-II*

Combined with Assumption 2.2(b), by a standard argument in Theorem 3.3, we have the following results of the uNA method for $\mu_\epsilon$ in global horizon.

**Theorem 4.3.** *Under Assumptions 2.1 and 2.2(b)-(c), for sufficiently small* $\epsilon$,

(i) *The IPM $\mu_\epsilon$ (resp., IPDF $\Psi_\epsilon(\cdot)$) can be globally approximated by the distribution $\mathbb{N}_n(\mathbf{X}^*, \Sigma_{[o]\epsilon})$ (resp., $\Phi_{[o]\epsilon}(\cdot)$) limited on $\mathbb{R}^n_+$. If $\Sigma_{[o]\epsilon} \succ \mathbb{O}$, then*

$$\Phi_{[o]\epsilon}(\mathbf{X}_\epsilon(t)) = (2\pi)^{-\frac{n}{2}} |\Sigma_{[o]\epsilon}|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{X}_\epsilon - \mathbf{X}^*)^\top \Sigma^{-1}_{[o]\epsilon} (\mathbf{X}_\epsilon - \mathbf{X}^*)}, \tag{4.17}$$

*and*

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n_+} \left| \mathbf{y} - \mathbf{X}^* \right|^\gamma \left| \Psi_\epsilon(\mathbf{y}) - \Phi_{[o]\epsilon}(\mathbf{y}) \right| d\mathbf{y} = 0, \quad \forall\, \gamma \in (0, 2].$$

(ii) *Under (3.44) and $\Sigma_{[o]\epsilon} \succ \mathbb{O}$, if there further exist constants $\widehat{K} > 0$, $\alpha_1 \in (0, 1)$ and a vector $\boldsymbol{\gamma}_2$ such that*

$$\sum_{i=1}^{n}\left(\left|f_i(\mathbf{x})\right|+\left|\partial_i f_i(\mathbf{x})\right|\right)\le \widehat{K}e^{\frac{1}{\epsilon^{\alpha_1}}|\ln\mathbf{x}-\boldsymbol{\gamma}_2|^2},\quad \forall\,\mathbf{x}\in\mathbb{R}^n_+. \tag{4.18}$$

*Then*

$$\sup_{\mathbf{x}\in\mathbb{R}^n_+}\left|\mathscr{L}_\epsilon\Phi_{[o]\epsilon}(\mathbf{x})\right|\le\wp_2(\epsilon), \tag{4.19}$$

*where* $\lim_{\epsilon\to 0}\frac{\wp_2(\epsilon)}{\sqrt{\epsilon}}=0$.

**Remark 7.** Similar to Remark 5, we let $\Phi^{\partial}_{[o]\epsilon}(X_{k_1},...,X_{k_l})$ be the marginal density of $\mathbb{N}_n(\mathbf{X}^*,\Sigma_{[o]\epsilon})$ with respect to the components $(X_{k_1},...,X_{k_l})$, where $1\le k_1<\cdots<k_l\le n$. Then by Theorem 4.3, for sufficiently small $\epsilon$, if $\Sigma_{[o]\epsilon}\succ\mathbb{O}$, then the measure $\mu^{\partial}_\epsilon(X_{\epsilon,i})$ approximately has a normal probability density

$$\Phi^{\partial}_{[o]\epsilon}(X_{\epsilon,i})=\frac{1}{\sqrt{2\pi\,\Sigma_{[o]\epsilon}(i,i)}}e^{-\frac{(X_{\epsilon,i}-X_i^*)^2}{2\Sigma_{[o]\epsilon}(i,i)}},\quad\forall\, i\in\mathbb{S}^0_n,$$

which is limited on $\mathbb{R}^n_+$.

The LNA method is considered to be a more biologically reasonable approximation for the IPDF $\Psi_\epsilon(\cdot)$ of (1.2) compared with the uNA method. Most generalized Kolmogorov systems are modeled to describe the dynamics of interacting populations that are required to be nonnegative. This requirement is in accordance with the result $\mathbb{P}(\mathbf{X}_\epsilon\in\mathbb{R}^n_+)=1$ under the distribution $\mathbb{LN}_n(\ln\overline{\mathbf{X}}^*_\epsilon,\Sigma_\epsilon)$. But if the distribution $\mathbb{N}_n(\mathbf{X}^*,\Sigma_{[o]\epsilon})$ is considered as a global approximation for $\mu_\epsilon$, it causes a unreasonable result $\mathbb{P}(\mathbf{X}_\epsilon\in\mathbb{R}^n\setminus\mathbb{R}^n_+)>0$. Fortunately, such probability is negligible for sufficiently small $\epsilon$. In this case, the distribution $\mathbb{N}_n(\mathbf{X}^*,\Sigma_{[o]\epsilon})$ limited on $\mathbb{R}^n_+$ becomes a viable approximation for $\mu_\epsilon$, as shown before in part (i) of Theorem 4.3.

## 5. Applications

This section presents a number of applications of our theoretical results. Our main aim is to approximately characterize the relevant IPMs and IPDFs by the LNA (or uNA) method.

**Remark 8.** Before proceeding further, let us make the following remarks.

- As is well known, the IPM $\mu_\epsilon$ denotes a long-time, stochastic, positive steady state of the generalized Kolmogorov system (1.2), and is a distribution function defined on $t\to\infty$. The existence and form of $\mu_\epsilon$ cannot be directly determined due to the finite number of iterations of computer simulation. By a standard argument in [8,13,79], for any finite time interval $[0,T_0]$, if Assumption 2.1(2) holds, system (1.2) will have an empirical normalized occupation measure $\overline{\mu}_\epsilon(T_0,\cdot)$ that relies on $T_0$. In addition, the measure $\overline{\mu}_\epsilon(T_0,\cdot)$ will converge weakly to $\mu_\epsilon(\cdot)$ in the sense that for every continuous and bounded function $h(\cdot)$,

$$\lim_{T_0\to\infty}\int_{\mathbb{R}^n_+}h(\mathbf{x})\overline{\mu}_\epsilon(T_0,d\mathbf{x})=\int_{\mathbb{R}^n_+}h(\mathbf{x})\mu(d\mathbf{x}).$$

This implies that for sufficiently large $T_0 > 0$, $\overline{\mu}_\epsilon(T_0, \cdot)$ can reflect most of the dynamic behavior and statistical properties of $\mu_\epsilon$. Thus, unless specifically stated, we use the empirical measure $\overline{\mu}_\epsilon(T_0, \cdot)$ with a large enough time interval and sufficient iterations as a viable alternative for $\mu_\epsilon$. Accordingly, let $\overline{\Psi}_\epsilon(T_0, \cdot)$ be the empirical density function of $\overline{\mu}_\epsilon(T_0, \cdot)$. Combined with Remark 5, the notations $\overline{\mu}^\partial_\epsilon(T_0, \cdot)$ and $\overline{\Psi}^\partial_\epsilon(T_0, \cdot)$ can be similarly understood.

- In each application example, $\Psi_\epsilon(\mathbf{X}_\epsilon)$ should be specifically written as $\Psi_\epsilon((X_{\epsilon,1}, ..., X_{\epsilon,n})^\top)$. For convenience, a short notation $\Psi_\epsilon(X_{\epsilon,1}, ..., X_{\epsilon,n})$ is used. Without causing ambiguity, we will adopt such abbreviation throughout Section 5. [Similarly, $\Psi^\partial_\epsilon(X_{\epsilon,k_1}, ..., X_{\epsilon,k_l})$, $\Phi_\epsilon(X_{\epsilon,1}, ..., X_{\epsilon,n})$ and $\Phi_{[o]\epsilon}(X_{\epsilon,1}, ..., X_{\epsilon,n})$ are well defined.]

### 5.1. Stochastic SIR epidemic models

Compartmental models, introduced first by Kermack and McKendrick [77,78], are an effective way to describe the transmission dynamics of infectious diseases such as avian influenza, cholera and syphilis. The main idea of these models is to subdivide a host population into several epidemiological distinct types of individuals (or called compartments), and the SIR (Susceptible-Infected-Recovered) model is one of the basic building blocks along this line [1], from which many epidemic systems are established. To begin, we consider the following stochastic equation with degenerate diffusion, which in the absence of $\sqrt{\epsilon}$ was studied in [57,83]:

$$\begin{cases} dS_\epsilon(t) = \big[\Lambda - aS_\epsilon(t)I_\epsilon(t) - pS_\epsilon(t)\big]dt + \sqrt{\epsilon}\sigma_1 S_\epsilon(t)dW(t), \\ dI_\epsilon(t) = \big[aS_\epsilon(t)I_\epsilon(t) - (p+\alpha+\gamma)I_\epsilon(t)\big]dt + \sqrt{\epsilon}\sigma_2 I_\epsilon(t)dW(t), \end{cases} \tag{5.1}$$

where $S_\epsilon(t)$ and $I_\epsilon(t)$ denote the population of susceptible and infected humans, respectively, at time $t$. $\Lambda$ is the intrinsic recruitment rate; $a$ is the incidence rate; $p$ and $\alpha$ are the natural death rate and disease-induced mortality rate, respectively; $\gamma$ denotes the recovery rate of infected individuals. All of the above parameters are positive. Normally the recovered individuals $R_\epsilon(t)$ have no influence on the underlying properties of infectious diseases, thus only the dynamics of individuals $S_\epsilon(t)$ and $I_\epsilon(t)$ are considered in (5.1).

By a similar argument in [57,83], one can conclude that Assumptions 2.1 and 2.2(b) are satisfied by system (5.1) if the following conditions hold:

$$\mathscr{R}^S_{0,\epsilon} = \frac{a\Lambda}{p(p+\alpha+\gamma+\frac{\epsilon\sigma_2^2}{2})} > 1, \quad p > \epsilon\sigma_1^2, \quad p+\alpha+\gamma > \epsilon\sigma_2^2. \tag{5.2}$$

Before applying Theorem 3.1, the procedures in Assumption 2.2(a) and (3.1)-(3.3) corresponding to system (5.1) should be provided. Denote $\mathscr{R}^S_{1,\epsilon} = \frac{a\Lambda}{(p+\frac{\epsilon\sigma_1^2}{2})(p+\alpha+\gamma+\frac{\epsilon\sigma_2^2}{2})}$, and let

$$\begin{cases} \dfrac{\Lambda}{\overline{S}^*_\epsilon} - a\overline{I}^*_\epsilon - \Big(p + \dfrac{\epsilon\sigma_1^2}{2}\Big) = 0, \\ a\overline{S}^*_\epsilon - \Big(p+\alpha+\gamma+\dfrac{\epsilon\sigma_2^2}{2}\Big) = 0. \end{cases}$$

If $\mathscr{R}_{1,\epsilon}^S > 1$, the solution of above equations is unique on $\mathbb{R}_+^2$, and it is $\overline{S}_\epsilon^* = \frac{p+\alpha+\gamma+\frac{\epsilon\sigma_2^2}{2}}{a}$, $\overline{I}_\epsilon^* = \frac{(p+\frac{\epsilon\sigma_1^2}{2})(\mathscr{R}_{1,\epsilon}^S-1)}{a}$. Then by calculation, we have

$$B_\epsilon = \begin{pmatrix} -\frac{\Lambda}{\overline{S}_\epsilon^*} & -a\overline{I}_\epsilon^* \\ a\overline{S}_\epsilon^* & 0 \end{pmatrix}, \quad \Theta_\epsilon\Theta_\epsilon^\top = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \mathcal{G}_\epsilon = \begin{pmatrix} \frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}} & \frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}} \\ \frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}} & -\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}} \end{pmatrix}.$$

This yields that $\mathcal{G}_\epsilon(\Theta_\epsilon\Theta_\epsilon^\top)\mathcal{G}_\epsilon^\top = (\sigma_1^2+\sigma_2^2)\amalg_{2,1}$, and

$$A_\epsilon = \frac{1}{\sigma_1^2+\sigma_2^2}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{11} = -\frac{\sigma_1^2\Lambda}{\overline{S}_\epsilon^*} + a\sigma_1\sigma_2(\overline{S}_\epsilon^* - \overline{I}_\epsilon^*), \quad a_{12} = -\frac{\sigma_1\sigma_2\Lambda}{\overline{S}_\epsilon^*} + a(\sigma_1^2\overline{I}_\epsilon^* + \sigma_2^2\overline{S}_\epsilon^*),$$
$$a_{21} = -\frac{\sigma_1\sigma_2\Lambda}{\overline{S}_\epsilon^*} - a(\sigma_1^2\overline{S}_\epsilon^* + \sigma_2^2\overline{I}_\epsilon^*), \quad a_{22} = -\frac{\sigma_2^2\Lambda}{\overline{S}_\epsilon^*} + a\sigma_1\sigma_2(\overline{I}_\epsilon^* - \overline{S}_\epsilon^*).$$

By Algorithm 1, we consider the algebraic equation

$$\Im_c(\Sigma_{1,\epsilon}, A_\epsilon, \amalg_{2,1}) = \mathbb{O}.$$

Clearly, $J_1 = \mathbf{I}_2$. Note that $a_{21} \neq 0$, then $\nu_{1(1)} = 2$ and $\eta_1 = 2$. Using Theorem 3.1, we can construct a matrix $M_{1,2}$ such that $M_{1,2}\Sigma_{1,\epsilon}M_{1,2}^\top = \Delta_{1,2} \succ \mathbb{O}$, where

$$M_{1,2} = \begin{pmatrix} \frac{a_{21}}{\sigma_1^2+\sigma_2^2} & \frac{a_{22}}{\sigma_1^2+\sigma_2^2} \\ 0 & 1 \end{pmatrix}, \quad \Delta_{1,2} = \begin{pmatrix} \frac{\overline{S}_\epsilon^*}{2\Lambda} & 0 \\ 0 & \frac{1}{2a^2\Lambda\overline{I}_\epsilon^*} \end{pmatrix}.$$

Hence,

$$\Sigma_\epsilon = \frac{\epsilon a_{21}^2}{\sigma_1^2+\sigma_2^2}\mathcal{G}_\epsilon^\top M_{1,2}^{-1}\Delta_{1,2}(M_{1,2}^{-1})^\top\mathcal{G}_\epsilon \succ \mathbb{O}. \tag{5.3}$$

Combining Theorem 3.3, (5.2) and $\mathscr{R}_{1,\epsilon}^S \le \mathscr{R}_{0,\epsilon}^S$, we obtain

(⊗-1) If $\mathscr{R}_{1,\epsilon}^S > 1$, $p > \epsilon\sigma_1^2$ and $p+\alpha+\gamma > \epsilon\sigma_2^2$, then the unique IPM $\mu_\epsilon$ of (5.1) around $(\overline{S}_\epsilon^*, \overline{I}_\epsilon^*)^\top$ is approximated by the distribution $\mathbb{LN}_2(\ln(\overline{S}_\epsilon^*, \overline{I}_\epsilon^*)^\top, \Sigma_\epsilon)$, with $\Sigma_\epsilon$ shown in (5.3). Moreover, for sufficiently small $\epsilon$, the IPDF $\Psi_\epsilon(S_\epsilon, I_\epsilon)$ can be globally approximated by

$$\Phi_\epsilon(S_\epsilon, I_\epsilon) = \frac{1}{2\pi\sqrt{|\Sigma_\epsilon|}S_\epsilon I_\epsilon}e^{-\frac{1}{2}(\ln\frac{S_\epsilon}{\overline{S}_\epsilon^*}, \ln\frac{I_\epsilon}{\overline{I}_\epsilon^*})\Sigma_\epsilon^{-1}(\ln\frac{S_\epsilon}{\overline{S}_\epsilon^*}, \ln\frac{I_\epsilon}{\overline{I}_\epsilon^*})^\top}.$$

To demonstrate, a numerical example is provided.

**Example 5.1.** Consider (5.1) with parameters $\lambda = 0.2$, $a = 0.65$, $p = 0.2$, $\alpha = 0.1$, $\gamma = 0.2$, $\sigma_1 = 0.05$, $\sigma_2 = 0.1$ and initial value $(S_\epsilon(0), I_\epsilon(0)) = (0.8, 0.1)$. By choosing $\epsilon = 10^{-2}$, we then compute $\mathscr{R}^S_{1,\epsilon} = 1.2998$, $p - \epsilon\sigma_1^2 > 0.199$, $p + \alpha + \gamma - \epsilon\sigma_2^2 = 0.4999$, $(\overline{S}^*_\epsilon, \overline{I}^*_\epsilon) = (0.7693, 0.0922)$, and

$$\Sigma_\epsilon = 10^{-5} \times \begin{pmatrix} 7.1144 & -9.9990 \\ -9.9990 & 190 \end{pmatrix}.$$

Thus,

$$\Phi_\epsilon(S_\epsilon, I_\epsilon) = \frac{454.9527}{S_\epsilon I_\epsilon} e^{-7602(\ln \frac{S_\epsilon}{\overline{S}^*_\epsilon})^2 - 290.6696(\ln \frac{I_\epsilon}{\overline{I}^*_\epsilon})^2 - 817.0503 \ln \frac{S_\epsilon}{\overline{S}^*_\epsilon} \ln \frac{I_\epsilon}{\overline{I}^*_\epsilon}},$$

with two marginal densities (MDs):

$$\Phi^\partial_\epsilon(S_\epsilon) = \frac{47.2978}{S_\epsilon} e^{-7028(\ln \frac{S_\epsilon}{0.7693})^2}, \quad \Phi^\partial_\epsilon(I_\epsilon) = \frac{9.1524}{I_\epsilon} e^{-263.158(\ln \frac{I_\epsilon}{0.0922})^2}.$$

We first plot the empirical marginal measures (MMs) $\overline{\mu}^\partial_\epsilon(T_0, S_\epsilon)$ and $\overline{\mu}^\partial_\epsilon(T_0, I_\epsilon)$ at iteration time $T_0 = 30000$ (i.e., the frequency histograms of $S_\epsilon$ and $I_\epsilon$ of the empirical measure $\overline{\mu}_\epsilon(30000, S_\epsilon, I_\epsilon)$), as shown in the right-hand column of Fig. 1. Intuitively, their outlines are close to the type of log-normal or normal distribution. To verify (⊗-1), we use the command "ksdensity(·, ·)" in MATLAB (MathWorks, 2022b) to plot the empirical density $\overline{\Psi}_\epsilon(T_0, S_\epsilon, I_\epsilon)$ of (5.1) in 2D setting at $T_0 = 10000$, $20000$ and $30000$, respectively in Fig. 2(a)-(c). Fig. 2(d) shows the function $\Phi_\epsilon(\cdot)$ in 2D setting. It is easily seen that these four density pictures are very similar. To further support the similarity, Fig. 3 depicts the empirical MDs $\overline{\Psi}^\partial_\epsilon(T_0, S_\epsilon)$ and $\overline{\Psi}^\partial_\epsilon(T_0, I_\epsilon)$ (i.e., the frequency histogram fitting curves of $S_\epsilon$ and $I_\epsilon$) at $T_0 = 10000$, $20000$ and $30000$, each in a different color. In this figure, $\Phi^\partial_\epsilon(S_\epsilon)$ and $\Phi^\partial_\epsilon(I_\epsilon)$ both almost coincide with the corresponding three fitting curves. To illustrate this quantitatively, we use the Kolmogorov–Smirnov test [84] to test the alternative hypothesis that $\Phi^\partial_\epsilon(S_\epsilon)$ (resp., $\Phi^\partial_\epsilon(I_\epsilon)$) and $\overline{\Psi}^\partial_\epsilon(T_0, S_\epsilon)$ (resp., $\overline{\Psi}^\partial_\epsilon(T_0, I_\epsilon)$) are from different distributions against the null hypothesis that they are from the same distribution for each component, where $T_0 \in \{10000, 20000, 30000\}$. With 5% significance level, the relevant Kolmogorov–Smirnov tests imply that we cannot reject the null hypothesis. Thus, the log-normal density $\Phi_\epsilon(S_\epsilon, I_\epsilon)$ approximates the IPDF $\Psi_\epsilon(S_\epsilon, I_\epsilon)$ very well, which verifies (⊗-1).

