Commun Nonlinear Sci Numer Simulat 132 (2024) 107936

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Double Hopf bifurcation induced by spatial memory in a diffusive predator–prey model with Allee effect and maturation delay of predator

Shuai Li [a], Sanling Yuan [a,*], Zhen Jin [b], Hao Wang [c]

[a] *College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China*
[b] *Complex Systems Research Center, Shanxi University, Taiyuan, Shanxi 030006, China*
[c] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada*

ARTICLE INFO

MSC:
92D25
35B32
35B10
35K57

Keywords:
Normal form
Memory-based diffusion
Maturation delay
Double Hopf bifurcation
Predator–prey model

ABSTRACT

In this paper, we delve into double Hopf bifurcation induced by memory-driven directed movement in a spatial predator–prey model with Allee effect and maturation delay of predators. We first adopt a novel technique to handle the associated characteristic equation and thus obtain the crossing curves as well as the double Hopf points. We then calculate explicit formulae of normal form regarding non-resonant double Hopf bifurcation. We thus divide the dynamics of the developed model into several categories near the double Hopf bifurcation points. Our numerical and theoretical results both demonstrate that the model can exhibit various complex phenomena when the parameters are near the double Hopf bifurcation points. For example, the transition from one stable spatially inhomogeneous periodic orbit with mode-5 to another with mode-4 and the coexistence of them can be observed.

1. Introduction

Understanding the spatiotemporal distribution of animals exerts a great influence on biodiversity conservation and is thought of as one of the top five ranked research fronts in ecology [1,2]. The spatial predator–prey model could elucidate the underlying mechanisms of some biological and abiotic processes and thus usually be thought of as an important tool to study spatial distribution. Recently, considerable models have been dedicated to revealing the intricate biological processes via incorporating various factors such as Allee effect [3,4] and maturation period of predators [5,6]. The Allee effect refers to the density-mediated intrinsic growth rate of species, which can be applied to illustrate mechanisms such as group defense enhancement [7,8]. Sun accommodated the fact that Allee effect in the prey also plays a critical role in pattern formation in predator–prey model [9]. Plenty of recent research pointed out that Allee effect can also exist in predators owing to reproductive facilitation mechanisms and also merits further consideration [10–12]. Maturation period of predators reflects the time that predators need to multiply offspring. Usually, it can be characterized by delay-dependent parameters in predator–prey models [13,14]. Both maturation period and Allee effect can independently induce complicated behaviors. However, the results on the models with considerations of two biotic processes simultaneously are rare. Therefore, it is essential to delve further into this topic.

Note also that the movement pattern of an animal exerts great influence on its distribution in space and time and shapes global biodiversity patterns [15]. It is acknowledged that random diffusions can induce the emergence of non-constant equilibria exhibiting complicated spatial structure. This phenomenon is often dubbed Turing pattern [16,17]. In addition to random diffusion, some

* Corresponding author.
E-mail address: sanling@usst.edu.cn (S. Yuan).

https://doi.org/10.1016/j.cnsns.2024.107936
Received 12 May 2023; Received in revised form 18 February 2024; Accepted 23 February 2024
Available online 24 February 2024
1007-5704/

abiotic processes (such as wind or water current) can induce the passive directed movements of animals [18,19]. These processes can be reflected by reaction–diffusion–advection equations and can alter population invasion outcomes [20]. Recently, some researchers pointed out that some animals can move according to their spatial memory and cognition and thus adopt tactic movements [21–25]. Shi et al. opened the possibility of disclosing the memory-driven movement through a dynamical model by incorporating time delay in the diffusion term [26]. Afterward, the authors in [27] generalized the aforementioned single species models to predator–prey models with memory delay in predators. The authors in [28] considered the maturation period and incorporated maturation delay into the reaction term of a model established in [26]. Wang et al. recently set up a diffusive predator–prey model incorporating both memory delay and pregnancy delay in the prey [29]. They coped with the difficulties induced by two delays to perform linear analysis and obtained the crossing curves in the delays plane.

However, results concerning the impact of memory-driven directed movement of the prey or predators along with Allee effect and maturation period of predators on the spatial patterns are few. To characterize these biological processes, we recently formulated the following spatial predator–prey model incorporating memory-based directed movement and delay-dependent parameters:

$$\begin{cases} \frac{\partial N}{\partial t} = \delta_{11} N_{xx} + \delta_{12}\left(N P_x(x,t-\tau)\right)_x + rN\left(1-\frac{N}{K}\right) - bNP, \\ \frac{\partial P}{\partial t} = \delta_{22} P_{xx} + \dfrac{\beta N_\sigma P_\sigma^2}{h+P_\sigma} \mathrm{e}^{-d\sigma} - \mu P,\ 0<x<\iota\pi,\ t>0, \\ N_x(x,t) = P_x(x,t) = 0,\ x=0,\iota\pi,\ t\ge 0, \end{cases} \tag{1.1}$$

