# Quiz 1 — MATH 334

## Fall 2025 – Solutions

1. [6 pts] Find the general solution to the equation

$$\frac{dy}{dx} + \frac{y}{x} = -4\frac{x}{y^2}.$$

Solution: This is a Bernoulli equation $y' + P(x)y = Q(x)y^n$ with $P(x) = \dfrac{1}{x}$, $Q(x) = -4x$, and $n = -2$. Multiply both sides by $y^2$ to get

$$y^2 y' + \frac{y^3}{x} = -4x.$$

Let $v = y^3$. Then $v' = 3y^2y'$, so $y^2y' = \dfrac{v'}{3}$. Substituting gives

$$\frac{v'}{3} + \frac{v}{x} = -4x \quad\Longrightarrow\quad v' + \frac{3}{x}v = -12x.$$

This linear ODE has integrating factor $\mu(x) = e^{\int (3/x)\,dx} = x^3$. Thus

$$\frac{d}{dx}\left(x^3 v\right) = -12x^4.$$

Integrating,

$$x^3 v = -\frac{12}{5}x^5 + C \quad\Longrightarrow\quad v = -\frac{12}{5}x^2 + Cx^{-3}.$$

Since $v = y^3$, the general solution is

$$\boxed{y^3 = Cx^{-3} - \frac{12}{5}x^2}, \qquad \text{equivalently} \qquad \boxed{y(x) = \left(Cx^{-3} - \tfrac{12}{5}x^2\right)^{1/3}}.$$

2. [8 pts] Solve the initial value problem

$$\frac{dy}{dx} = -\frac{3y^2 + 8x}{2xy},$$

with the initial condition $y(1) = 2$.

Solution: The given differential equation

$$\frac{dy}{dx} = -\frac{3y^2 + 8x}{2xy}$$

is equivalent to

$$(3y^2 + 8x)\,dx + (2xy)\,dy = 0.$$

Let $M(x, y) = 3y^2 + 8x$ and $N(x, y) = 2xy$. The equation $M\,dx + N\,dy = 0$ is not exact since

$$M_y = 6y \neq N_x = 2y.$$

Check for an integrating factor depending only on $x$:

$$\frac{M_y - N_x}{N} = \frac{4y}{2xy} = \frac{2}{x} \quad \Rightarrow \quad \mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2.$$

Multiplying the differential equation by $x^2$ yields

$$(3x^2y^2 + 8x^3)\,dx + (2x^3y)\,dy = 0,$$

which is exact, because $\partial_y(3x^2y^2 + 8x^3) = 6x^2y = \partial_x(2x^3y)$. Using Method 1 of exact differential equations, we have

$$\psi_x = 3x^2y^2 + 8x^3.$$

Integrate with respect to $x$:

$$\psi(x, y) = x^3y^2 + 2x^4 + g(y).$$

Differentiate with respect to $y$ and match $\psi_y$ to $2x^3y$:

$$F_y = 2x^3y + g'(y) = 2x^3y \quad \Rightarrow \quad g'(y) = 0 \Rightarrow g(y) = C_0.$$

Therefore the general solution is

$$x^3y^2 + 2x^4 = C.$$

Apply the initial condition $y(1) = 2$: $1 \cdot 4 + 2 \cdot 1 = 6$, so $C = 6$. Hence

$$\boxed{x^3y^2 + 2x^4 = 6}.$$

Solving for $y$ near $x = 1$ (where $x > 0$),

$$\boxed{y(x) = +\sqrt{\frac{6}{x^3} - 2x}},$$

where the positive branch is chosen to satisfy $y(1) = 2$.

Note: It is OK to use Method 2 of exact differential equations, we start from $\psi_y = 2x^3y$ and integrate with respect to $y$ to obtain $\psi = x^3y^2 + h(x)$. Then we differential with respect to $x$ and match $\psi_x$ to $3x^2y^2 + 8x^3$:

$$3x^2y^2 + h'(x) = 3x^2y^2 + 8x^3 \Rightarrow h'(x) = 8x^3 \Rightarrow h(x) = 2x^4.$$

Therefore the general solution is

$$x^3y^2 + 2x^4 = C,$$

exactly same as using Method 1.

3. [6 pts] Solve the initial value problem

$$y''(x) - 10y'(x) + 25y(x) = 0, \quad y(0) = -2, \quad y'(0) - 5y(0) = 7.$$

Solution: The characteristic polynomial is $r^2 - 10r + 25 = (r-5)^2$, giving a repeated root $r = 5$. The general solution is

$$y(x) = (C_1 + C_2x)e^{5x}.$$

From $y(0) = -2$ we get $C_1 = -2$. Also $y'(x) = (C_2 + 5C_1 + 5C_2x)e^{5x}$, hence $y'(0) = C_2 + 5C_1$. Using $y'(0) - 5y(0) = 7$,

$$(C_2 + 5C_1) - 5C_1 = C_2 = 7.$$

Therefore

$$\boxed{y(x) = (-2 + 7x)\,e^{5x}}.$$