### 5.2. *Stochastic delayed chemostat models*

The chemostat is a laboratory apparatus used for the continuous culture of microorganisms. The researchers continuously add fresh substrate while simultaneously removing culture liquid at the same rate, including metabolic end products and leftover nutrients, to maintain a constant culture volume [27]. Chemostat models are considered to be the best idealization for biological systems [2], and are widely used in fermentation processes, wastewater treatment, and microbial dynamics research [96]. It has been discovered that there is a delay between the time nutrients are consumed and the time they are converted to available energy. Remarkably, chemostat experiments have fully confirmed the hypothesis that an infinite (distributed) delay with the general

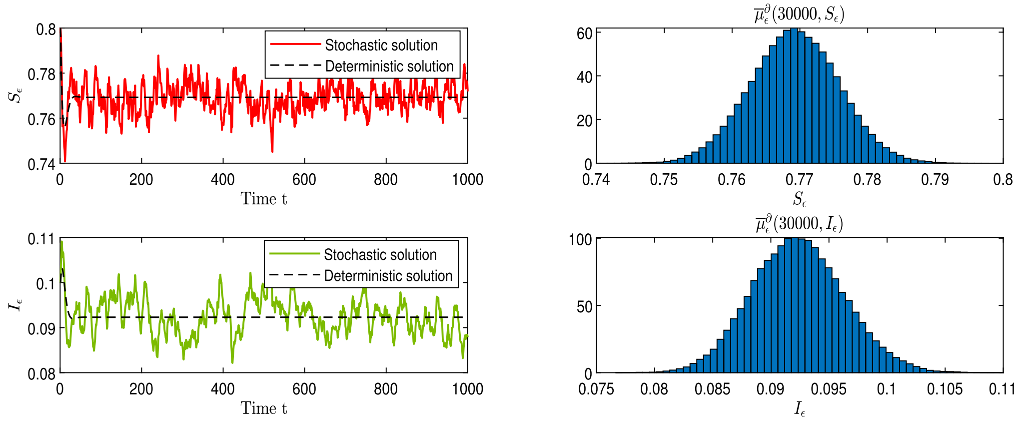


Fig. 1. The left-hand column presents the trajectories of $S_\epsilon(t)$ and $I_\epsilon(t)$ of (5.1), and of its deterministic system on $t \in [0, 1000]$. The right-hand column shows the empirical MMs $\overline{\mu}^{\partial}_{\epsilon}(T_0, S_\epsilon)$ and $\overline{\mu}^{\partial}_{\epsilon}(T_0, I_\epsilon)$ of (5.1) on the iteration interval [0, 30000]. All the iteration step sizes are $\Delta t = 10^{-3}$.

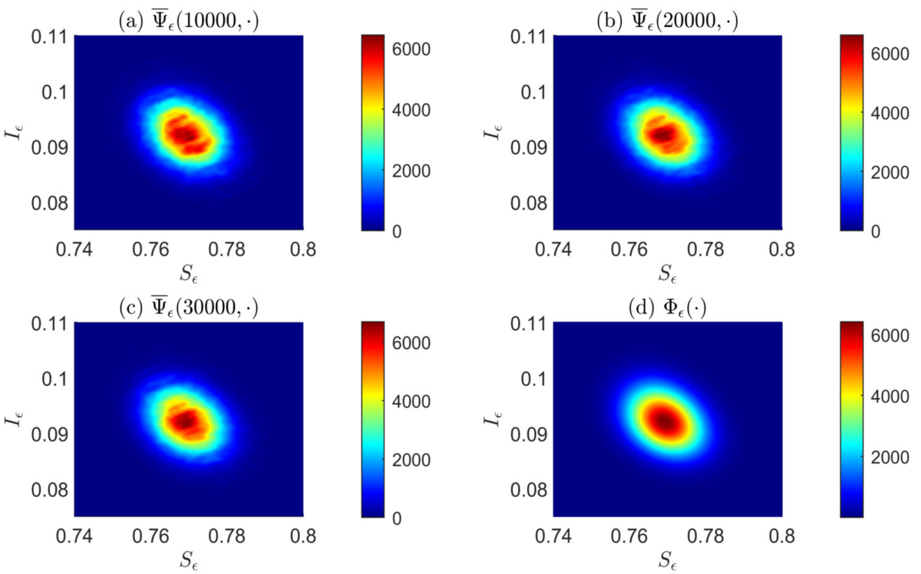


Fig. 2. (a)-(c) The empirical density $\overline{\Psi}_\epsilon(T_0, S_\epsilon, I_\epsilon)$ of (5.1) in 2D setting at iteration time $T_0$ equals to 10000, 20000 and 30000, respectively; (d) The function $\Phi_\epsilon(\cdot)$ in 2D setting. All of the parameter values and step size are the same as in Fig. 1. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

concept of a response time kernel can reflect the cumulative effect of past states [85]. Therefore, in this study, we focus on a class of chemostat models that incorporate delay in uptake conversion, which was first proposed in [86] without $\sqrt{\epsilon}$. Precisely,

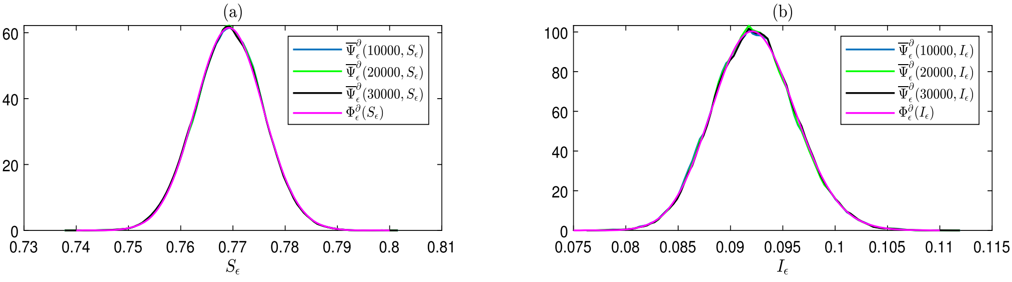


Fig. 3. The blue, green and black lines represent the empirical MDs $\overline{\Psi}^{\partial}_{\epsilon}(T_0, S_\epsilon)$ and $\overline{\Psi}^{\partial}_{\epsilon}(T_0, I_\epsilon)$ of (5.1) at iteration time $T_0 = 10000$, 20000 and 30000, respectively. The purple lines denote the MDs of $\Phi_\epsilon(S_\epsilon, I_\epsilon)$ (i.e., $\Phi^{\partial}_{\epsilon}(S_\epsilon)$ and $\Phi^{\partial}_{\epsilon}(I_\epsilon)$). All of the parameter values and step size are the same as in Fig. 1.

$$\begin{cases} dS_\epsilon(t) = \Big[\big(S^0 - S_\epsilon(t)\big)D - \dfrac{x_\epsilon(t)}{\hbar(S_\epsilon(t))}\Big]dt + \sqrt{\epsilon}\sigma S_\epsilon(t)dW(t), \\ dx_\epsilon(t) = \Big[-(D+\theta)x_\epsilon(t) + \displaystyle\int_{-\infty}^{t} \dfrac{x_\epsilon(\tau)}{\hbar(S_\epsilon(\tau))} e^{-D(t-\tau)}\Gamma(t-\tau)d\tau\Big]dt, \end{cases} \tag{5.4}$$

where $S_\epsilon(t)$ and $x_\epsilon(t)$ are the concentrations of substrate and microbial species, respectively, at time $t$. $D$ is the dilution rate (or equivalently, $\frac{1}{D}$ is the mean residence time), $S^0$ is the input concentration of nutrient, and $\theta$ is the specific death rate of microorganism. Due to the outflow in the chemostat, $x_\epsilon(\tau)e^{-D(t-\tau)}$ stands for the biomass of $x_\epsilon$ that consumes nutrient at time $t-\tau$ and survives so that it can complete the conversion process of the substrate at time $t$. The kernel $\Gamma(\cdot)$ depicts the distribution of time delay involved in the conversion of nutrient to viable cells, and is usually simulated by a Gamma distribution [52], i.e.,

$$\Gamma(\cdot) = \frac{t^m \alpha^{m+1} e^{-\alpha t}}{m!}, \quad \forall\, t \geq 0, \tag{5.5}$$

where $m$ is a nonnegative integer, and the constant $\alpha > 0$ denotes the decay rate. Two common types of (5.5) in the literature are: (i) The weak kernel ($m = 0$) (ii) The strong kernel ($m = 1$). $\frac{1}{\hbar(S_\epsilon)}$ denotes the nutrient consumption capacity, and the function $\hbar(\cdot) \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{R}_+)$ is generally assumed to satisfy

$$\hbar'(s) \leq 0, \quad \frac{1}{s\hbar(s)} \leq c \ \text{ and } \ \hbar''(s)s^3 \leq m_0, \quad \forall\, s > 0, \tag{5.6}$$

where $c > 0$ and $m_0$ are two constants.

In what follows, we mainly analyze the case of the weak kernel. Clearly, system (5.4) does not have the exact form as in (1.2). To this end, we give an equivalent transformation for (5.4) by means of the linear chain trick [87]. Let

$$U_\epsilon(t) = \int_{-\infty}^{t} \frac{x_\epsilon(\tau)}{\hbar(S_\epsilon(\tau))} e^{-D(t-\tau)}\Gamma(t-\tau)d\tau.$$

Then (5.4) is equivalent to

$$\begin{cases} dS_\epsilon(t) = \Big[\big(S^0 - S_\epsilon(t)\big)D - \dfrac{x_\epsilon(t)}{\hbar(S_\epsilon(t))}\Big]dt + \sqrt{\epsilon}\sigma S_\epsilon(t)dW(t), \\ dx_\epsilon(t) = \big[-(D+\theta)x_\epsilon(t) + U_\epsilon(t)\big]dt, \\ dU_\epsilon(t) = \Big[\dfrac{\alpha x_\epsilon(t)}{\hbar(S_\epsilon(t))} - (D+\alpha)U_\epsilon(t)\Big]dt, \end{cases} \tag{5.7}$$

which falls into our general setting. Thus, we need only to discuss the IPDF of (5.7). Denote

$$\mathscr{R}^S_{2,\epsilon} = \frac{\alpha}{(D+\alpha)(D+\theta)\hbar((1+\frac{\epsilon\sigma^2}{2D})^{-1}S^0)}, \quad \mathscr{R}^S_{3,\epsilon} = \frac{\alpha}{(D+\alpha)(D+\theta+\frac{c_1 S^0\epsilon\sigma^2}{2})\hbar(S^0)},$$

where $c_1 = \max\{0, \frac{\alpha m_0}{2D(D+\alpha)(S^0\hbar(S^0))^2}\}$.

A summary of the theoretical results in [83,86] indicates that Assumptions 2.1 and 2.2(b) are satisfied by (5.7) if $\mathscr{R}^S_{3,\epsilon} > 1$. Taking steps along the procedures in Assumption 2.2(a) and (3.1)-(3.3), we define a quasi-positive equilibrium $(\overline{S}^*_\epsilon, \overline{x}^*_\epsilon, \overline{U}^*_\epsilon) \in \mathbb{R}^3_+$ by

$$\begin{cases} \Big(\dfrac{S^0}{\overline{S}^*_\epsilon} - 1\Big)D - \dfrac{\overline{x}^*_\epsilon}{\overline{S}^*_\epsilon\hbar(\overline{S}^*_\epsilon)} - \dfrac{\epsilon\sigma^2}{2} = 0, \\ -(D+\theta) + \dfrac{\overline{U}^*_\epsilon}{\overline{x}^*_\epsilon} = 0, \\ \dfrac{\alpha\overline{x}^*_\epsilon}{\overline{U}^*_\epsilon\hbar(\overline{S}^*_\epsilon)} - (D+\alpha) = 0. \end{cases}$$

By calculation, the equilibrium $(\overline{S}^*_\epsilon, \overline{x}^*_\epsilon, \overline{U}^*_\epsilon)$ exists and is unique when $\mathscr{R}^S_{2,\epsilon} > 1$. In this case, we have $\Theta_\epsilon\Theta_\epsilon^\top = \sigma^2 \amalg_{3,1}$, $\mathcal{G}_\epsilon = \mathbf{I}_3$, $\lambda_1^+ = \sigma^2$, and

$$A_\epsilon = B_\epsilon = \begin{pmatrix} -a_{11} & -a_{12} & 0 \\ 0 & -a_{22} & a_{22} \\ a_{31} & a_{33} & -a_{33} \end{pmatrix}, \tag{5.8}$$

where $a_{11} = (D + \frac{\epsilon\sigma^2}{2}) - \frac{\overline{x}^*_\epsilon\hbar'(\overline{S}^*_\epsilon)}{\hbar^2(\overline{S}^*_\epsilon)} > 0$, $a_{12} = \frac{\overline{x}^*_\epsilon}{\overline{S}^*_\epsilon\hbar(\overline{S}^*_\epsilon)}$, $a_{22} = D+\theta$, $a_{31} = -\frac{(D+\alpha)\overline{S}^*_\epsilon\hbar'(\overline{S}^*_\epsilon)}{\hbar(\overline{S}^*_\epsilon)} \geq 0$ and $a_{33} = D+\alpha$.

In view of

$$\begin{aligned} \psi_{B_\epsilon}(\lambda) =& \lambda^3 + (a_{11}+a_{22}+a_{33})\lambda^2 + a_{11}(a_{22}+a_{33})\lambda + a_{12}a_{22}a_{31} \\ :=& \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \end{aligned}$$

we compute $|\mathscr{H}^{(1)}_{3,B_\epsilon}| = a_1 > 0$,

$$|\mathscr{H}^{(2)}_{3,B_\epsilon}| > a_{22}(a_{11}a_{33} - a_{12}a_{31}) = (D+\alpha)\Big(D + \frac{\epsilon\sigma^2}{2}\Big) > 0,$$

and $|\mathscr{H}_{3,B_\epsilon}| = a_3|\mathscr{H}^{(2)}_{3,B_\epsilon}| > 0$. Thus, $B_\epsilon \in \overline{\mathbf{RH}}(3)$.

Consider the algebraic equation

$$\Im_c(\Sigma_{1,\epsilon}, A_\epsilon, Ш_{3,1}) = \mathbb{O},$$

where $A_\epsilon$ is shown in (5.8). By Algorithm 1 and (3.18), we determine that $Q_{1,1} = \mathbf{I}_3$ and

$$P_{1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{(i.e., } \nu_{1(1)} = 3).$$

Then, $\eta_1 = 3$ and $\overline{A}_{1,2} \in \mathcal{U}_q(3)$. Combined with Corollary 3.1 and Theorem 3.1, one has

$$\Delta_{1,3} = \begin{pmatrix} \frac{a_2}{2|\mathscr{H}^{(2)}_{3,B_\epsilon}|} & 0 & -\frac{1}{2|\mathscr{H}^{(2)}_{3,B_\epsilon}|} \\ 0 & \frac{1}{2|\mathscr{H}^{(2)}_{3,B_\epsilon}|} & 0 \\ -\frac{1}{2|\mathscr{H}^{(2)}_{3,B_\epsilon}|} & 0 & \frac{|\mathscr{H}^{(1)}_{3,B_\epsilon}|}{2|\mathscr{H}_{3,B_\epsilon}|} \end{pmatrix} \succ \mathbb{O}, \quad M_{1,3} = \begin{pmatrix} \boldsymbol{\beta}_3 \overline{A}^2_{1,2} \\ \boldsymbol{\beta}_3 \overline{A}_{1,2} \\ \boldsymbol{\beta}_3 \end{pmatrix},$$

and $\Sigma_{1,\epsilon} = (a_{22}a_{31})^2 (M_{1,3}P_{1,1})^{-1} \Delta_{1,3} [(M_{1,3}P_{1,1})^{-1}]^\top$. Therefore,

$$\Sigma_\epsilon = \epsilon (a_{22}a_{31}\sigma)^2 (M_{1,3}P_{1,1})^{-1} \Delta_{1,3} [(M_{1,3}P_{1,1})^{-1}]^\top \succ \mathbb{O}. \tag{5.9}$$

In summary, we conclude

($\otimes$-2) If $\min_{i=2,3}\{\mathscr{R}^S_{i,\epsilon}\} > 1$, then the distribution $\mathbb{LN}_3(\ln(\overline{S}^*_\epsilon, \overline{x}^*_\epsilon, \overline{U}^*_\epsilon)^\top, \Sigma_\epsilon)$ is a local approximation for the unique IPM $\mu_\epsilon$ of (5.7) around $(\overline{S}^*_\epsilon, \overline{x}^*_\epsilon, \overline{U}^*_\epsilon)^\top$, where $\Sigma_\epsilon$ is given in (5.9). Moreover, for sufficiently small $\epsilon$, the IPDF $\Psi_\epsilon(S_\epsilon, x_\epsilon, U_\epsilon)$ is globally approximated by

$$\Phi_\epsilon(S_\epsilon, x_\epsilon, U_\epsilon) = \frac{1}{(2\pi)^{\frac{3}{2}} \sqrt{|\Sigma_\epsilon|} S_\epsilon x_\epsilon U_\epsilon} e^{-\frac{1}{2}(\ln \frac{S_\epsilon}{\overline{S}^*_\epsilon}, \ln \frac{x_\epsilon}{\overline{x}^*_\epsilon}, \ln \frac{U_\epsilon}{\overline{U}^*_\epsilon}) \Sigma_\epsilon^{-1} (\ln \frac{S_\epsilon}{\overline{S}^*_\epsilon}, \ln \frac{x_\epsilon}{\overline{x}^*_\epsilon}, \ln \frac{U_\epsilon}{\overline{U}^*_\epsilon})^\top}.$$

Below we provide a numerical example for illustration.

**Example 5.2.** Let the consumption capacity $\frac{1}{\hbar(S_\epsilon)}$ be the Michaelis–Menten response type [88], i.e., $\frac{1}{\hbar(S_\epsilon)} = \frac{m_c S_\epsilon}{a + S_\epsilon}$, where the constants $a, m_c > 0$, then condition (5.6) is satisfied. According to [88], we choose initial value $(S_\epsilon(0), x_\epsilon(0), U_\epsilon(0)) = (0.4, 0.3, 0.6)$ and the following parameters:

$$S^0 = 1, \ D = 1.2, \ m_c = 15, \ a = 2, \ \theta = 0.4, \ \alpha = 3, \ \sigma = 0.5.$$

By letting $\epsilon = 2 \times 10^{-2}$, we obtain $\mathscr{R}^S_{2,\epsilon} = 2.2290$ and $\mathscr{R}^S_{3,\epsilon} = 2.2252$. In view of ($\otimes$-2), $\Phi_\epsilon(S_\epsilon, x_\epsilon, U_\epsilon)$ (resp., $\mathbb{LN}_3(\ln(\overline{S}^*_\epsilon, \overline{x}^*_\epsilon, \overline{U}^*_\epsilon)^\top, \Sigma_\epsilon)$) is a good global fit for $\Psi_\epsilon(S_\epsilon, x_\epsilon, U_\epsilon)$ (resp., $\mu_\epsilon$). To verify this, the relevant MDs of $\Phi_\epsilon(S_\epsilon, x_\epsilon, U_\epsilon)$ are first presented in Table 2, where $(\overline{S}^*_\epsilon, \overline{x}^*_\epsilon, \overline{U}^*_\epsilon) = (0.3511, 0.3472, 0.5556)$.