where $N = N(x,t)$ and $P = P(x,t)$ respectively refer to the densities of the prey and adult predators at time t and location x; N_σ and P_σ respectively represent $N(x,t-\sigma)$ and $P(x,t-\sigma)$; all parameters in (1.1) are positive and their biological meanings are provided in [30]. We generalized the crossing curves method to spatial model (1.1) and discerned the linear stable and unstable regions in the delays plane. Noted that for a specific species, the maturation period is fixed. But the period of spatial memory and the ability of directed movement are usually acquired and can be changed. We focus on the effects of tactical directed dispersal on the dynamics of model (1.1) in the current work. Of course, we can also consider the scenario of spatial memory in predators, i.e., model (5.1). In this paper, we will devote our main attention to the mathematical analysis of model (1.1). We can use the similar logic to cope with model (5.1) with spatial memory in predators. The dynamics of model (5.1) will be numerically presented later in the discussion Section 5 for the brevity of the presentation.

Model (1.1) falls into the category of partial functional differential equations (PFDEs) [31]. Recently, considerable research efforts have been devoted to normal form of various bifurcations for PFDEs since the occurrences of these bifurcations are the precursors to spatiotemporal patterns [32–34]. For example, Wu refined the method established in [35] to establish the calculation procedures of normal form for reaction–diffusion equations with delays [31]. Also, Faria put forward a general framework to calculate normal form for various bifurcations through parameter perturbation [36]. Afterward, An and Jiang performed the computation of normal form with regard to Hopf-zero bifurcation [37]. Du et al. in [38] derived the formulae regarding the normal form of non-resonant double Hopf bifurcation for PFDEs. Here non-resonance refers to the case that there are no positive integers q_1 and q_2 such that the two pairs of purely imaginary roots $\pm i\varpi_{n_1}$ and $\pm i\varpi_{n_2}$ of the associated characteristic equation satisfy $q_1\varpi_{n_1} = q_2\varpi_{n_2},\ q_1+q_2 \le 4$ [39].

The above procedures to calculate normal forms cannot be directly applied to diffusive models with spatial memory since there exist nonlinearity and delay in the diffusion terms. More recently, Song et al. generalized the method in [36]. They established the algorithms to compute normal forms of non-resonant double Hopf bifurcation for the diffusive model with memory-based directed dispersal [40]. Liu et al. presented the formulae of normal form for Turing–Hopf bifurcation in a predator–prey model with memory-driven diffusion [41]. However, some topics in the memory-based diffusion model with two different types of delays remain unsolved [28,40]. For instance, the effects of nonlinear terms can be considered. Also, the computation of the normal form concerning double Hopf bifurcation induced by directed movement can be completed. Besides, it is worth noting that the two delays incorporated in model (1.1) are of different types, and hence it is impossible to obtain the explicit expression of frequency ϖ directly. This means that we cannot obtain the double Hopf bifurcation points in the usual way as in the literature, which presents a striking difference from models in [38,40]. The dynamics around the double Hopf bifurcation points may exhibit the coexistence of stable spatially heterogeneous periodic solutions with different modes. These dynamics are not considered in the previous spatial models with two delays [28,29]. In the present paper, we will dedicate our research efforts to investigating the normal form of double Hopf bifurcation in a memory-driven diffusive predator–prey model with delays.

We organize the rest of this paper as follows. In Section 2, we obtain crossing curves in the (δ_{12},τ) plane and then based on which determine the double Hopf bifurcation points of model (1.1). In Section 3, we complete the calculation of the normal form for the non-resonant double Hopf bifurcation. In Section 4, a numerical example is provided to verify the correctness of our obtained results. In Section 5, we consider the scenario of spatial memory in predators and end our paper with some discussions.

2. Crossing curves and double Hopf bifurcation points

In this section, we perform the linear analysis of model (1.1) and determine double Hopf bifurcation points in the (δ_{12},τ) plane. From the ecological context, we are concerned with the coexistence of the prey and predators. We just consider the positive constant equilibria of model (1.1) accordingly.