Below by a similar argument in Example 5.1, we provide the empirical MMs $\overline{\mu}^\partial_\epsilon(T_0, S_\epsilon)$, $\overline{\mu}^\partial_\epsilon(T_0, x_\epsilon)$ and $\overline{\mu}^\partial_\epsilon(T_0, U_\epsilon)$ at $T_0 = 20000$; see the right-hand column of Fig. 4. Obviously, it is quite possible to use some log-normal or normal distributions to approximate the theoretical

Table 2
List of some MDs $\Phi_\epsilon^\partial(\cdot)$ of $\Phi_\epsilon(S_\epsilon, x_\epsilon, U_\epsilon)$ in Example 5.2.

| MDs | Mean | Variance | Correlation coefficient |
|---|---|---|---|
| $\Phi_\epsilon^\partial(S_\epsilon, x_\epsilon)$ | $(\ln 0.3511, \ln 0.3472)$ | $(0.8799, 0.3914) \times 10^{-3}$ | $-0.0976 \times 10^{-3}$ |
| $\Phi_\epsilon^\partial(S_\epsilon)$ | $\ln 0.3511$ | $0.8799 \times 10^{-3}$ | $--$ |
| $\Phi_\epsilon^\partial(x_\epsilon)$ | $\ln 0.3472$ | $0.3914 \times 10^{-3}$ | $--$ |
| $\Phi_\epsilon^\partial(U_\epsilon)$ | $\ln 0.5556$ | $0.6093 \times 10^{-3}$ | $--$ |

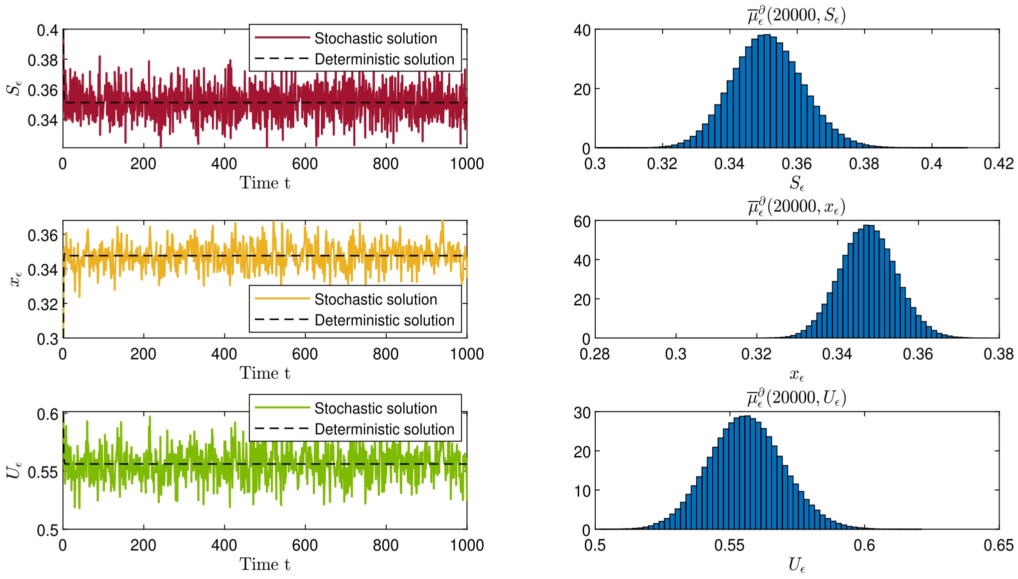


Fig. 4. The left-hand column shows the sample paths of $S_\epsilon(t)$, $x_\epsilon(t)$ and $U_\epsilon(t)$ of (5.7), and of its deterministic system on $t \in [0, 1000]$. The right-hand column presents the empirical MMs $\overline{\mu}_\epsilon^\partial(20000, S_\epsilon)$, $\overline{\mu}_\epsilon^\partial(20000, x_\epsilon)$ and $\overline{\mu}_\epsilon^\partial(20000, U_\epsilon)$ of (5.7). All the iteration step sizes are $\Delta t = 10^{-3}$.

MDs of $\Psi_\epsilon^\partial(S_\epsilon, x_\epsilon, U_\epsilon)$. Fig. 5(a)-(c) depicts the empirical MD $\overline{\Psi}_\epsilon^\partial(T_0, S_\epsilon, x_\epsilon)$ of (5.7) (or equivalently, the empirical density $\overline{\Psi}_\epsilon(T_0, S_\epsilon, x_\epsilon)$ of (5.4)) in 2D setting at iteration time $T_0$ equals to 5000, 10000 and 20000, respectively, and the function $\Phi_\epsilon^\partial(S_\epsilon, x_\epsilon)$ in 2D setting is shown in Fig. 5(d). Clearly, the four density pictures are very similar. [In fact, $U_\epsilon(t)$ is only an accompaniment to the transformed equation (5.7), our main focus is the dynamics of $S_\epsilon(t)$ and $x_\epsilon(t)$ in (5.4), and that is why we only study $\overline{\Psi}_\epsilon^\partial(T_0, S_\epsilon, x_\epsilon)$ and $\Phi_\epsilon^\partial(S_\epsilon, x_\epsilon)$. Such idea is also adopted in Example 5.4 later.] Furthermore, Fig. 6 presents the empirical MDs $\overline{\Psi}_\epsilon^\partial(T_0, S_\epsilon)$, $\overline{\Psi}_\epsilon^\partial(T_0, x_\epsilon)$ and $\overline{\Psi}_\epsilon^\partial(T_0, U_\epsilon)$) at $T_0 = 10000$, 20000 and 30000, each in a different color. It is clear that $\Phi_\epsilon^\partial(S_\epsilon)$, $\Phi_\epsilon^\partial(x_\epsilon)$ and $\Phi_\epsilon^\partial(U_\epsilon)$ in Table 2 all almost coincide with the corresponding three density curves. Using the Kolmogorov–Smirnov test, for any $T_0 \in \{10000, 20000, 30000\}$ and $\triangleright \in \{S_\epsilon, x_\epsilon, U_\epsilon\}$, we determine that the null hypothesis that $\Phi_\epsilon^\partial(\triangleright)$ and $\overline{\Psi}_\epsilon^\partial(T_0, \triangleright)$ are from the same distribution will be accepted at 2% significance level.

To summarize, the similarity between $\Phi_\epsilon(S_\epsilon, x_\epsilon, U_\epsilon)$ and the IPDF $\Psi_\epsilon(S_\epsilon, x_\epsilon, U_\epsilon)$ is significant. Thus, ($\otimes$-2) is well verified.

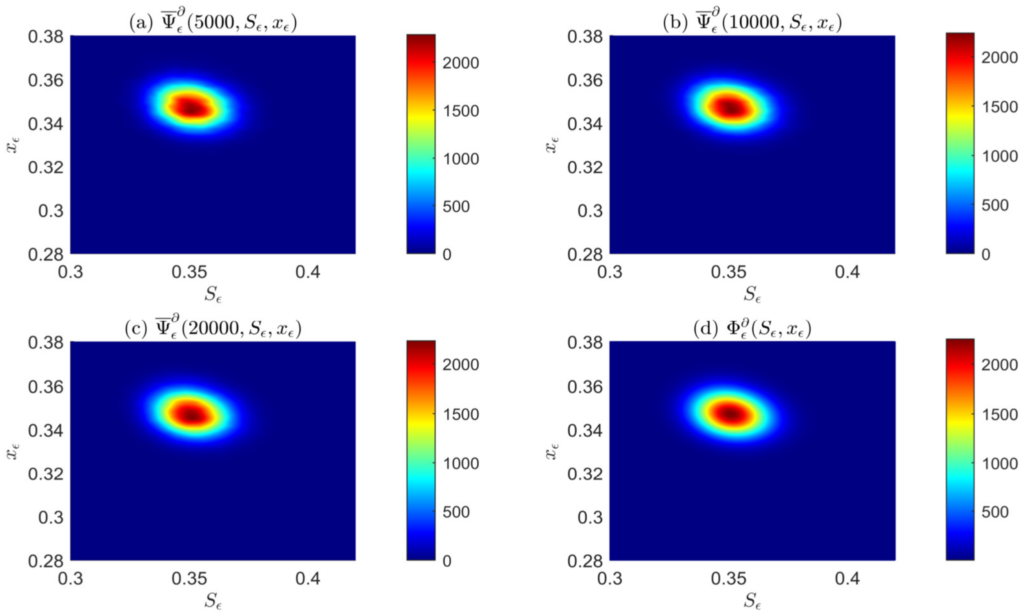


Fig. 5. (a)-(c) The empirical MD $\overline{\Psi}_\epsilon^\partial(T_0, S_\epsilon, x_\epsilon)$ of (5.7) in 2D setting at iteration time $T_0 = 5000,\ 10000$ and 20000; (d) The function $\Phi_\epsilon^\partial(S_\epsilon, x_\epsilon)$ in 2D setting. All of the parameter values and step size are the same as in Fig. 4.

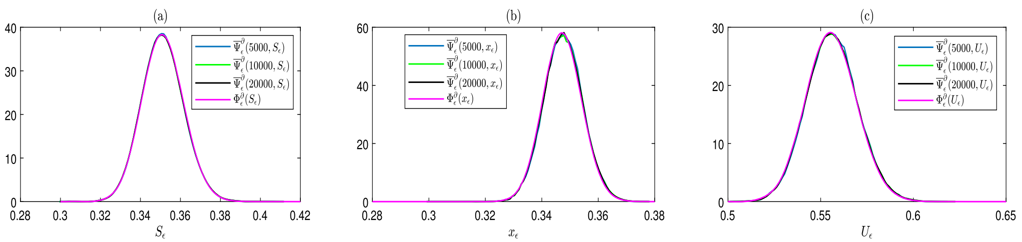


Fig. 6. The blue, green and black lines represent the empirical MDs $\overline{\Psi}_\epsilon^\partial(T_0, S_\epsilon)$, $\overline{\Psi}_\epsilon^\partial(T_0, x_\epsilon)$ and $\overline{\Psi}_\epsilon^\partial(T_0, U_\epsilon)$ of (5.7) at iteration time $T_0$ equals to 5000, 10000 and 20000, respectively. The purple lines denote the one-dimensional MDs of $\Phi_\epsilon(S_\epsilon, x_\epsilon, U_\epsilon)$. All of the parameter values and iteration step size are the same as in Fig. 4.

**Remark 9.** Combining Sections 5.1 and 5.2 yields that, if $\mathrm{rank}(\Theta_\epsilon \Theta_\epsilon^\top) = 1$, i.e., $\xi = 1$, then the unique approach for verifying $\Sigma_\epsilon \succ \mathbb{O}$ is to obtain $\eta_1 = n$, which means $\{\nu_{1(1)}, \ldots, \nu_{1(n)}\} = \mathbb{S}_n^0$. According to (3.6) and the expression of $Q_{\phi_1,i}$, a feasible practice to choose $\nu_{1(i)}$ is to ensure that the form of $\overline{A}_{\phi_1,i+1}$ is close to $\mathscr{S}(n)$. Thus, we present the first rule for the word '**suitable**" mentioned in Algorithm 1.

**Rule 1:** If $\xi = 1$, for any $i \geq 1$, the choice of $\nu_{1(i)}$ can maximize the following formula

$$\sum_{j \in \mathbb{S}_{n-2-i}} \mathbf{1}_{\{\ell_{k,n-2-i}^{(j)} = 0\}},$$

where $\mathbf{1}_{(\cdot)}$ is the indicator function.

To proceed, we will introduce the other two rules for the word '**suitable**" in Algorithm 1. Letting $\mathbf{X}^\top \Sigma_\epsilon \mathbf{X} = 0$, we easily deduce from (3.6) that $(Y_k)_{k\in\boldsymbol{\phi}} = \mathbf{0}$. Combining this with Remark 3, an equivalent result of $\Sigma_\epsilon \succ \mathbb{O}$ is that $(Y_l)_{l\in\mathbb{S}_n^0\setminus\boldsymbol{\phi}} = \mathbf{0}$. In view of the sequence $\{\mathbf{H}_{\phi_k,i}\}_{i=1}^{\eta_k}$ and the definition of $\boldsymbol{\ell}_{k,n-1-i}$, we have

$$\mathbf{H}_{\phi_k,i+1}^{(i+1)} = Y_{[0\sim\nu_{k(i)}]} + \sum_{j\in\mathbb{S}_{n-1-i}^0} \boldsymbol{\ell}_{k,n-1-i}^{(j)} Y_{[j\sim\nu_{k(i)}]}, \quad \forall\, i \in \mathbb{S}_{\eta_k-1}^0, \tag{5.10}$$

where $[j\sim\nu_{k(i)}]$ denotes the subscript value dependent on $\nu_{k(i)}$. Then we conclude:

**Rule 2:** If $\xi \in (\frac{n}{2}, n]$, i.e., there are many random fluctuations, for any $i \in \mathbb{S}_{\eta_k-1}^0$, the choice of $\nu_{k(i)}$ is based on two restrictions including (i) $[0\sim\nu_{k(i)}] \in \mathbb{S}_n^0 \setminus \boldsymbol{\phi}$, and (ii) the following formula

$$\sum_{[j\sim\nu_{k(i+1)}]\in\mathbb{S}_n^0\setminus\boldsymbol{\phi}} \mathbf{1}_{\{\boldsymbol{\ell}_{k,n-2-i}^{(j)}=0\}}$$

should be small.

In the case of $\xi \in (1, \frac{n}{2}]$, based on the relationship among the components of $\mathbf{X}_\epsilon(t)$ of (1.2) in practical terms, and the one-to-one match between $Y_l$ and $X_{\epsilon,l}$, $\forall\, l \in \mathbb{S}_n^0$, we first divide $(Y_i)_{i\in\mathbb{S}_n^0}$ into $\xi$ chains "$(Y_{\phi_k}, \bigvee_{\phi_k}^{\text{asso}} Y_\bullet)$", $\forall\, k \in \mathbb{S}_\xi^0$, where $\bigvee_{\phi_k}^{\text{asso}}$ denotes a index set dependent on $\phi_k$, satisfying: (i) every $Y_j$ ($j \in \bigvee_{\phi_k}^{\text{asso}}$) is strongly associated with $Y_{\phi_k}$, and (ii) $\{\bigvee_{\phi_k}^{\text{asso}}\}_{k\in\mathbb{S}_\xi^0}$ is a finite covering of $\mathbb{S}_n^0 \setminus \boldsymbol{\phi}$. Then,

**Rule 3:** If $\xi \in (1, \frac{n}{2}]$, for any chain $(Y_{\phi_k}, \bigvee_{\phi_k}^{\text{asso}} Y_\bullet)$ and $i \in \mathbb{S}_{\eta_k-1}^0$, the choice of $\nu_{k(i)}$ requires the condition $[0\sim\nu_{k(c)}] \in (\mathbb{S}_n^0 \setminus \boldsymbol{\phi}) \cap \bigvee_{\phi_k}^{\text{asso}}$, $\forall\, c = i,\ i+1$.

### 5.3. Stochastic Lotka–Volterra predator–prey models

Competition, predation and cooperation, as three primary interactions among species in ecosystems, affect largely the structure of animal and plant communities, as well as the evolution process of population dynamics. Traditionally, these interactions are modeled by a class of systems of ordinary differential equations known as the Lotka–Volterra models. To better characterize the underlying asymptotic dynamics of interacting species, the impact of some abiotic factors (e.g., random noise, seasonal variation, age structure) on the original systems has been widely studied [87,89,91,92]. This section, together with the next one, will further derive explicit approximations for the IPDFs of Lotka–Volterra models. Consider a stochastic Lotka–Volterra prey-predator system with one prey and two competing predators:

$$\begin{cases} dX_{\epsilon,1}(t) = X_{\epsilon,1}(t)\big[r_1 - b_{11}X_{\epsilon,1}(t) - b_{12}X_{\epsilon,2}(t) - b_{13}X_{\epsilon,3}(t)\big]dt, \\ dX_{\epsilon,2}(t) = X_{\epsilon,2}(t)\big[-r_2 + b_{21}X_{\epsilon,1}(t) - b_{22}X_{\epsilon,2}(t) - b_{23}X_{\epsilon,3}(t)\big]dt + \sqrt{\epsilon}\sigma_1 X_{\epsilon,2}(t)dW_1(t), \\ dX_{\epsilon,3}(t) = X_{\epsilon,3}(t)\big[-r_3 + b_{31}X_{\epsilon,1}(t) - b_{32}X_{\epsilon,2}(t) - b_{33}X_{\epsilon,3}(t)\big]dt + \sqrt{\epsilon}\sigma_2 X_{\epsilon,3}(t)dW_2(t), \end{cases} \tag{5.11}$$

where $X_{\epsilon,j}(t)$ ($j \in \mathbb{S}_3^0$) denotes the population density of prey and two predators, respectively, at time $t$. $r_1 > 0$ is the growth rate of prey $X_{\epsilon,1}$, and $b_{ii} > 0$ is the intraspecific competition coefficient, $\forall\, i \in \mathbb{S}_3^0$. $r_l, b_{1l}, b_{l1} > 0$ are the natural death rate, capture rate and food conversion rate of predator $X_{\epsilon,l}$, respectively, $l = 2, 3$. $b_{23}, b_{32} \geq 0$ depict the interspecific competitions of two predators. According to Connell [93], we assume that the intraspecific competitions are stronger than the interactions among different species, i.e.,

$$b_{ii} > \sum_{j\neq i} b_{ij}, \quad \forall\, i \in \mathbb{S}_3^0. \tag{5.12}$$

Unlike the similar model in [53], (5.11) is established under the degenerate setting whereby the intensity of random fluctuations of prey is 0. The motivation stems from the argument in Benaim et al. [76]. Biologically, this assumption implies that the sources of environmental randomness have negligible effects on the prey population under strong competition between predators.

For simplicity, let

$$\nabla_4 = b_{21}r_1 - b_{11}r_2, \quad \nabla_5 = b_{31}r_1 - b_{11}r_3, \quad \nabla_6 = b_{21}b_{33} - b_{23}b_{31}, \quad \nabla_7 = b_{21}b_{32} - b_{22}b_{31},$$

$$\nabla_0 = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ -b_{21} & b_{22} & b_{23} \\ -b_{31} & b_{32} & b_{33} \end{vmatrix}, \quad \nabla_8 = \begin{vmatrix} b_{11} & r_1 & 0 \\ -b_{21} & -r_2 & \frac{\epsilon\sigma_1^2}{2} \\ -b_{31} & -r_3 & \frac{\epsilon\sigma_2^2}{2} \end{vmatrix}.$$

Moreover, $\nabla_i$ (resp., $\widetilde{\nabla}_i$) is defined by only replacing the $i$th column of $\nabla_0$ with vector $(r_1, -r_2, -r_3)^\top$ (resp., $(0, \frac{\epsilon\sigma_1^2}{2}, \frac{\epsilon\sigma_2^2}{2})^\top$), $\forall\ i \in \mathbb{S}_3^0$. Next, by a similar procedure in Assumption 2.2(a) and (3.1)-(3.3), we first define an equilibrium $\overline{\mathbf{X}}_\epsilon^* = (\overline{X}_{\epsilon,1}^*, \overline{X}_{\epsilon,2}^*, \overline{X}_{\epsilon,3}^*)^\top$, which satisfies:

$$\begin{cases} r_1 - b_{11}\overline{X}_{\epsilon,1}^* - b_{12}\overline{X}_{\epsilon,2}^* - b_{13}\overline{X}_{\epsilon,3}^* = 0, \\ -\Big(r_2 + \dfrac{\epsilon\sigma_1^2}{2}\Big) + b_{21}\overline{X}_{\epsilon,1}^* - b_{22}\overline{X}_{\epsilon,2}^* - b_{23}\overline{X}_{\epsilon,3}^* = 0, \\ -\Big(r_3 + \dfrac{\epsilon\sigma_2^2}{2}\Big) + b_{31}\overline{X}_{\epsilon,1}^* - b_{32}\overline{X}_{\epsilon,2}^* - b_{33}\overline{X}_{\epsilon,3}^* = 0. \end{cases} \tag{5.13}$$

It follows from Cramer's rule that Eq. (5.13) has a unique positive solution $\overline{\mathbf{X}}_\epsilon^* = (\frac{\nabla_1 - \widetilde{\nabla}_1}{\nabla_0}, \frac{\nabla_2 - \widetilde{\nabla}_2}{\nabla_0}, \frac{\nabla_3 - \widetilde{\nabla}_3}{\nabla_0})$ if $\nabla_0 > 0$ and $\nabla_i > \widetilde{\nabla}_i,\ \forall\imath \in \mathbb{S}_3^0$. Then we have

$$\Theta_\epsilon \Theta_\epsilon^\top = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_2^2 \end{pmatrix}, \quad A_\epsilon = B_\epsilon = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} \\ a_{21} & -a_{22} & -a_{23} \\ a_{31} & -a_{32} & -a_{33} \end{pmatrix},$$

where $a_{ij} = b_{ij}\overline{X}_{\epsilon,j}^*$, $\forall\, i, j \in \mathbb{S}_3^0$. By calculation,

$$\begin{aligned} \psi_{B_\epsilon}(\lambda) =& \lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 + [a_{11}(a_{22} + a_{33}) + a_{12}a_{21} + a_{13}a_{31} + (a_{22}a_{33} - a_{23}a_{32})]\lambda \\ & + a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{13}a_{22} - a_{12}a_{23}) \\ :=& \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3. \end{aligned}$$

Clearly, $|\mathscr{H}_{3,B_\epsilon}^{(1)}| = a_1 > 0$ and $a_3 = \overline{X}_{\epsilon,1}^*\overline{X}_{\epsilon,2}^*\overline{X}_{\epsilon,3}^* > 0$. By (5.12), we obtain $a_{22}a_{33} - a_{23}a_{32} = (b_{22}b_{33} - b_{23}b_{32})\overline{X}_{\epsilon,2}^*\overline{X}_{\epsilon,3}^* > 0$. Thus,

$$|\mathscr{H}_{3,B_\epsilon}^{(2)}| > a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} > 0, \quad |\mathscr{H}_{3,B_\epsilon}| = a_3|\mathscr{H}_{3,B_\epsilon}^{(2)}| > 0,$$

implying that $B_\epsilon \in \overline{\mathbf{RH}}(3)$.

Below we will study two algebraic equations $\Im_c(\Sigma_{i,\epsilon}, A_\epsilon, \amalg_{3,i}) = \mathbb{O}$, $i = 2, 3$. In view of **Rule 2** and $\mathcal{G}_\epsilon = \mathbf{I}_3$, an equivalent result of $\Sigma_\epsilon \succ \mathbb{O}$ is that $\mathbf{X}^\top \Sigma_\epsilon \mathbf{X} = 0$ holds if and only if $X_1 = 0$. Thus, we need to choose a $\nu_{i(2)}$ such that $[0 \sim \nu_{i(2)}] = 1$. In other words, the position of $X_1$ should be transformed to before $X_j$ by similarity transformation when solving the equation $\Im_c(\Sigma_{i,\epsilon}, A_\epsilon, \amalg_{3,i}) = \mathbb{O}$, where $\{i, j\} = \mathbb{S}_2^0$. More specifically,

Step 1. For the Lyapunov equation

$$\Im_c(\Sigma_{2,\epsilon}, A_\epsilon, \amalg_{3,2}) = \mathbb{O}.$$

According to Algorithm 1 and the form of $B_\epsilon$, we let $\nu_{1(1)} = 2$, and

$$J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{a_{32}}{a_{12}} & 1 \end{pmatrix}.$$

Obviously, $\mathbf{H}_{2,1}^{(1)} = X_2$ and $\mathbf{H}_{2,2}^{(2)} = X_1 + \frac{a_{32}}{a_{12}} X_3$. Hence, $\eta_1 \geq 2$.

Step 2. Consider the Lyapunov equation

$$\Im_c(\Sigma_{3,\epsilon}, A_\epsilon, \amalg_{3,3}) = \mathbb{O}.$$

By the similar argument in Step 1, we choose $\nu_{2(1)} = 2$ and

$$J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q_{3,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{a_{23}}{a_{13}} & 1 \end{pmatrix}.$$

Then, $\mathbf{H}_{3,1}^{(1)} = X_3$, $\mathbf{H}_{3,2}^{(2)} = X_1 + \frac{a_{23}}{a_{13}} X_2$ and $\eta_2 \geq 2$. Using Theorem 3.1, we determine that

$$\begin{aligned} \mathbf{X}^\top \Sigma_\epsilon \mathbf{X} =& \rho_\epsilon \sum_{k=1}^{2} \sum_{j=1}^{\eta_k} \left(\mathbf{H}_{\phi_k, j}^{(j)}\right)^2 \\ \geq& \rho_\epsilon \Big[ X_2^2 + (\mathbf{H}_{2,2}^{(2)})^2 + X_3^2 + (\mathbf{H}_{3,2}^{(2)})^2 \Big] \\ =& \rho_\epsilon \Big[ X_2^2 + X_3^2 + \Big( X_1 + \frac{a_{32}}{a_{12}} X_3 \Big)^2 + \Big( X_1 + \frac{a_{23}}{a_{13}} X_2 \Big)^2 \Big]. \end{aligned}$$

It is clear that $\{\mathbf{X} \in \mathbb{R}^3 | \mathbf{X}^\top \Sigma_\epsilon \mathbf{X} = 0\} = \{\mathbf{0}\}$. Thus, $\Sigma_\epsilon \succ \mathbb{O}$. The special expression of $\Sigma_\epsilon$ can be obtained by further analysis in Steps 1 and 2, and is then omitted.

By slightly modifying the proofs of [53, Theorem 2.1 (iv), Lemma 3.11] and [62, Theorem 5.1], we have if (5.12) and the following conditions are satisfied,

$$\text{(i)}\ \nabla_i > 0,\ \forall\, i \in \mathbb{S}_8^{-1}, \quad \text{(ii)}\ \frac{\nabla_3}{\widetilde{\nabla}_3} > 1, \quad \text{and} \quad \text{(iii)}\ b_{kk} > \frac{\epsilon \sigma_{k-1}^2}{2},\ k = 2, 3, \tag{5.14}$$

then Assumptions 2.1 and 2.2(b) corresponding to system (5.11) hold, and its solution $(X_{\epsilon,1}(t), X_{\epsilon,2}(t), X_{\epsilon,3}(t))^\top$ is globally attractive. Combining Theorems 3.1 and 3.3 yields that,

(⊗-3) Under (5.14), $\Phi_\epsilon(X_{\epsilon,1}, X_{\epsilon,2}, X_{\epsilon,3})$ (resp., $\mathbb{LN}_3(\ln(\overline{X}^*_{\epsilon,1}, \overline{X}^*_{\epsilon,2}, \overline{X}^*_{\epsilon,3})^\top, \Sigma_\epsilon)$) is a local approximation for the IPDF $\Psi_\epsilon(X_{\epsilon,1}, X_{\epsilon,2}, X_{\epsilon,3})$ (resp., IPM $\mu_\epsilon$) of (5.11) around $(\overline{X}^*_{\epsilon,1}, \overline{X}^*_{\epsilon,2}, \overline{X}^*_{\epsilon,3})^\top$, where $\Sigma_\epsilon \succ \mathbb{O}$. Such approximation has a significant global fitting effect for sufficiently small $\epsilon$.

We present a numerical example for verification.

**Example 5.3.** Consider (5.11) with the same initial values and parameters as in [53], i.e.,

$$(X_{\epsilon,1}(0), X_{\epsilon,2}(0), X_{\epsilon,3}(0)) = (0.6, 0.1, 0.05),\ r_1 = 1.2,\ r_2 = 0.15,\ r_3 = 0.01,\ b_{11} = 1.6,$$

$$b_{12} = 1.2,$$

$$b_{13} = 0.3,\ b_{21} = 0.85,\ b_{22} = 1.9,\ b_{23} = 0.6,\ b_{31} = 0.4,\ b_{32} = 1,\ b_{33} = 2.1,\ \sigma_1 = 0.2,\ \sigma_2 = 0.2.$$

We choose $\epsilon = 2 \times 10^{-2}$. Direct calculation shows that $\nabla_0 = 7.251$, $\nabla_1 = 4.3995$, $\nabla_2 = 1.3441$, $\nabla_3 = 0.1634$, $\nabla_4 = 0.78$, $\nabla_5 = 0.464$, $\nabla_6 = 1.545$, $\nabla_7 = 0.09$, $\nabla_8 = 1.264 \times 10^{-4}$, $\frac{\nabla_3}{\tilde{\nabla}_3} = 206.3131$, $b_{22} - \frac{\epsilon\sigma_1^2}{2} = 1.8996$, $b_{33} - \frac{\epsilon\sigma_2^2}{2} = 2.0996$ and $(\overline{X}^*_{\epsilon,1}, \overline{X}^*_{\epsilon,2}, \overline{X}^*_{\epsilon,3}) = (0.6069, 0.1852, 0.0224)$. Condition (5.14) holds and hence, $\Phi_\epsilon(X_{\epsilon,1}, X_{\epsilon,2}, X_{\epsilon,3})$ has a global approximation for the IPDF $\Psi_\epsilon(X_{\epsilon,1}, X_{\epsilon,2}, X_{\epsilon,3})$. To support this deeply, we first provide in Table 3 the MDs of $\Phi_\epsilon(X_{\epsilon,1}, X_{\epsilon,2}, X_{\epsilon,3})$. Inspired by the ideas in Examples 5.1 and 5.2, Fig. 7 presents all the empirical MMs $\overline{\mu}^\partial_\epsilon(30000, X_{\epsilon,i})$ of (5.11). Furthermore, all the MDs of $\Phi_\epsilon(X_{\epsilon,1}, X_{\epsilon,2}, X_{\epsilon,3})$ and $\overline{\Psi}_\epsilon(T_0, X_{\epsilon,1}, X_{\epsilon,2}, X_{\epsilon,3})$ under different large iteration time $T_0$ are shown in Figs. 8 and 9. Obviously, the similarity between the corresponding MDs (or MMs) is significant. Based on the Kolmogorov–Smirnov test, for each $i \in \mathbb{S}^0_3$ and $T_0 \in \{10000, 20000, 30000\}$, we further consider the hypothesis testing problem with its null hypothesis $H^0_i$ that $\Phi^\partial_\epsilon(X_{\epsilon,i})$ and $\overline{\Psi}^\partial_\epsilon(T_0, X_{\epsilon,i})$ are from the same distribution. It is shown that the hypothesis $H^0_i$ cannot be rejected with 2% significance level, $\forall\, i \in \mathbb{S}^0_3$. More quantitative results are shown in Table 3. Hence, these greatly verify (⊗-3) and Theorem 3.3 from the side.

## 5.4. Stochastic delayed Lotka–Volterra cooperative models

Our aim in this section is to keep studying the Lotka–Volterra models, with a focus on cooperative interactions in ecosystems. Let $X_{\epsilon,1}(t)$ and $X_{\epsilon,2}(t)$ be the population size of two cooperation species at time $t$, which satisfy the following delayed equations:

$$\begin{cases} dX_{\epsilon,1}(t) = X_{\epsilon,1}(t)\big[r_1 + a_{12}\int_{-\infty}^t \Gamma_1(t-s)X_{\epsilon,2}(s)ds - a_{11}X_{\epsilon,1}(t)\big]dt \\ \qquad\qquad + \sqrt{\epsilon}X_{\epsilon,1}(t)\sum_{i=1}^2 \sigma_{1i}dW_i(t), \\ dX_{\epsilon,2}(t) = X_{\epsilon,2}(t)\big[r_2 + a_{21}\int_{-\infty}^t \Gamma_2(t-s)X_{\epsilon,1}(s)ds - a_{22}X_{\epsilon,2}(t)\big]dt \\ \qquad\qquad + \sqrt{\epsilon}X_{\epsilon,2}(t)\sum_{i=1}^2 \sigma_{2i}dW_i(t), \end{cases} \tag{5.15}$$

Table 3
List of the MDs of $\Phi_\epsilon(X_{\epsilon,1}, X_{\epsilon,2}, X_{\epsilon,3})$ in Example 5.3.

| MDs | Mean | Variance | Correlation coefficient |
|---|---|---|---|
| $\Phi_\epsilon^\partial(X_{\epsilon,1}, X_{\epsilon,2})$ | $(\ln 0.6069, \ln 0.1852)$ | $(0.32329, 9.5390) \times 10^{-4}$ | $-1.4497 \times 10^{-4}$ |
| $\Phi_\epsilon^\partial(X_{\epsilon,1}, X_{\epsilon,3})$ | $(\ln 0.6069, \ln 0.0224)$ | $(3.2329 \times 10^{-5}, 0.0122)$ | $1.2419 \times 10^{-4}$ |
| $\Phi_\epsilon^\partial(X_{\epsilon,2}, X_{\epsilon,3})$ | $(\ln 0.1852, \ln 0.0224)$ | $(9.5390 \times 10^{-4}, 0.0122)$ | $-7.8217 \times 10^{-4}$ |
| $\Phi_\epsilon^\partial(X_{\epsilon,1})$[a] | $\ln 0.6069$ | $3.2329 \times 10^{-5}$ | $--$ |
| $\Phi_\epsilon^\partial(X_{\epsilon,2})$[a] | $\ln 0.1852$ | $9.5390 \times 10^{-4}$ | $--$ |
| $\Phi_\epsilon^\partial(X_{\epsilon,3})$[a] | $\ln 0.0224$ | $0.0122$ | $--$ |

[a] Let $p_{i,T_0}^H$ be the minimum significance level that can reject the above hypothesis $H_i^0$ at iteration time $T_0$. In fact, the hypothesis $H_i^0$ is equivalent to the one that there is no difference between the distributions $\overline{\mu}_\epsilon^\partial(T_0, X_{\epsilon,i})$ and $\mathbb{LN}(\ln \overline{X}_{\epsilon,i}^*, \Sigma_\epsilon(i,i))$. Using several Kolmogorov–Smirnov tests, we have $p_{1,T_0}^H \le 1.51\%$, $p_{2,T_0}^H \le 0.69\%$, $p_{3,T_0}^H \le 1.72\%$, $\forall\ T_0 \in \{10000, 20000, 30000\}$. The relevant analysis of other numerical examples in Section 5 is exactly carried out along this line.

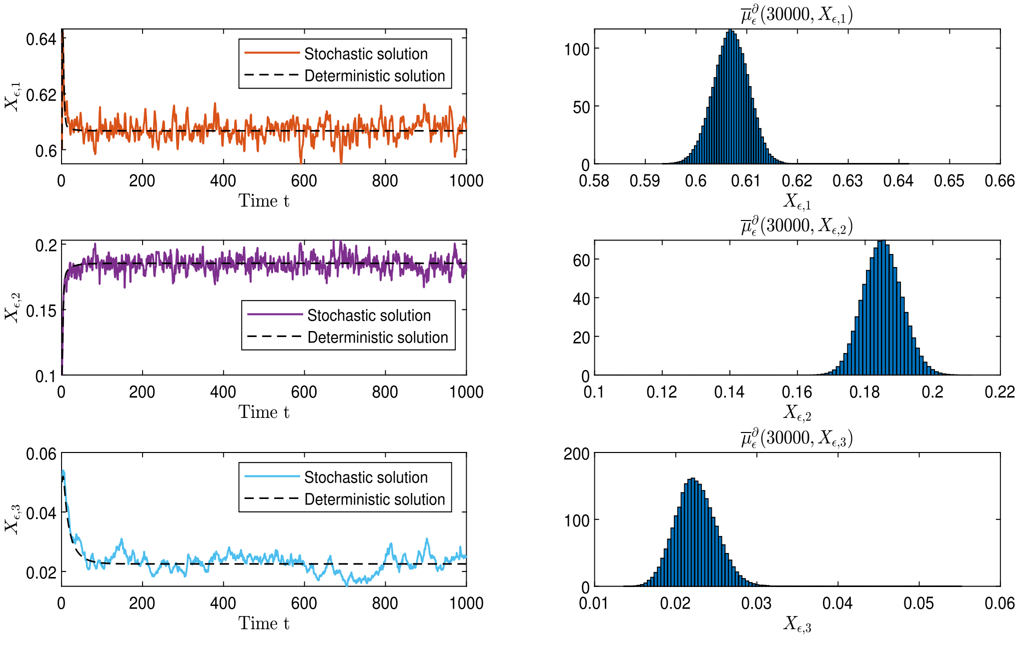


Fig. 7. The left-hand column depicts the variation trends of $X_{\epsilon,i}(t)$ $(i \in \mathbb{S}_3^0)$ of (5.11), and of its deterministic system on $t \in [0, 1000]$. The right-hand column shows the empirical MMs $\overline{\mu}_\epsilon^\partial(30000, X_{\epsilon,j})$ $(j \in \mathbb{S}_3^0)$ of (5.11). All the iteration step sizes are $\Delta t = 10^{-3}$.

where $r_i > 0$ and $a_{ii} > 0$ are the growth rate and intraspecific competition rate of species $X_{\epsilon,i}$, respectively, $i = 1, 2$; $a_{12}, a_{21} \ge 0$ represent the natural interspecific cooperation rates. As stated before in Section 5.2, the cumulative cooperation effect of the past state of $X_{\epsilon,j}$ on the current species $X_{\epsilon,i}(t)$ is described by the distributed delay function $\int_{-\infty}^t \Gamma_i(t-s)X_{\epsilon,j}(s)ds$, where $\{i, j\} = \mathbb{S}_2^0$, and the response kernel $\Gamma_i(\cdot)$ is simulated by the general Gamma distribution, which has the same form as (5.5) by replacing $(m, \alpha)$ with $(m_i, \alpha_i)$, i.e.,