The positive equilibrium $E^* = (N^*, P^*)$ meets $N^* = \dfrac{\mu(h+P^*)\mathrm{e}^{d\sigma}}{\beta P^*}$ and P^* is a positive real root of the following equation:

$$\beta bKP^2 + (r\mu \mathrm{e}^{d\sigma} - rK\beta)P + r\mu h\mathrm{e}^{d\sigma} = 0. \tag{2.1}$$

We can deduce that Eq. (2.1) admits two positive roots provided

$$\mu < \beta K e^{-d\sigma} \text{ and } h < \frac{1}{4}\frac{r(K\beta - \mu e^{d\sigma})^2 e^{-d\sigma}}{bK\beta\mu}.$$

We denote the two positive equilibria respectively as $E_1 = (N_1, P_1)$ and $E_2 = (N_2, P_2)$ with $P_1 < P_2$. Linearizing model (1.1) at E_i we obtain

$$\begin{cases} \frac{\partial N(x,t)}{\partial t} = \delta_{11} N_{xx}(x,t) + \delta_{12} N_i P_{xx}(x,t-\tau) + \alpha_{11} N(x,t) + \alpha_{12} P(x,t), \ x \in (0, \iota\pi), \\ \frac{\partial P(x,t)}{\partial t} = \delta_{22} P_{xx}(x,t) + \alpha_{22} P(x,t) + \beta_{21} N(x,t-\sigma) + \beta_{22} P(x,t-\sigma), \ x \in (0, \iota\pi), \\ N_x(x,t) = P_x(x,t) = 0, \ x = 0, \iota\pi, \ t \ge 0, \end{cases} \tag{2.2}$$

where

$$\alpha_{11} = -\frac{rN_i}{K}, \quad \alpha_{12} = -bN_i, \quad \alpha_{22} = -\mu,$$
$$\beta_{21} = \frac{\beta P_i^2}{h+P_i} e^{-d\sigma}, \quad \beta_{22} = \frac{\beta N_i P_i^2 + 2\beta h N_i P_i}{(h+P_i)^2} e^{-d\sigma}.$$

We then obtain the following characteristic equation of linear system (2.2):

$$\mathfrak{g}_0^n(\lambda) + \mathfrak{g}_1^n(\lambda) e^{-\lambda\sigma} + \mathfrak{g}_2^n(\delta_{12}) e^{-\lambda(\tau+\sigma)} = 0, \tag{2.3}$$

where

$$\begin{cases} \mathfrak{g}_0^n(\lambda) = \lambda^2 + \mathfrak{g}_{01}^n \lambda + \mathfrak{g}_{00}^n, \\ \mathfrak{g}_1^n(\lambda) = \mathfrak{g}_{11}^n \lambda + \mathfrak{g}_{10}^n, \\ \mathfrak{g}_2^n(\delta_{12}) = \delta_{12} \mathfrak{g}_{20}^n, \end{cases}$$

with

$$\begin{cases} \mathfrak{g}_{01}^n = \delta_{11}\left(\frac{n}{\iota}\right)^2 + \delta_{22}\left(\frac{n}{\iota}\right)^2 - \alpha_{11} - \alpha_{22}, \\ \mathfrak{g}_{00}^n = \delta_{11}\delta_{22}\left(\frac{n}{\iota}\right)^4 - \delta_{11}\alpha_{22}\left(\frac{n}{\iota}\right)^2 - \delta_{22}\alpha_{11}\left(\frac{n}{\iota}\right)^2 + \alpha_{11}\alpha_{22}, \\ \mathfrak{g}_{11}^n = -\beta_{22}, \\ \mathfrak{g}_{10}^n = -\delta_{11}\beta_{22}\left(\frac{n}{\iota}\right)^2 + \alpha_{11}\beta_{22} - \alpha_{12}\beta_{21}, \\ \mathfrak{g}_{20}^n = N_i \beta_{21}\left(\frac{n}{\iota}\right)^2. \end{cases}$$

It is easy to check that E_1 is a saddle when memory delay and diffusions are absent, i.e., $n = 0$ and $\tau = 0$. Thus, we devote ourselves to the stability of E_2 in the sequel. We are now in a position to state how to obtain the crossing curves in the (δ_{12}, τ) plane. Multiplying both sides of Eq. (2.3) by $e^{\lambda\sigma}$ leads to

$$\mathfrak{g}_0^n(\lambda) e^{\lambda\sigma} + \mathfrak{g}_1^n(\lambda) + \mathfrak{g}_2^n(\delta_{12}) e^{-\lambda\tau} = 0. \tag{2.4}$$

Substituting $\lambda = i\varpi (\varpi > 0)$ into Eq. (2.4) and doing some simple calculations yield

$$\begin{cases} \cos(\varpi\tau) = \dfrac{(\varpi^2 - \mathfrak{g}_{00}^n)\cos(\varpi\sigma) + \mathfrak{g}_{01}^n \varpi \sin(\varpi\sigma) - \mathfrak{g}_{10}^n}{\delta_{12}\mathfrak{g}_{20}^n} := F_n(\delta_{12}, \varpi), \\ \sin(\varpi\tau) = \dfrac{(\mathfrak{g}_{00}^n - \varpi^2)\sin(\varpi\sigma) + \mathfrak{g}_{01}^n \varpi \cos(\varpi\sigma) + \mathfrak{g}_{11}^n \varpi}{\delta_{12}\mathfrak{g}_{20}^n} := G_n(\delta_{12}, \varpi). \end{cases} \tag{2.5}$$