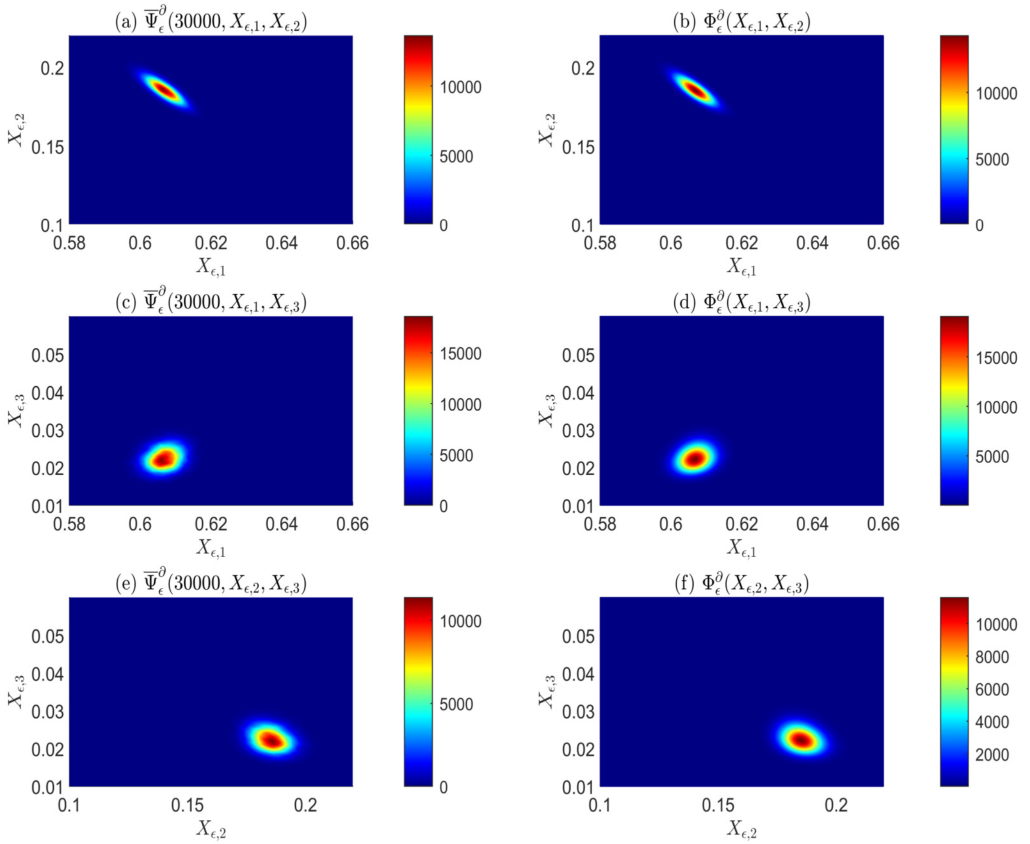


Fig. 8. (a), (c), (e): The empirical MDs $\overline{\Psi}_\epsilon^\partial(T_0, X_{\epsilon,1}, X_{\epsilon,2})$, $\overline{\Psi}_\epsilon^\partial(T_0, X_{\epsilon,1}, X_{\epsilon,3})$ and $\overline{\Psi}_\epsilon^\partial(T_0, X_{\epsilon,2}, X_{\epsilon,3})$ of (5.11) in 2D setting at iteration time $T_0 = 30000$; (b), (d), (f): The functions $\Phi_\epsilon^\partial(X_{\epsilon,1}, X_{\epsilon,2})$, $\Phi_\epsilon^\partial(X_{\epsilon,1}, X_{\epsilon,3})$ and $\Phi_\epsilon^\partial(X_{\epsilon,2}, X_{\epsilon,3})$ in 2D setting. All of the parameter values and step size are the same as in Fig. 7.

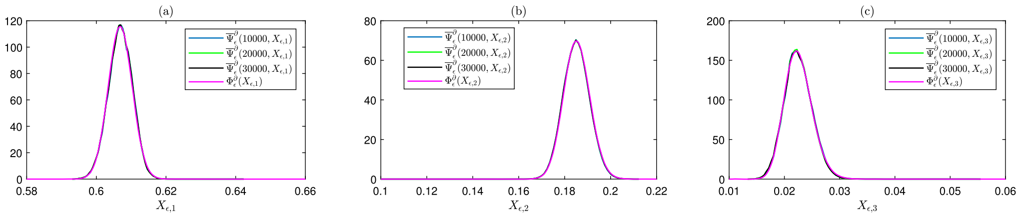


Fig. 9. The blue, green and black lines represent the empirical MDs $\overline{\Psi}_\epsilon^\partial(T_0, X_{\epsilon,i})$ ($i \in \mathbb{S}_3^0$) of (5.11) at iteration time $T_0 = 10000$, 20000 and 30000, respectively. The purple lines denote the one-dimensional MDs of $\Phi_\epsilon(X_{\epsilon,1}, X_{\epsilon,2}, X_{\epsilon,3})$. All of the parameter values and iteration step size are the same as in Fig. 7.

$$\Gamma_i(t) = \frac{t^{m_i} \alpha_i^{m_i+1} e^{-\alpha_i t}}{m_i!}, \quad t \geq 0.$$

To the best of our knowledge, no studies concerning the impact of generally distributed delay on cooperative interactions have been reported yet. To this end, we will do some work to fill this gap. It is worth noting that (5.15) does not fall into our general setting (1.2). By letting

$$u_{\epsilon,k}(t) = \int_{-\infty}^{t} \frac{(t-s)^{k-1}\alpha_2^k e^{-\alpha_2(t-s)}}{(k-1)!} X_{\epsilon,1}(s)ds, \quad \forall\, k \in \mathbb{S}^0_{m_2+1},$$

$$v_{\epsilon,l}(t) = \int_{-\infty}^{t} \frac{(t-s)^{l-1}\alpha_1^l e^{-\alpha_1(t-s)}}{(l-1)!} X_{\epsilon,2}(s)ds, \quad \forall\, k \in \mathbb{S}^0_{m_1+1},$$

system (5.15) can then be transformed into the following equivalent $(m_1+m_2+4)$-dimensional equations

$$\begin{cases} dX_{\epsilon,1}(t) = X_{\epsilon,1}(t)\big[r_1 - a_{11}X_{\epsilon,1}(t) + a_{12}v_{\epsilon,m_1+1}(t)\big]dt + \sqrt{\epsilon}X_{\epsilon,1}(t)\sum_{i=1}^{2}\sigma_{1i}dW_i(t), \\ du_{\epsilon,1}(t) = \alpha_1\big(X_{\epsilon,1}(t) - u_{\epsilon,1}(t)\big)dt, \\ du_{\epsilon,j}(t) = \alpha_1\big(u_{\epsilon,j-1}(t) - u_{\epsilon,j}(t)\big)dt, \quad \forall\, j \in \mathbb{S}^1_{m_2+1}, \\ dX_{\epsilon,2}(t) = X_{\epsilon,2}(t)\big[r_2 - a_{22}X_{\epsilon,2}(t) + a_{21}u_{\epsilon,m_2+1}(t)\big]dt + \sqrt{\epsilon}X_{\epsilon,2}(t)\sum_{i=1}^{2}\sigma_{2i}dW_i(t), \\ dv_{\epsilon,1}(t) = \alpha_2\big(X_{\epsilon,2}(t) - v_{\epsilon,1}(t)\big)dt, \\ dv_{\epsilon,j}(t) = \alpha_2\big(v_{\epsilon,j-1}(t) - v_{\epsilon,j}(t)\big)dt, \quad \forall\, j \in \mathbb{S}^1_{m_1+1}. \end{cases} \tag{5.16}$$

In this sense, we mainly analyze the IPM and IPDF of (5.16). Consider the following conditions

$$a_{11}a_{22} > a_{12}a_{21}, \quad r_1 > \frac{\epsilon(\sigma_{11}^2+\sigma_{12}^2)}{2}, \quad r_2 > \frac{\epsilon(\sigma_{21}^2+\sigma_{22}^2)}{2}. \tag{5.17}$$

It is shown that Assumptions 2.1 and 2.2(b) corresponding to (5.16) are satisfied under (5.17). The proof can refer to [94, Theorems 2.1 and 2.4] and [51, Lemmas 3.2 and 4.5] with a slight modification, and is thus omitted.

Repeating the procedures in Assumption 2.2(a) and (3.1)-(3.3), we define an equilibrium $(\overline{X}^*_{1,\epsilon}, \overline{\mathbf{u}}^*_\epsilon, \overline{X}^*_{2,\epsilon}, \overline{\mathbf{v}}^*_\epsilon)$ by

$$\begin{cases} \left(r_1 - \frac{\epsilon(\sigma_{11}^2+\sigma_{12}^2)}{2}\right) - a_{11}\overline{X}^*_{\epsilon,1} + a_{12}\overline{v}^*_{\epsilon,m_1+1} = 0, \\ \alpha_1\big(\overline{X}^*_{\epsilon,1} - \overline{u}^*_{\epsilon,1}\big) = 0, \\ \alpha_1\big(\overline{u}^*_{\epsilon,j-1} - \overline{u}^*_{\epsilon,j}\big) = 0, \quad \forall\, j \in \mathbb{S}^1_{m_2+1}, \\ \left(r_2 - \frac{\epsilon(\sigma_{21}^2+\sigma_{22}^2)}{2}\right) - a_{22}\overline{X}^*_{\epsilon,2} + a_{21}\overline{u}^*_{\epsilon,m_2+1} = 0, \\ \alpha_2\big(\overline{X}^*_{\epsilon,2} - \overline{v}^*_{\epsilon,1}\big) = 0, \\ \alpha_2\big(\overline{v}^*_{\epsilon,j-1} - \overline{v}^*_{\epsilon,j}\big) = 0, \quad \forall\, j \in \mathbb{S}^1_{m_1+1}, \end{cases} \tag{5.18}$$

where $\overline{\mathbf{u}}^*_\epsilon = (\overline{u}^*_{\epsilon,1}, ..., \overline{u}^*_{\epsilon,m_2+1})$ and $\overline{\mathbf{v}}^*_\epsilon = (\overline{v}^*_{\epsilon,1}, ..., \overline{v}^*_{\epsilon,m_1+1})$. In fact, if (5.17) holds, the solution of Eq. (5.18) is unique, positive, and it is

$$\overline{u}^*_{\epsilon,i}=\overline{X}^*_{\epsilon,1}=\frac{a_{22}(r_1-\frac{\epsilon(\sigma_{11}^2+\sigma_{12}^2)}{2})+a_{12}(r_2-\frac{\epsilon(\sigma_{21}^2+\sigma_{22}^2)}{2})}{a_{11}a_{22}-a_{12}a_{21}},\quad \forall\, i\in\mathbb{S}^0_{m_2+1},$$

$$\overline{v}^*_{\epsilon,j}=\overline{X}^*_{\epsilon,2}=\frac{a_{11}(r_2-\frac{\epsilon(\sigma_{21}^2+\sigma_{22}^2)}{2})+a_{21}(r_1-\frac{\epsilon(\sigma_{11}^2+\sigma_{12}^2)}{2})}{a_{11}a_{22}-a_{12}a_{21}},\quad \forall\, j\in\mathbb{S}^0_{m_1+1}.$$

Then we have

$$A_\epsilon=B_\epsilon:=\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix},\tag{5.19}$$

where

$$K_{ij}=\begin{pmatrix}a_{ij}\overline{X}^*_{\epsilon,j}\boldsymbol{\beta}_{m_i+2}\\ \mathbb{O}\end{pmatrix},\quad K_{ii}=\begin{pmatrix}-a_{ii}\overline{X}^*_{\epsilon,i} & & & & \\ \sigma_i & -\sigma_i & & & \\ & \sigma_i & -\sigma_i & & \\ & & \ddots & \ddots & \\ & & & \sigma_i & -\sigma_i\end{pmatrix}$$

with $\{i,j\}=\mathbb{S}^0_2$. Clearly, $K_{ii}\in\mathcal{U}_q(m_j+2)$. By calculation,

$$\psi_{B_\epsilon}(\lambda)=\prod_{i=1}^{2}(\lambda+a_{ii}\overline{X}^*_{\epsilon,i})(\lambda+\alpha_i)^{m_i+1}-a_{12}a_{21}\prod_{i=1}^{2}\overline{X}^*_{\epsilon,i}\alpha_i^{m_i+1}.\tag{5.20}$$

Below we prove $B_\epsilon\in\overline{\mathbf{RH}}(m_1+m_2+4)$. Using the contradiction method, it is assumed that $B_\epsilon$ has at least an eigenvalue $\lambda_0$ with positive real component, i.e., $\lambda_0:=a+b\mathrm{i}$ ($a>0$). By fundamental theorem of algebra, $\overline{\lambda}_0=a-b\mathrm{i}$ is a root of equation $\psi_{B_\epsilon}(\lambda)=0$ (if $b=0$, then $\overline{\lambda}_0=\lambda_0$). As a consequence of (5.20),

$$\begin{cases}\prod_{i=1}^{2}\Big(1+\frac{\lambda_0}{a_{ii}\overline{X}^*_{\epsilon,i}}\Big)\Big(1+\frac{\lambda_0}{\alpha_i}\Big)^{m_i+1}=\frac{a_{12}a_{21}}{a_{11}a_{22}},\\ \prod_{i=1}^{2}\Big(1+\frac{\overline{\lambda}_0}{a_{ii}\overline{X}^*_{\epsilon,i}}\Big)\Big(1+\frac{\overline{\lambda}_0}{\alpha_i}\Big)^{m_i+1}=\frac{a_{12}a_{21}}{a_{11}a_{22}}.\end{cases}$$

This leads to the contradiction

$$\begin{aligned}\Big(\frac{a_{12}a_{21}}{a_{11}a_{22}}\Big)^2&=\prod_{i=1}^{2}\Big(1+\frac{\lambda_0}{a_{ii}\overline{X}^*_{\epsilon,i}}\Big)\Big(1+\frac{\overline{\lambda}_0}{a_{ii}\overline{X}^*_{\epsilon,i}}\Big)\Big(1+\frac{\lambda_0}{\alpha_i}\Big)^{m_i+1}\Big(1+\frac{\overline{\lambda}_0}{\alpha_i}\Big)^{m_i+1}\\ &>\prod_{i=1}^{2}\Big(1+\frac{2a}{a_{ii}\overline{X}^*_{\epsilon,i}}\Big)\Big(1+\frac{2a}{\alpha_i}\Big)^{m_i+1}>1.\end{aligned}$$

Hence, $B_\epsilon\in\overline{\mathbf{RH}}(m_1+m_2+4)$.

Based on the LNA method, we need to obtain the special expression and positive definiteness of $\Sigma_\epsilon$ in Eq. (3.8) corresponding to (5.16). The former can be derived by Algorithm 1. Then for the latter, let $\lambda_0^+$ be the minimal eigenvalue of the following matrix

$$\begin{pmatrix} \sigma_{11}^2+\sigma_{12}^2 & \sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22} \\ \sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22} & \sigma_{21}^2+\sigma_{22}^2 \end{pmatrix} := \Upsilon_0.$$

If $\sigma_{11}\sigma_{22} > \sigma_{12}\sigma_{21}$, then $\Upsilon_0 \succ \mathbb{O}$, which implies $\lambda_0^+ > 0$ and

$$\Theta_\epsilon\Theta_\epsilon^\top \succeq \lambda_0^+\big(\amalg_{m_1+m_2+4,1} + \amalg_{m_1+m_2+4,m_2+3}\big).$$

By virtue of Algorithm 2, we consider the following algebraic equations:

$$\begin{cases} \Im_c(\Sigma_{1,\epsilon}^\circ, B_\epsilon, \amalg_{2m+4,1}) = \mathbb{O}, \\ \Im_c(\Sigma_{m_2+3,\epsilon}^\circ, B_\epsilon, \amalg_{m_1+m_2+4,m_2+3}) = \mathbb{O}. \end{cases}$$

As in (5.19), one has $B_\epsilon \in \mathcal{U}_q(m_1+m_2+4)$. Note that $\phi_1^\circ = 1$, we obtain $\nu_{1(i)}^\circ = i+1$ and $\underline{J}_1 = \underline{Q}_{1,i} = \mathbf{I}_{m_1+m_2+4}$, $\forall\, i \in \mathbb{S}_{m_1+m_2+3}^0$. That is, $\eta_1^\circ = m_1+m_2+4$ and $\underline{\mathbf{H}}_{1,j}^{(j)} = X_j$, $\forall\, j \in \mathbb{S}_{m_1+m_2+4}^0$. Applying Theorem 3.2 and $\eta_2^\circ \geq 1$ leads to

$$\mathbf{X}^\top\Sigma_\epsilon\mathbf{X} \geq \rho_\epsilon^\circ\Big(X_{\phi_1^\circ}^2 + X_{\phi_2^\circ}^2 + \sum_{j=2}^{\eta_1^\circ}(\underline{\mathbf{H}}_{\phi_1^\circ,j}^{(j)})^2\Big) \geq \rho_\epsilon^\circ\sum_{j=1}^{2m+4} X_j^2,$$

where $\rho_\epsilon^\circ > 0$ is a constant dependent on $\lambda_0^+$. Thus, $\Sigma_\epsilon \succ \mathbb{O}$.

To summarize, using Theorems 3.1-3.3, we have the following result:

($\otimes$-4) Under (5.17) and $\sigma_{11}\sigma_{22} > \sigma_{12}\sigma_{21}$, $\Phi_\epsilon(X_{\epsilon,1}, \mathbf{u}_\epsilon, X_{\epsilon,2}, \mathbf{v}_\epsilon)$ (resp., $\mathbb{LN}_{m_1+m_2+2}(\ln(\overline{X}_{1,\epsilon}^*, \overline{\mathbf{u}}_\epsilon^*, \overline{X}_{2,\epsilon}^*, \overline{\mathbf{v}}_\epsilon^*)^\top, \Sigma_\epsilon)$) is a local approximation for the unique IPDF $\Psi_\epsilon(\cdot)$ (resp., IPM $\mu_\epsilon$) of (5.16) around $(\overline{X}_{1,\epsilon}^*, \overline{\mathbf{u}}_\epsilon^*, \overline{X}_{2,\epsilon}^*, \overline{\mathbf{v}}_\epsilon^*)^\top$, where $\Sigma_\epsilon \succ \mathbb{O}$, $\mathbf{u}_\epsilon(t) = (u_{\epsilon,1}(t), ..., u_{\epsilon,m_2+1}(t))_{t\geq 0}$ and $\mathbf{v}_\epsilon(t) = (v_{\epsilon,1}(t), ..., v_{\epsilon,m_1+1}(t))_{t\geq 0}$. In addition, the fitting effect of such approximation is global for sufficiently small $\epsilon$.

To demonstrate this, we present a numerical example. We adopt the weak kernels to describe the cumulative cooperation effect in (5.15) for simplicity, i.e., $m_i = 0$, $\forall\, i = 1, 2$.

**Example 5.4.** Let the initial value $(X_{\epsilon,1}(0), u_{\epsilon,1}(0), X_{\epsilon,2}(0), v_{\epsilon,1}(0)) = (0.5, 0.4, 0.6, 0.5)$. Following Qi et al. [51], we consider

$$r_1 = 0.295,\ r_2 = 0.3,\ a_{11} = 0.75,\ a_{12} = 0.05,\ a_{21} = 0.05,\ a_{22} = 0.65,$$

$$\alpha_1 = 0.1,\ \alpha_2 = 0.2,\ (\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}) = (0.3, 0.2, 0.1, 0.2).$$

By choosing $\epsilon = 2\times 10^{-2}$, we compute $\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} = 0.4$, $a_{11}a_{22} - a_{12}a_{21} = 0.485$, $r_1 - \frac{\epsilon(\sigma_{11}^2+\sigma_{12}^2)}{2} = 0.2937$, $r_2 - \frac{\epsilon(\sigma_{21}^2+\sigma_{22}^2)}{2} = 0.2995$, $(\overline{X}_{\epsilon,1}^*, \overline{X}_{\epsilon,2}^*) = (0.4245, 0.4934)$, and

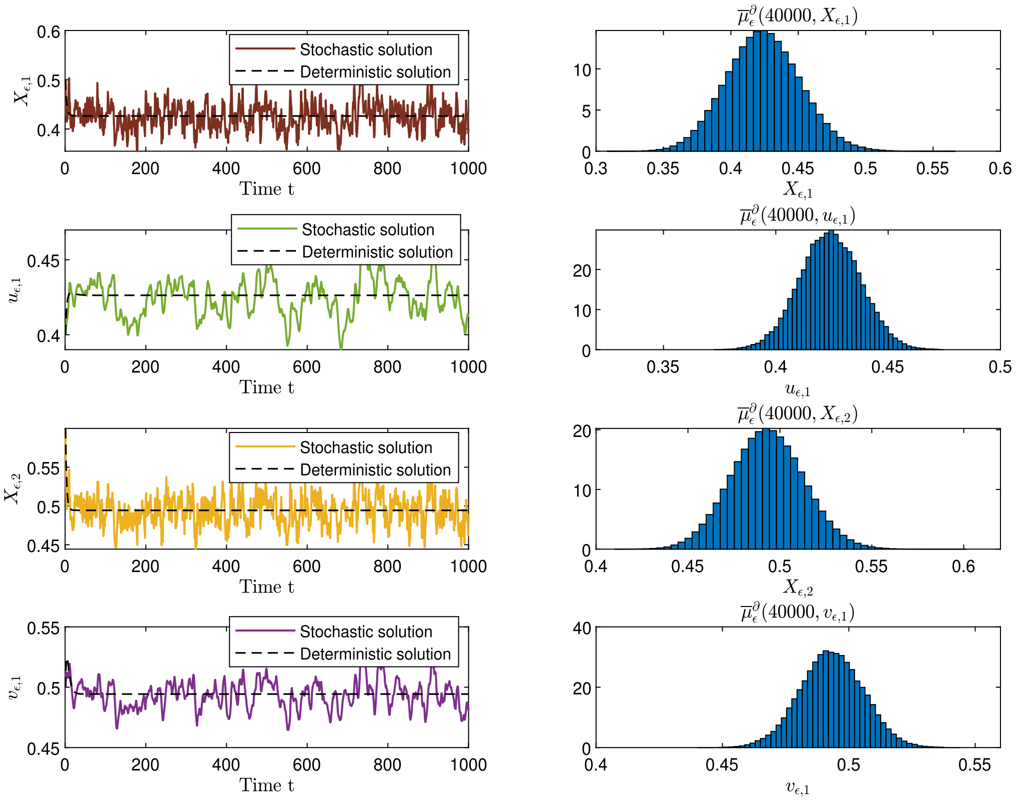


Fig. 10. The left-hand column presents the sample paths of $X_{\epsilon,1}(t)$, $u_{\epsilon,1}(t)$, $X_{\epsilon,2}(t)$ and $v_{\epsilon,1}(t)$ of (5.16), and of its deterministic model on $t \in [0, 1000]$. The right-hand column shows all the one-dimensional MMs of $\overline{\mu}_\epsilon(40000, X_{\epsilon,1}, u_{\epsilon,1}, X_{\epsilon,2}, v_{\epsilon,1})$. All the iteration step sizes are $\Delta t = 10^{-3}$.

$$\Sigma_\epsilon = \begin{pmatrix} 0.0042 & 0.0010 & 0.0022 & 8.9853 \times 10^{-4} \\ 0.0010 & 0.0010 & 5.8685 \times 10^{-4} & 6.9074 \times 10^{-4} \\ 0.0022 & 5.8685 \times 10^{-4} & 0.0016 & 6.4184 \times 10^{-4} \\ 8.9853 \times 10^{-4} & 6.9074 \times 10^{-4} & 6.4184 \times 10^{-4} & 6.4184 \times 10^{-4} \end{pmatrix}.$$

According to ($\otimes$-4), the IPDF $\Psi_\epsilon(X_{\epsilon,1}, u_{\epsilon,1}, X_{\epsilon,2}, v_{\epsilon,1})$ can be globally approximated by $\Phi_\epsilon(X_{\epsilon,1}, u_{\epsilon,1}, X_{\epsilon,2}, v_{\epsilon,1})$. To verify this, by a standard argument in Example 5.2, we mainly focus on the similarities between: (i) $\overline{\Psi}_\epsilon^\partial(T_0, X_{\epsilon,1}, X_{\epsilon,2})$ and $\Phi_\epsilon^\partial(X_{\epsilon,1}, X_{\epsilon,2})$, (ii) all the corresponding one-dimensional MDs of $\Phi_\epsilon(\cdot)$ and $\overline{\Psi}_\epsilon^\partial(T_0, \cdot)$, at large iteration time $T_0$. As shown in Figs. 10–12, these corresponding density curves are very similar. Moreover, using several Kolmogorov–Smirnov tests with 5% significance level, the similarities regarding (ii) are significant. Thus, ($\otimes$-4) and Theorem 3.3 are well verified.

### 5.5. *Stochastic HIV/AIDS infection models*

In Sections 5.1-5.4, we use the LNA method to approximate the IPDFs of some common biomathematical models with small diffusion in the literature. Certainly, the relevant analysis of

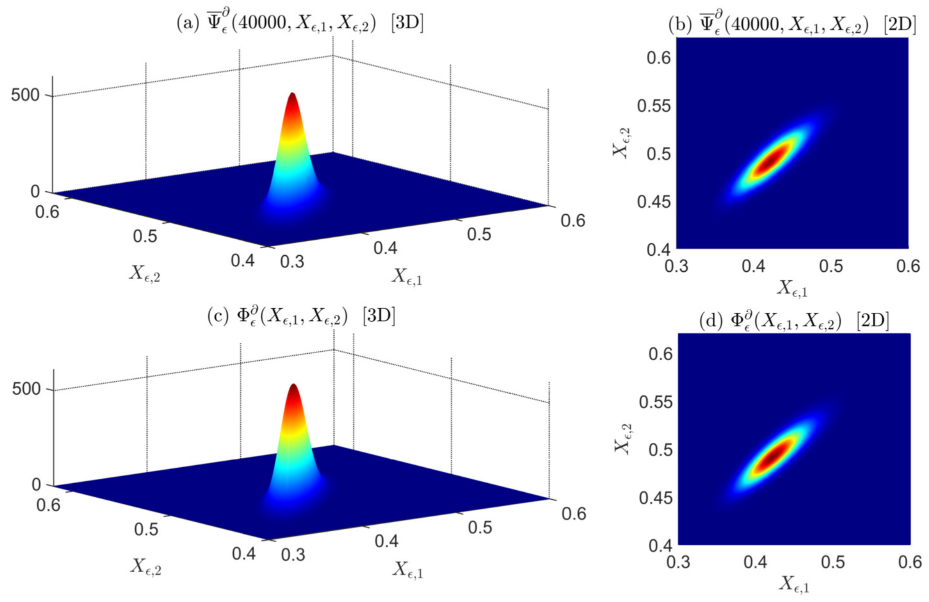


Fig. 11. (a), (b): The empirical MD $\overline{\Psi}^{\partial}_{\epsilon}(T_0, X_{\epsilon,1}, X_{\epsilon,2})$ of (5.16) (or equivalently, the empirical density $\overline{\Psi}_{\epsilon}(T_0, X_{\epsilon,1}, X_{\epsilon,2})$ of (5.15)) in 2D and 3D settings at iteration time $T_0 = 40000$; (c), (d): The function $\Phi^{\partial}_{\epsilon}(X_{\epsilon,1}, X_{\epsilon,2})$ in 2D and 3D settings. All of the parameter values and step size are the same as in Fig. 10.

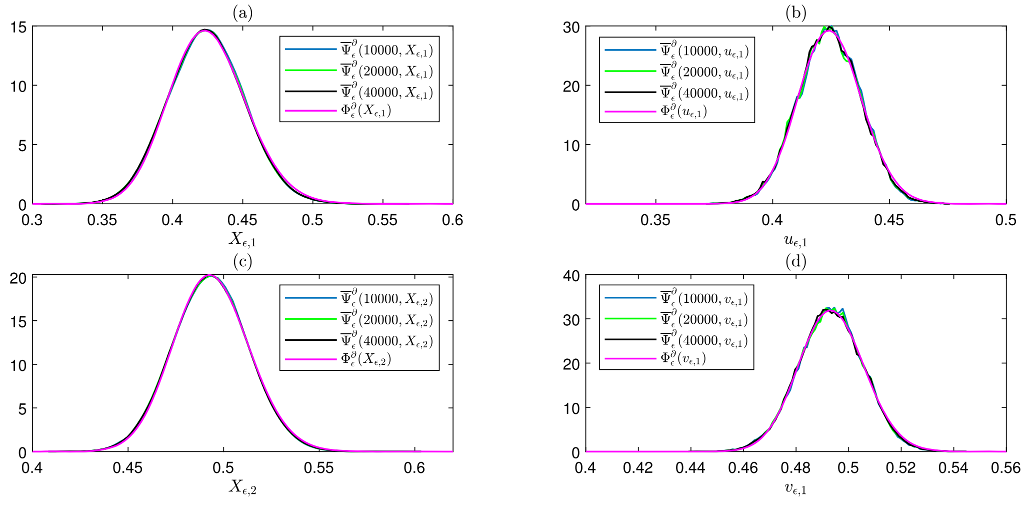


Fig. 12. The blue, green and black lines represent the empirical MDs $\overline{\Psi}^{\partial}_{\epsilon}(T_0, X_{\epsilon,1})$, $\overline{\Psi}^{\partial}_{\epsilon}(T_0, u_{\epsilon,1})$, $\overline{\Psi}^{\partial}_{\epsilon}(T_0, X_{\epsilon,2})$ and $\overline{\Psi}^{\partial}_{\epsilon}(T_0, v_{\epsilon,1})$ of (5.16) at iteration time $T_0 = 10000$, 20000 and 40000, respectively. The purple lines denote the one-dimensional MDs of $\Phi_{\epsilon}(X_{\epsilon,1}, u_{\epsilon,1}, X_{\epsilon,2}, v_{\epsilon,1})$. All of the parameter values and iteration step size are the same as in Fig. 10.

the uNA method can be similarly carried out. To supplement, we provide a new application of the uNA method in this section, where a detailed account of how to use Algorithm 3 and (4.4) to study the normal approximation is mainly presented.

Similar to [67], we focus on a high-dimensional stochastic HIV/AIDS infection model that takes into account virus carrier screening and the active search for treatment by infected individuals. Inspired by the idea of the SIR model, the population at time $t$ is divided into six compartments, which include susceptible individuals $S_\epsilon(t)$, infectious and symptomatic primary HIV-infected individuals $I_\epsilon(t)$, asymptomatic and disease carriers $C_\epsilon(t)$, randomly screened disease carriers $C_{s,\epsilon}(t)$, individuals under treatment $T_\epsilon(t)$ and individuals with full-blown AIDS $L_\epsilon(t)$. The corresponding infection dynamics can be described by

$$\begin{cases} dS_\epsilon(t) = [\Lambda - k_0(a_1 I_\epsilon(t) + a_2 C_\epsilon(t) + a_3 T_\epsilon(t) + a_4 L_\epsilon(t))S_\epsilon(t) - pS_\epsilon(t)]dt \\ \qquad\qquad + \sqrt{\epsilon}\sigma_1 S_\epsilon(t) dW_1(t), \\ dI_\epsilon(t) = [k_0(a_1 I_\epsilon(t) + a_2 C_\epsilon(t) + a_3 T_\epsilon(t) + a_4 L_\epsilon(t))S_\epsilon(t) - (p+\sigma+\vartheta_1+\gamma_1)I_\epsilon(t)]dt, \\ dC_\epsilon(t) = [\sigma I_\epsilon(t) - (p+\vartheta_2+\alpha_s)C_\epsilon(t)]dt, \\ dC_{s,\epsilon}(t) = [\alpha_s C_\epsilon(t) - (p+\vartheta_3+\gamma_2)C_{s,\epsilon}(t)]dt, \\ dT_\epsilon(t) = [\gamma_1 I_\epsilon(t) + \gamma_2 C_{s,\epsilon}(t) - (p+\vartheta_4)T_\epsilon(t)]dt + \sqrt{\epsilon}\sigma_2 T_\epsilon(t) dW_2(t), \\ dL_\epsilon(t) = [\vartheta_1 I_\epsilon(t) + \vartheta_2 C_\epsilon(t) + \vartheta_3 C_{s,\epsilon}(t) + \vartheta_4 T_\epsilon(t) - (p+\gamma_3+p_1)L_\epsilon(t)]dt, \end{cases} \tag{5.21}$$

where $\Lambda$ is the recruitment rate, and the average number of sexual partners $k_0$ measures risk behavior. $p$ and $p_1$ are the natural death rate and disease-induced mortality rate, respectively. The infection probabilities for different infectious individuals $I_\epsilon(t)$, $C_\epsilon(t)$, $T_\epsilon(t)$ and $L_\epsilon(t)$ are $a_i$ $(i \in \mathbb{S}_4^0)$ in sequence. The individuals $I_\epsilon(t)$, $C_\epsilon(t)$, $C_{s,\epsilon}(t)$ and $T_\epsilon(t)$ become AIDS patients $L_\epsilon(t)$ at the rate of $\vartheta_i$ $(i \in \mathbb{S}_4^0)$, respectively. $\gamma_j$ $(j \in \mathbb{S}_3^0)$ are the rates at which the infected seek treatment, $\sigma$ stands for the rate from the infected individuals $I_\epsilon(t)$ to carriers $C_\epsilon(t)$, and $\alpha_s$ is the rate at which carriers are screened. Similar to the idea of (5.11), it is assumed that the environmental randomness mainly affects the individuals $S_\epsilon(t)$ and $T_\epsilon(t)$.

Taking steps along the procedures in Assumption 2.2(c) and (4.1), we first consider the positive equilibrium $\mathbf{X}_{hiv}^* = (S^*, I^*, C^*, C_s^*, T^*, L^*)^\top$ of the deterministic counterpart of (5.21), which satisfies

$$\begin{cases} \Lambda - k_0(a_1 I^* + a_2 C^* + a_3 T^* + a_4 L^*)S^* - pS^* = 0, \\ k_0(a_1 I^* + a_2 C^* + a_3 T^* + a_4 L^*)S^* - (p+\sigma+\vartheta_1+\gamma_1)I^* = 0, \\ \sigma I^* - (p+\vartheta_2+\alpha_s)C^* = 0, \\ \alpha_s C^* - (p+\vartheta_3+\gamma_2)C_s^* = 0, \\ \gamma_1 I^* + \gamma_2 C_s^* - (p+\vartheta_4)T^*, \\ \vartheta_1 I^* + \vartheta_2 C^* + \vartheta_3 C_s^* + \vartheta_4 T^* - (p+\gamma_3+p_1)L^* = 0. \end{cases}$$

Denote

$$\mathscr{R}_0 = \frac{\Lambda k_0(a_1 + a_2 w_1 + a_3 w_3 + a_4 w_4)}{p(p+\sigma+\vartheta_1+\gamma_1)}, \quad \mathscr{R}_{4,\epsilon}^S = \frac{\Lambda k_0(a_1 + a_2 w_1 + a_3 \widetilde{w}_3 + a_4 \widetilde{w}_4)}{(p+\frac{\epsilon\sigma_1^2}{2})(p+\sigma+\vartheta_1+\gamma_1)},$$

where

$$w_1=\frac{\sigma}{p+\vartheta_2+\alpha_s},\quad w_2=\frac{\alpha_s w_1}{p+\gamma_2+\vartheta_3},\quad w_3=\frac{\gamma_1+\gamma_2 w_2}{p+\vartheta_4},\quad w_4=\frac{\vartheta_1+\vartheta_2 w_1+\vartheta_3 w_2+\vartheta_4 w_3}{p+\gamma_3+p_1}.$$

$$\widetilde{w}_3=\frac{\gamma_1+\gamma_2 w_2}{p+\vartheta_4+\frac{\epsilon\sigma_2^2}{2}},\quad \widetilde{w}_4=\frac{\vartheta_1+\vartheta_2 w_1+\vartheta_3 w_2+\vartheta_4 \widetilde{w}_3}{p+\gamma_3+p_1}.$$

We have the following Lemma.

**Lemma 5.1.** *For any initial value* $(S_\epsilon(0), I_\epsilon(0), C_\epsilon(0), C_{s,\epsilon}(0), T_\epsilon(0), L_\epsilon(0))^\top := \mathbf{X}_{\epsilon,h}(0) \in \mathbb{R}^6_+$,

- *If* $\mathscr{R}_0 > 1$, *then the equilibrium* $\mathbf{X}^*_{hiv} = (S^*, I^*, C^*, C^*_s, T^*, L^*)^\top$ *exists and is unique on* $\mathbb{R}^6_+$, *where* $S^* = \frac{\Lambda}{p\mathscr{R}_0}$, $I^* = \frac{\Lambda(\mathscr{R}_0-1)}{\mathscr{R}_0(p+\sigma+\vartheta_1+\gamma_1)}$, $C^* = w_1 I^*$, $C^*_s = w_2 I^*$, $T^* = w_3 I^*$ *and* $L^* = w_4 I^*$. *Moreover,* $A_{[o]} \in \overline{\mathbf{RH}}(6)$, *where*

$$A_{[o]}=C_{[o]}=\begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & 0 & -a_{15} & -a_{16}\\ a_{21} & -a_{22} & a_{23} & 0 & a_{25} & a_{26}\\ 0 & a_{32} & -a_{33} & 0 & 0 & 0\\ 0 & 0 & a_{43} & -a_{44} & 0 & 0\\ 0 & a_{52} & 0 & a_{54} & -a_{55} & 0\\ 0 & a_{62} & a_{63} & a_{64} & a_{65} & -a_{66} \end{pmatrix},$$

  *with* $a_{11} = p + k_0(a_1 I^* + a_2 C^* + a_3 T^* + a_4 L^*)$, $a_{12} = k_0 a_1 S^*$, $a_{13} = k_0 a_2 S^*$, $a_{15} = k_0 a_3 S^*$, $a_{16} = k_0 a_4 S^*$, $a_{21} = k_0(a_1 I^* + a_2 C^* + a_3 T^* + a_4 L^*)$, $a_{22} = p + \sigma + \vartheta_1 + \gamma_1 - k_0 a_1 S^*$, $a_{2i} = a_{1i}$ $(i = 3, 5, 6)$, $a_{32} = \sigma$, $a_{33} = p + \vartheta_2 + \alpha_s$, $a_{43} = \alpha_s$, $a_{44} = p + \vartheta_3 + \gamma_2$, $a_{52} = \gamma_1$, $a_{54} = \gamma_2$, $a_{55} = p + \vartheta_4$, $a_{62} = \vartheta_1$, $a_{63} = \vartheta_2$, $a_{64} = \vartheta_3$, $a_{65} = \vartheta_4$ *and* $a_{66} = p + \gamma_3 + p_1$.
- *System (5.21) has a unique solution* $\mathbf{X}_{\epsilon,h}(t)$ *on* $t \geq 0$ *and it will remain in* $\mathbb{R}^6_+$ *a.s. If* $\mathscr{R}^S_{4,\epsilon} > 1$, *then (5.21) has at least one IPM* $\mu_{\epsilon,h}(\cdot)$ *supported on* $\mathbb{R}^6_+$.

The proof can completely refer to [67, Section 2.2 and Theorem 4.1] with a slight modification.

We consider the linearized system of (5.21) around $\mathbf{X}^*_{hiv}$:

$$\begin{cases} d\mathbf{Z}_{\epsilon,s}(t) = A_{[o]}\mathbf{Z}_{\epsilon,s}(t)dt + \sqrt{\epsilon}\Gamma_s d\mathbf{W}_s(t),\\ \mathbf{Z}_{\epsilon,s}(0) = \mathbf{X}_{\epsilon,h}(0) - \mathbf{X}^*_{hiv}, \end{cases} \tag{5.22}$$

where $\Gamma_s\Gamma_s^\top = \mathrm{diag}\{(\sigma_1 S^*)^2, 0, 0, (\sigma_2 T^*)^2, 0, 0\}$ and $\mathbf{W}_s(t) = (W_1(t), W_2(t))^\top$. In view of Lemma 5.1, (5.22) has a unique IPM $\mathbb{N}_6(\mathbf{0}, \Sigma_{[o]\epsilon})$ if $\mathscr{R}_0 > 1$, where $\Sigma_{[o]\epsilon}$ is determined by equation $\Im_c(\Sigma_{[o]\epsilon}, A_{[o]}, \epsilon\Gamma_s\Gamma_s^\top) = \mathbb{O}$. Although Assumption 2.1(2) corresponding to (5.21) is difficult to verify currently, Algorithm 3 and (4.4) can be still used to treat $\Sigma_{[o]\epsilon}$. Clearly, $\overline{\xi} = 2$ and $\overline{\boldsymbol{\phi}} = \{1, 5\}$. Using **Rule 3**, we choose $\bigvee_1^{\text{asso}} = \{1, 2, 3, 4\}$ and $\bigvee_1^{\text{asso}} = \{5, 6, 1\}$. Then,

Step 1. Consider the algebraic equation

$$\Im_c(\Sigma_{[o]1,\epsilon}, A_{[o]}, Ш_{6,1}) = \mathbb{O}.$$

By Algorithm 3 and the chain $(Y_1, \bigvee_1^{\text{asso}} Y_\bullet)$, we determine $\overline{\nu}_{1(1)} = 2$ and $Q_{[o]1,1} = \mathbf{I}_6$. Thus, $\mathbf{G}_{1,2}^{(2)} = X_2$. Moreover, we select $\overline{\nu}_{1(2)} = 3$ and

$$Q_{[o]1,2} = \begin{pmatrix} \mathbf{I}_2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & 1 & \mathbb{O} \\ \mathbb{O} & \boldsymbol{\ell}_{[o]1,3} & \mathbf{I}_3 \end{pmatrix},$$

where $\boldsymbol{\ell}_{[o]1,3} = (0, -\frac{a_{52}}{a_{32}}, -\frac{a_{62}}{a_{32}})^\top$. In this sense,

$$\mathbf{G}_{1,3}^{(3)} = X_3 + \frac{a_{52}}{a_{32}} X_5 + \frac{a_{62}}{a_{32}} X_6.$$

By further letting $\overline{\nu}_{1(3)} = 4$ and $\boldsymbol{\ell}_{[o]1,4} = (-\frac{a_{52}(a_{33}-a_{55})}{a_{32}a_{43}}, -\frac{a_{62}(a_{33}+a_{66})+a_{52}a_{65}+a_{32}a_{63}}{a_{32}a_{43}})^\top$, then

$$\mathbf{G}_{1,4}^{(4)} = X_4 + \frac{a_{52}(a_{33} - a_{55})}{a_{32}a_{43}} X_5 + \frac{a_{62}(a_{33} + a_{66}) + a_{52}a_{65} + a_{32}a_{63}}{a_{32}a_{43}} X_6,$$

which implies $\overline{\eta}_1 \geq 4$.

Step 2. Consider the algebraic equation

$$\Im_c(\Sigma_{[o]5,\epsilon}, A_{[o]}, \amalg_{6,5}) = \mathbb{O}.$$

In view of the chain $(Y_5, \bigvee_5^{\text{asso}} Y_\bullet)$, we let $\overline{\nu}_{2(1)} = 6$ and $\boldsymbol{\ell}_{[o]2,4} = (\frac{a_{15}}{a_{65}}, -\frac{a_{25}}{a_{65}}, 0, 0)^\top$. That means

$$\mathbf{G}_{5,2}^{(2)} = X_6 - \frac{a_{15}}{a_{65}} X_1 + \frac{a_{25}}{a_{65}} X_2.$$

Hence, $\overline{\eta}_2 \geq 2$.

Combining the above two steps with (4.4), we obtain

$$\begin{aligned}
\mathbf{X}^\top \Sigma_{[o]\epsilon} \mathbf{X} \geq & \varrho_\epsilon \sum_{k=1}^{2} \Big( X_{\overline{\phi}_k}^2 + \sum_{j=1}^{\overline{\eta}_k} (\mathbf{G}_{\overline{\phi}_k, j}^{(j)})^2 \Big) \\
\geq & \varrho_\epsilon \Big[ \Big( X_1^2 + \sum_{j=2}^{4} (\mathbf{G}_{1,j}^{(j)})^2 \Big) + \Big( X_5^2 + (\mathbf{G}_{5,2}^{(2)})^2 \Big) \Big] \\
= & \varrho_\epsilon \Big\{ X_1^2 + X_2^2 + X_5^2 + \Big( X_3 + \frac{a_{52}}{a_{32}} X_5 + \frac{a_{62}}{a_{32}} X_6 \Big)^2 + \Big( X_6 - \frac{a_{15}}{a_{65}} X_1 + \frac{a_{25}}{a_{65}} X_2 \Big)^2 \\
& + \Big[ X_4 + \frac{a_{52}(a_{33} - a_{55})}{a_{32}a_{43}} X_5 + \frac{a_{62}(a_{33} + a_{66}) + a_{52}a_{65} + a_{32}a_{63}}{a_{32}a_{43}} X_6 \Big]^2 \Big\}.
\end{aligned}$$

Clearly, $\{\mathbf{X} \in \mathbb{R}^6 | \mathbf{X}^\top \Sigma_\epsilon \mathbf{X} = 0\} = \{\mathbf{0}\}$. Thus, $\Sigma_{[o]\epsilon} \succ \mathbb{O}$. The explicit form of $\Sigma_{[o]\epsilon}$ can be further derived by Algorithm 3 and (4.3).

For convenience, let $\Phi_{[o]\epsilon,s}(\cdot)$ and $\Phi_{[o]\epsilon,h}(\cdot)$ be the density functions of the distributions $\mathbb{N}_6(\mathbf{0}, \Sigma_{[o]\epsilon})$ and $\mathbb{N}_6(\mathbf{X}_{hiv}^*, \Sigma_{[o]\epsilon})$, respectively. Then by Theorem 4.1 and Remark 6, we determine:

Table 4
List of values of biological parameters of (5.21).

| Parameters | Value (Unit: year$^{-1}$) | Source | Parameters | Value (Unit: year$^{-1}$) | Source |
|---|---|---|---|---|---|
| $\Lambda$ | 25000 people | [67,95] | $\gamma_1$ | 0.2 | [67,95] |
| $k_0$ | 1.2 | [67,95] | $\vartheta_2$ | 0.09 | [67,95] |
| $a_1$ | $0.8 \times 10^{-5}$ | [67] | $\alpha$ | 0.2 | [67,95] |
| $a_2$ | $0.6 \times 10^{-5}$ | [67] | $\vartheta_3$ | 0.4 | [67,95] |
| $a_3$ | $0.4 \times 10^{-5}$ | [67] | $\gamma_2$ | 0.15 | [67,95] |
| $a_4$ | $0.7 \times 10^{-5}$ | [67] | $\vartheta_4$ | 0.2 | [95] |
| $p$ | 0.015 | [67], CSZ data | $\gamma_3$ | 0.3 | [67,95] |
| $\sigma$ | 0.2 | [95] | $p_1$ | 0.33 | [67,95] |
| $\vartheta_1$ | 0.001 | Estimated | $\sigma_i$ $(i = 1, 2)$ | 0.5 | [67] |

($\otimes$-5) If $\mathscr{R}_0 > 1$, system (5.22) has a unique IPDF $\Phi_{[o]\epsilon,s}(\mathbf{Z}_{\epsilon,s})$. Moreover, the probability density of $\mu_{\epsilon,h}(\cdot)$ of (5.21) around $\mathbf{X}^*_{hiv}$ can be approximated by $\Phi_{[o]\epsilon,h}(\mathbf{X}_{\epsilon,h})$, which is given by

$$\Phi_{[o]\epsilon,h}(\mathbf{X}_{\epsilon,h}) = (2\pi)^{-3}|\Sigma_{[o]\epsilon}|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{X}_{\epsilon,h}-\mathbf{X}^*_{hiv})^\top \Sigma^{-1}_{[o]\epsilon}(\mathbf{X}_{\epsilon,h}-\mathbf{X}^*_{hiv})}.$$

Although the uniqueness of $\mu_{\epsilon,h}(\cdot)$ is unknown, we can numerically study whether the distribution $\mathbb{N}_6(\mathbf{X}^*_{hiv}, \Sigma_{[o]\epsilon})$ (resp., $\Phi_{[o]\epsilon,h}(\mathbf{X}_{\epsilon,h})$) limited on $\mathbb{R}^6_+$ can well approximate the empirical probability measure (resp., density function) of (5.21) under small diffusion and large iteration time.

**Example 5.5.** Based on the actual data from [67,95] and the Central Statistical Office of Zimbabwe (CSZ), the corresponding values of the biological parameters in (5.21) are shown in Table 4. We choose $\epsilon = 10^{-3}$ and initial value $\mathbf{X}_{\epsilon,h}(0) = (350000, 6000, 7500, 8500, 12000, 7500)^\top$ (unit: people), it is calculated that $\mathscr{R}^S_{4,\epsilon} = 96.927$, $\mathbf{X}^*_{hiv} = 10^4 \times (1.7050, 5.9481, 3.9004, 1.3807, 6.4964, 3.4241)^\top$ and

$$\Sigma_{[o]\epsilon} = 10^6 \times \begin{pmatrix} 0.04117 & 0.004262 & -0.007024 & -0.002498 & -0.18918 & -0.058545 \\ 0.004262 & 0.081447 & 0.026892 & 0.005528 & 0.096841 & 0.021921 \\ -0.007024 & 0.026892 & 0.017634 & 0.005325 & 0.049126 & 0.018898 \\ -0.002498 & 0.005528 & 0.005325 & 0.001885 & 0.014376 & 0.006524 \\ -0.18918 & 0.096841 & 0.049126 & 0.014376 & 2.5538 & 0.61208 \\ -0.058545 & 0.021921 & 0.018898 & 0.006524 & 0.61208 & 0.19651 \end{pmatrix}.$$

For simplicity, the empirical marginal measure and density function of (5.21) at large iteration time $T_0$ are denoted by $\mathrm{eMM}_\epsilon(T_0, \cdot)$ and $\mathrm{eMD}_\epsilon(T_0, \cdot)$, respectively. Fig. 13 shows the six one-dimensional $\mathrm{eMM}_\epsilon(T_0, \cdot)$ at $T_0 = 6$ years. Furthermore, we plot the curves $\mathrm{eMD}_\epsilon(T_0, \cdot)$ of different types of individuals at iteration time $T_0 = 4,\ 6$ and 8 years, each in a different color. It is clear that every one-dimensional function $\Phi^{\partial}_{[o]\epsilon,h}(\cdot)$ almost coincides with the corresponding three density curves; see Fig. 14. Such significant similarity can be quantitatively verified by several Kolmogorov–Smirnov tests with 5% significance level, where the null hypothesis is set the same as in Examples 5.1-5.4. Thus, the function $\Phi_{[o]\epsilon,h}(\mathbf{X}_{\epsilon,h})$ limited on $\mathbb{R}^6_+$ is a good global fit for the $\mathrm{eMD}_\epsilon(T_0, \cdot)$ of (5.21) under large iteration time.

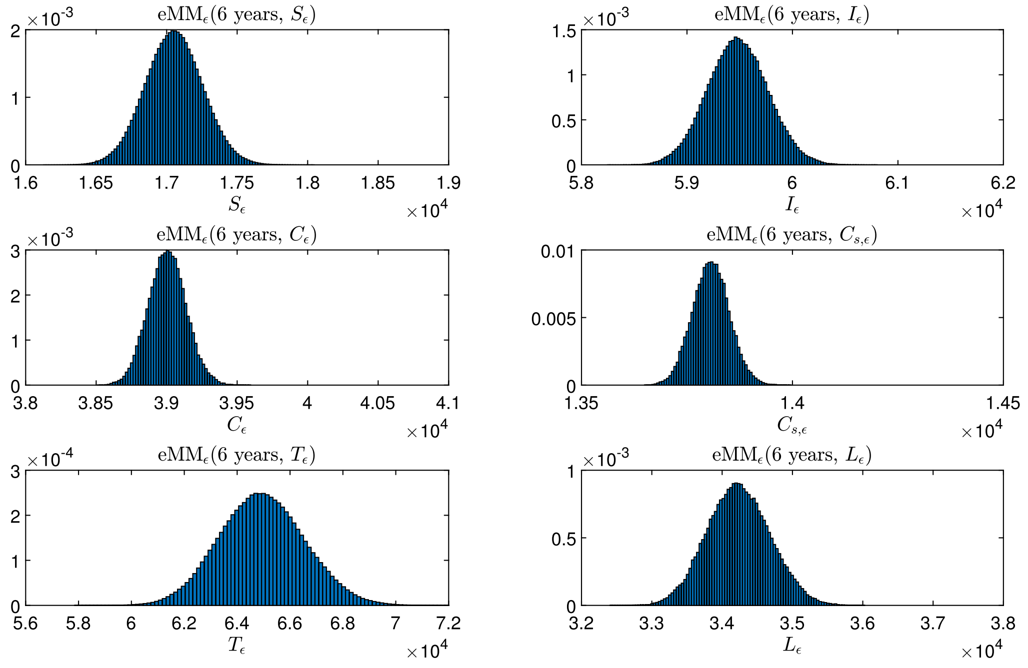


Fig. 13. The diagrams represent the eMM$_\epsilon(T_0, \cdot)$ of different types of individuals in (5.21) at $T_0 = 6$ years. The iteration step size is $\Delta t = 10^{-3}$ years.

**Remark 10.** Based on Figs. 13 and 14, we propose a conjecture that the IPM $\mu_{\epsilon,h}(\cdot)$ is unique. To investigate this conjecture, we performed several computer simulations of the one-dimensional eMM$_\epsilon(T_0, \cdot)$ using MATLAB R2022b software, varying the initial values and iteration times. The numerical results obtained provide support for this conjecture. However, to establish it rigorously, we plan to explore more theoretical approaches in future work.

## Data availability

No data was used for the research described in the article.

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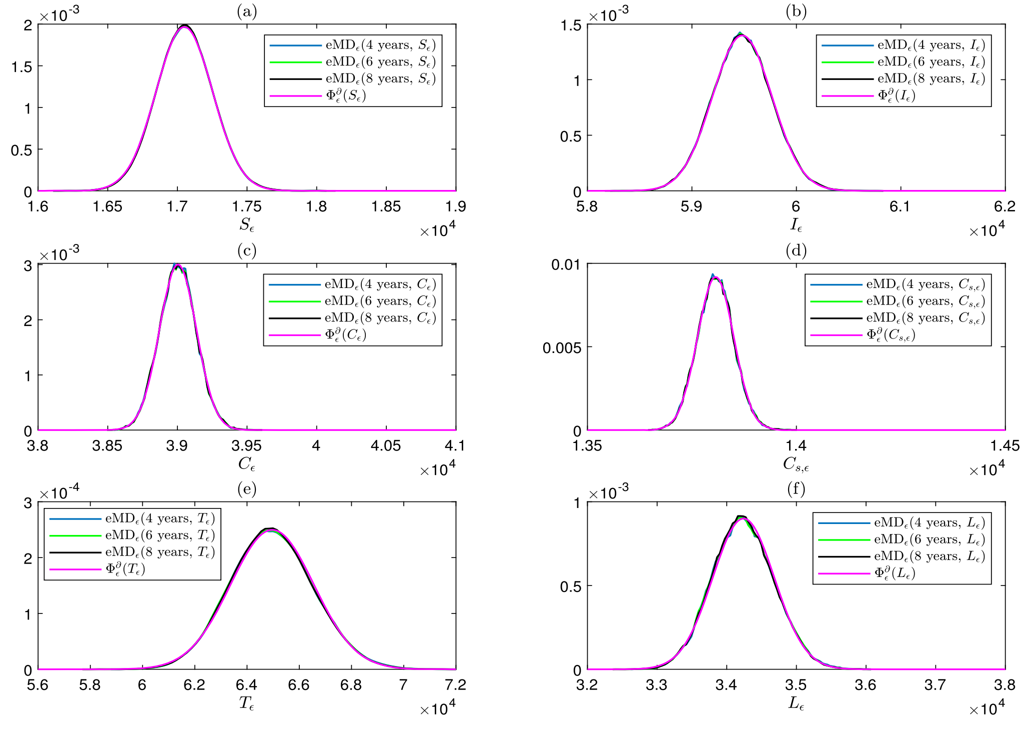


Fig. 14. The blue, green and black lines represent the curves $\mathrm{eMD}_\epsilon(T_0, S_\epsilon)$, $\mathrm{eMD}_\epsilon(T_0, I_\epsilon)$, $\mathrm{eMD}_\epsilon(T_0, C_\epsilon)$, $\mathrm{eMD}_\epsilon(T_0, C_{s,\epsilon})$, $\mathrm{eMD}_\epsilon(T_0, T_\epsilon)$ and $\mathrm{eMD}_\epsilon(T_0, L_\epsilon)$ of (5.21) at iteration time $T_0 = 4$, 6 and 8 years, respectively. The purple lines denote the one-dimensional MDs of $\Phi_{[o]\epsilon,h}(\mathbf{X}_{\epsilon,h})$. All of the parameter values and iteration step size are the same as in Fig. 13.

## Appendix A. Proof of Proposition 2.1

Throughout this appendix, let $\psi_A(\lambda) = \sum_{i=0}^{l} a_i \lambda^{l-i}$ $(a_0 = 1)$. Using Definition 2.1, Lemma 2.1 and $A \in \mathscr{S}(l)$, we determine that $(a_1, ..., a_l)^\top := \mathbf{a}^\top \in \mathbb{R}_+^l$ and

$$A = \begin{pmatrix} -\mathbf{a}^{\langle l-1\rangle} & -a_l \\ \mathbf{I}_{l-1} & \mathbb{O} \end{pmatrix}. \tag{A.1}$$

Moreover, let $\lambda_i$ $(i \in \mathbb{S}_l^0)$ be the roots of equation $\psi_A(\lambda) = 0$. Below we divide the proof of Proposition 2.1 into three steps.

**Step 1.** (Proof of (i) in Proposition 2.1): It is clear that $\mathbf{Re}(\lambda_j) < 0$ and $\lambda_j + \lambda_k \neq 0$ for any $j, k \in \mathbb{S}_l^0$. This together with Lemma 2.2 yields that $\Xi_l$ is unique. We define two matrices $C$ and $D$ by

$$C = \int_0^\infty e^{At}\, \text{Ш}_{l,1}\, e^{A^\top t} dt, \quad D = \left(\text{Ш}_{l,1}, A\text{Ш}_{l,1}, ..., A^{l-1}\text{Ш}_{l,1}\right).$$

Note that $C = C^\top$ and

$$AC + CA^\top = \int_0^\infty \left(Ae^{At}\,\amalg_{l,1}\,e^{A^\top t} + e^{At}\,\amalg_{l,1}\,e^{A^\top t}A^\top\right)dt$$
$$= -\int_0^\infty \frac{d}{dt}\left(e^{At}\,\amalg_{l,1}\,e^{A^\top t}\right)dt = -\amalg_{l,1}\,. \tag{A.2}$$

Thus, $\Xi_l = \int_0^\infty e^{At}\,\amalg_{l,1}\,e^{A^\top t}dt$. By direct calculation, we have $D = \left(\boldsymbol{\zeta}_1, \mathbb{O}_{l,l-1}, \boldsymbol{\zeta}_2, \mathbb{O}_{l,l-1}, ..., \boldsymbol{\zeta}_l, \mathbb{O}_{l,l-1}\right)$, where the $\mathbb{R}^{l\times 1}$-valued vectors $\boldsymbol{\zeta}_j$ ($j \in \mathbb{S}_l^0$) satisfy

$$\left(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, ..., \boldsymbol{\zeta}_l\right) = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{l-1} \\ 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{l-2} \\ 0 & 0 & 1 & \alpha_1 & \cdots & \alpha_{l-3} \\ 0 & 0 & 0 & 1 & \cdots & \alpha_{l-4} \\ \vdots & \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

with $\alpha_k$ ($k \in \mathbb{S}_{l-1}^0$) determined by the iterative scheme $\alpha_k = -\sum_{i=1}^k a_i\alpha_{k-i}$ ($\alpha_0 = 1$). Clearly, $|\left(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, ..., \boldsymbol{\zeta}_l\right)| = 1$, then $\operatorname{rank}(D) = l$.

For any $\mathbf{X} \in \mathbb{R}^l$, by $\amalg_{l,1} = \amalg_{l,1}^\top \amalg_{l,1}$, one has

$$\mathbf{X}^\top \Xi_l \mathbf{X} = \int_0^\infty \mathbf{X}^\top e^{At}\,\amalg_{l,1}\,\amalg_{l,1}^\top e^{A^\top t}\mathbf{X}dt$$
$$= \int_0^\infty \left|\amalg_{l,1}^\top e^{A^\top t}\mathbf{X}\right|^2 dt \geq 0. \tag{A.3}$$

Thus, $\Xi_l \succeq \mathbb{O}$. Below we verify $\Xi_l \succ \mathbb{O}$ by a contradiction argument. Suppose that there is a $\mathbf{X}_\phi \in \mathbb{R}^l \setminus \{\mathbf{0}\}$ such that $\mathbf{X}_\phi^\top \Xi_l \mathbf{X}_\phi = 0$, then

$$\amalg_{l,1}^\top e^{A^\top t}\mathbf{X}_\phi = \mathbf{0}, \quad \forall\, t \geq 0. \tag{A.4}$$

Consider a function $\mathcal{S}_\phi(t) = \amalg_{l,1}^\top e^{A^\top t}\mathbf{X}_\phi$ defined on $[0, \infty)$. Using (A.4), we determine

$$\frac{d^k \mathcal{S}_\phi(0)}{dt^k} = (A^k \amalg_{l,1})^\top \mathbf{X}_\phi = \mathbf{0}, \quad \forall\, k = 0, 1, ... \tag{A.5}$$

By Cayley–Hamilton theorem [98], (A.5) is equivalent to

$$\mathbf{X}_\phi^\top (A^k \amalg_{l,1}) = \mathbf{0}^\top, \quad \forall\, k \in \mathbb{S}_{l-1}^{-1},$$

i.e., $\mathbf{X}_\phi^\top D = \mathbf{0}^\top$. Using $\operatorname{rank}(D) = l$ yields $\mathbf{X}_\phi = \mathbf{0}$, which leads to a contradiction. Hence, $\Xi_l \succ \mathbb{O}$.

**Step 2.** (Proof of (ii) in Proposition 2.1): Let $\Xi_l = (\sigma_{ij})_{l\times l}$, combining (A.1) and Eq. (2.1), we obtain the following $\frac{l(l-1)}{2}$ equalities:

$$\begin{cases} 1-2\sum_{k=1}^{l} a_i\sigma_{i1}=0, \\ \sigma_{k,k+1}=0, \quad \forall\, k\in\mathbb{S}_{l-1}^{0}, \\ \sigma_{i-1,1}-\sum_{j=1}^{l} a_j\sigma_{ji}=0, \quad \forall\, i\in\mathbb{S}_{l}^{1}, \\ \sigma_{p,q}=\sigma_{q-1,p+1}, \quad \forall\, q>p+1. \end{cases} \tag{A.6}$$

Consider an algebraic equation

$$\mathscr{H}_{l,A}\boldsymbol{\theta}=\frac{1}{2}\mathbf{e}_l, \tag{A.7}$$

where $\boldsymbol{\theta}=(\theta_1,-\theta_2,...,(-1)^{l-1}\theta_l)^\top$. Applying the first and third sets of equalities in (A.6) to the second and fourth sets of equalities yields that $\sigma_{ii}=\theta_i$ for any $i\in\mathbb{S}_l^0$, and $\Xi_l$ takes the form as (2.2).

To summarize, all the properties of $\Xi_l$ are determined by $\boldsymbol{\theta}$. To this end, some results of $\boldsymbol{\theta}$ are shown below. By Lemma 2.1 and Cramer's rule, we have $|\mathscr{H}_{l,A}|>0$ and

$$\theta_j=\frac{|\mathscr{H}_{l,A}^{\xi}(1,j)|}{2|\mathscr{H}_{l,A}|}, \quad \forall\, j\in\mathbb{S}_l^0,$$

where $\mathscr{H}_{l,A}^{\xi}(1,j)$ is the matrix obtained by crossing out the first row and $j$-th column of $\mathscr{H}_{l,A}$. By a standard argument in [97, Theorem 2], we can obtain $|\mathscr{H}_{l,A}^{\xi}(1,k)|>0$ for any $k\in\mathbb{S}_l^0$. Thus, $\theta_k>0$.

Furthermore, if all the roots of equation $\psi_A(\lambda)=0$ are simple, i.e., $\lambda_p\neq\lambda_j,\ \forall\ p\neq j$, we consider a closed path $\Gamma_R$ in the complex plane:

$$\Gamma_R\triangleq[L_R+C_R],$$

where $L_R$ is the directed segment from $(0,-R\mathrm{i})$ to $(0,R\mathrm{i})$, and $C_R$ is a semicircle line with analytical formula $z=Re^{\mathrm{i}t},\ t\in[\frac{\pi}{2},\frac{3\pi}{2}]$, see Fig. 15 in detail.

Since $\mathbf{Re}(\lambda_j)<0,\ \forall\ j\in\mathbb{S}_l^0$, we can determine a sufficiently large $R_0$ such that each $\lambda_j$ lies in the interior of $\Gamma_R$ when $R>R_0$. Now we construct some complex integrals by

$$\omega_k=\frac{1}{2\pi\mathrm{i}}\int\limits_{\Gamma_R}\frac{z^k dz}{\psi_A(z)\psi_A(-z)}, \quad k\in\mathbb{S}_{2l-1}^{-1};\ R>R_0. \tag{A.8}$$

Clearly, $\psi_A(-\lambda_j)\neq 0,\ \forall\ j\in\mathbb{S}_l^0$. Combining Cauchy's residue theorem,

$$\omega_k=\frac{1}{2\pi\mathrm{i}}\times 2\pi\mathrm{i}\sum_{j=1}^{l}\mathbf{Res}_{\lambda_j}\Big(\frac{z^k}{\psi_A(z)\psi_A(-z)}\Big)$$

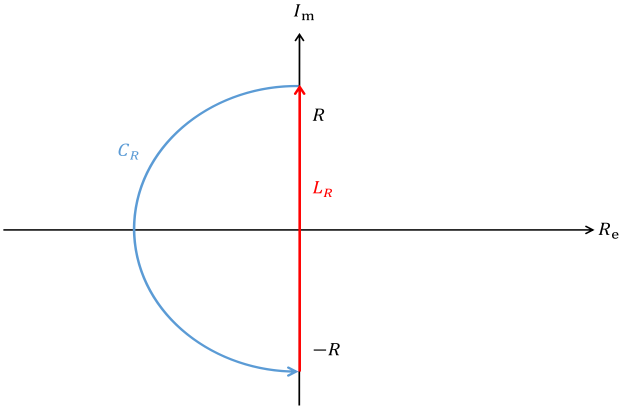


Fig. 15. The graph of closed path $\Gamma_R$.

$$=\sum_{j=1}^{l}\lim_{z\to\lambda_j}\frac{z^k}{\psi_A(z)\psi_A(-z)}=\sum_{j=1}^{l}\frac{\lambda_j^k}{\psi_A'(\lambda_j)\psi_A(-\lambda_j)},\quad \forall\, k\in\mathbb{S}_{2l-1}^{-1}. \tag{A.9}$$

Furthermore, applying Cauchy integral theorem to (A.8) yields

$$\begin{aligned}\omega_{2p+1}=&\frac{1}{2\pi\mathrm{i}}\int_{L_R}\frac{z^{2p+1}dz}{\psi_A(z)\psi_A(-z)}+\frac{1}{2\pi\mathrm{i}}\int_{C_R}\frac{z^{2p+1}dz}{\psi_A(z)\psi_A(-z)}\\ =&\frac{1}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{C_R}\frac{z^{2p+1}dz}{\psi_A(z)\psi_A(-z)}\\ =&\frac{1}{2\pi\mathrm{i}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\lim_{R\to\infty}\frac{(R\mathrm{e}^{\mathrm{i}t})^{2p+1}R\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\psi_A(R\mathrm{e}^{\mathrm{i}t})\psi_A(-R\mathrm{e}^{\mathrm{i}t})}dt,\quad \forall\, p\in\mathbb{S}_{l-1}^{-1}.\end{aligned} \tag{A.10}$$

Note that

$$\lim_{R\to\infty}\frac{(R\mathrm{e}^{\mathrm{i}t})^{2p+1}R\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\psi_A(R\mathrm{e}^{\mathrm{i}t})\psi_A(-R\mathrm{e}^{\mathrm{i}t})}=\begin{cases}(-1)^l\mathrm{i}, & \text{for } p=l-1,\\ 0, & \text{for } p\in\mathbb{S}_{l-2}^{-1}.\end{cases}$$

This together with (A.10) leads to

$$\omega_{2p+1}=\begin{cases}\dfrac{(-1)^l}{2}, & \text{for } p=l-1,\\ 0, & \text{for } p\in\mathbb{S}_{l-2}^{-1}.\end{cases} \tag{A.11}$$

Using (A.9) and $\psi_A(\lambda_j)=0$ ($\forall\, j\in\mathbb{S}_l^0$), we obtain

$$
\begin{aligned}
0 &= \sum_{j=1}^{l} \frac{\lambda_j^k \psi_A(\lambda_j)}{\psi_A'(\lambda_j)\psi_A(-\lambda_j)} = \sum_{j=1}^{l} \frac{\lambda_j^k \sum_{i=0}^{l} a_i \lambda_j^{l-i}}{\psi_A'(\lambda_j)\psi_A(-\lambda_j)} \\
&= \sum_{j=0}^{l} a_j \omega_{n+k-j}, \quad \forall\, k \in \mathbb{S}_{l-1}^{-1}.
\end{aligned} \tag{A.12}
$$

Let $\boldsymbol{\omega} = (\omega_{2l-2}, \omega_{2l-4}, \ldots, \omega_0)^\top$. It follows from (A.11) and (A.12) that $\boldsymbol{\omega}$ satisfies the following equation

$$
\mathscr{H}_{l,A}\boldsymbol{\omega} = \frac{(-1)^l}{2}\mathbf{e}_l. \tag{A.13}
$$

Combining (A.7) and (A.13),

$$
\theta_k = (-1)^{l-k}\omega_{2(l-k)} = (-1)^{l-k}\sum_{j=1}^{l} \frac{\lambda_j^{2(l-k)}}{\psi_A'(\lambda_j)\psi_A(-\lambda_j)}, \quad \forall\, k \in \mathbb{S}_l^0.
$$

Hence, (2.3) is verified.

**Step 3.** (Proof of (iii) in Proposition 2.1): We only discuss the case where $l$ is odd, and the even $l$ case can be similarly analyzed. Without loss of generality, we assume that $l = 2k + 1$. Denote

$$
\mathcal{I}_{1,k} = \begin{vmatrix} \theta_1 & \cdots & (-1)^{k+2}\theta_{k+1} \\ \vdots & \ddots & \vdots \\ (-1)^{k+2}\theta_{k+1} & \cdots & (-1)^{2k+2}\theta_{2k+1} \end{vmatrix}, \quad \mathcal{I}_{2,k} = \begin{vmatrix} \theta_2 & \cdots & (-1)^{k+1}\theta_{k+1} \\ \vdots & \ddots & \vdots \\ (-1)^{k+1}\theta_{k+1} & \cdots & (-1)^{2k}\theta_{2k} \end{vmatrix}.
$$

As in (2.2), an application of Laplace theorem for the $(2i+1)$-th ($i \in \mathbb{S}_k^{-1}$) rows and columns of $\Xi_l$ lead to

$$
|\Xi_l| = \mathcal{I}_{1,k}\mathcal{I}_{2,k}. \tag{A.14}
$$

Combining (2.2) and (A.6), we obtain

$$
\begin{cases}
(-1)^{k+1+j}\theta_{k+j} = -\dfrac{1}{a_{2k+1}}\sum\limits_{i=1}^{k}(-1)^{j+i}\theta_{j+i-1}a_{2i-1}, \quad \forall\, j \in \mathbb{S}_{k+1}^1, \\
(-1)^{k+1}\theta_{k+1} = \dfrac{1}{a_{2k+1}}\Big[\dfrac{1}{2} - \sum\limits_{i=1}^{k}(-1)^{i+1}\theta_i a_{2i-1}\Big], \\
(-1)^{k+1+j}\theta_{k+j} = -\dfrac{1}{a_{2k}}\sum\limits_{i=1}^{m}(-1)^{j+i}\theta_{j+i-1}a_{2i-2}, \quad \forall\, j \in \mathbb{S}_k^0.
\end{cases} \tag{A.15}
$$

Applying the first two sets of equality in (A.15) to the last row of $\mathcal{I}_{1,k}$, we compute

$$
\mathcal{I}_{1,k} = \frac{(-1)^k}{2a_{2k+1}} \times \begin{vmatrix} -\theta_2 & \cdots & (-1)^{k+2}\theta_{k+1} \\ \vdots & \ddots & \vdots \\ (-1)^{k+2}\theta_{k+1} & \cdots & (-1)^{2k+1}\theta_{2k} \end{vmatrix} = \frac{\mathcal{I}_{2,k}}{2a_{2k+1}}. \tag{A.16}
$$

Substituting the third set of equality of (A.15) into the last row of $\mathcal{I}_{2,k}$, by the notation $[\frac{l-1}{2}]^\star = k$, one similarly has $\mathcal{I}_{2,k} = \frac{(-1)^k}{a_{2k}}\varphi_{ol}$. This together with (A.14) and (A.16) yields the desired result.

## Appendix B. Proof of Proposition 2.2

Using Remark 1, Proposition 2.2 is evidently true when $l = 1$. Below we only discuss the case of $l \geq 2$. Suppose that the vector $\mathbf{X}(t) = (X_1(t), ..., X_l(t))^\top$ is determined by equation $d\mathbf{X} = C\mathbf{X}dt$. We construct a vector $\mathbf{Y}(t) = (Y_1(t), ..., Y_l(t))^\top$ which satisfies:

$$Y_l(t) = X_l(t), \text{ and } Y_j(t) = Y'_{j+1}(t), \ \forall\, j \in \mathbb{S}^0_{l-1}. \tag{B.1}$$

To proceed, we stipulate that $c_{1,0} = 1$. An application of recursion method coupled with (B.1), Definition 2.2 and $C \in \mathcal{U}_q(l)$ yields that

$$Y_j = \boldsymbol{\beta}_l C^{l-j}\mathbf{X}, \quad \forall\, j \in \mathbb{S}^0_l,$$

and

$$\begin{cases} (\boldsymbol{\beta}_l C^j)^{\langle l-j-1\rangle} = \mathbf{0}, & \forall\, j \in \mathbb{S}^0_{l-2}, \\ (\boldsymbol{\beta}_l C^j)^{(l-j)} = \prod_{i=l-j}^{l-1} c_{i+1,i} \neq 0, & \forall\, j \in \mathbb{S}^0_{l-2}, \\ (\boldsymbol{\beta}_l C^{l-1})^{(1)} = \prod_{i=0}^{l-1} c_{i+1,i} \neq 0, \end{cases} \tag{B.2}$$

where $C = (c_{ij})_{l\times l}$. Thus,

$$\mathbf{Y}(t) = \begin{pmatrix} \boldsymbol{\beta}_l C^{l-1} \\ \boldsymbol{\beta}_l C^{l-2} \\ \cdots \\ \boldsymbol{\beta}_l \end{pmatrix} \mathbf{X}(t) = M\mathbf{X}(t). \tag{B.3}$$

Moreover, $M$ is an upper triangular matrix. Note that $|M| = \prod_{j=1}^{l-1}(\boldsymbol{\beta}_l C^{l-j})^{(j)} \neq 0$, then $M \in \mathcal{U}(l)$. Using (B.2), we obtain that $\mathbf{X}(t) = M^{-1}\mathbf{Y}(t)$ and $d\mathbf{Y} = MCM^{-1}\mathbf{Y}dt$. Then by (B.1), we have

$$d\mathbf{Y} = \begin{pmatrix} -\overline{\mathbf{c}}^{\langle l-1\rangle} & -\overline{c}_l \\ \mathbf{I}_{l-1} & \mathbb{O} \end{pmatrix} \mathbf{Y}dt,$$

where $\overline{\mathbf{c}} = (\overline{c}_1, ..., \overline{c}_l)$. Combining Definition 2.1, one gets $MCM^{-1} \in \mathscr{S}(l)$. This completes the proof.

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