It follows from Eq. (2.5) that

$$F_n^2(\delta_{12}, \varpi) + G_n^2(\delta_{12}, \varpi) = 1. \tag{2.6}$$

From this, the expression of δ_{12} about ϖ is

$$\delta_{12}^n(\varpi) = \frac{\sqrt{\left((\varpi^2 - \mathfrak{g}_{00}^n)\cos(\varpi\sigma) + \mathfrak{g}_{01}^n \varpi \sin(\varpi\sigma) - \mathfrak{g}_{10}^n\right)^2 + \left((\mathfrak{g}_{00}^n - \varpi^2)\sin(\varpi\sigma) + \mathfrak{g}_{01}^n \varpi \cos(\varpi\sigma) + \mathfrak{g}_{11}^n \varpi\right)^2}}{\mathfrak{g}_{20}^n}. \tag{2.7}$$

We can then obtain the following expression of $\tau_j^n(\varpi) (j \ge 0)$ by plugging Eq. (2.7) into (2.5):

$$\tau_j^n(\varpi) = \begin{cases} \dfrac{\arccos(F_n(\delta_{12}^n(\varpi), \varpi)) + 2j\pi}{\varpi}, & \text{if } G_n(\delta_{12}^n(\varpi), \varpi) > 0, \\ \dfrac{2\pi - \arccos(F_n(\delta_{12}^n(\varpi), \varpi)) + 2j\pi}{\varpi}, & \text{if } G_n(\delta_{12}^n(\varpi), \varpi) < 0. \end{cases}$$

We thus plot the curves $\Gamma_j^n = (\delta_{12}(\varpi), \tau_j^n(\varpi))$ on the (δ_{12}, τ) plane by treating Γ_j^n as parametric curves about $\varpi \ge 0$ (see Fig. 1). The characteristic Eq. (2.3) has two pairs of purely imaginary roots at the points where the two curves $\Gamma_{j_1}^{n_1}$ and $\Gamma_{j_2}^{n_2}$ interact. Once

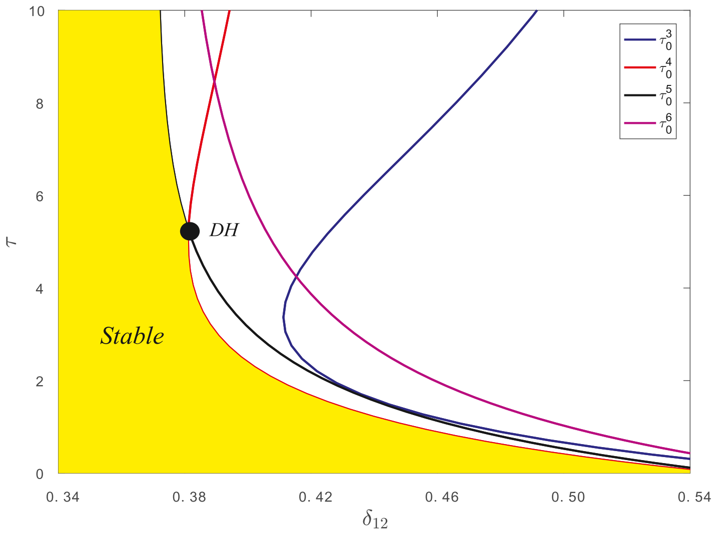

Fig. 1. Crossing curves and double Hopf bifurcation point.

the values of δ_{12} and τ at these points are determined, we can then calculate the critical frequencies ϖ_{n_1} and ϖ_{n_2} according to (2.5). We thus have the following theorem:

Theorem 2.1. *If there exists δ_{12}^H such that $\tau_{j_1}^{n_1} = \tau_{j_2}^{n_2}$ and Eq. (2.3) has two pairs of purely imaginary roots $\pm i\varpi_{n_1}$, $\pm i\varpi_{n_2}$, the associated transversality condition is matched and all other roots have negative real parts. Then model (1.1) undergoes a double Hopf bifurcation at E_2 provided $\delta = \delta_{12}^H$, $\tau = \tau_{j_1}^{n_1} = \tau_{j_2}^{n_2} = \tau^H$.*

Remark 2.2. The condition that all other roots have negative real parts is necessary for the following two reasons. The first reason is that the equilibrium E_2 is already unstable if there exist roots with positive real parts. The stability switch behaviors cannot be observed in this case. Also, it follows from [42] that the dynamics of the normal form could not be topologically equivalent to those of the original model if there exist characteristic roots with positive real parts. For instance, the bifurcating orbits projected in the center manifold are stable but could be unstable in the whole phase space.

Remark 2.3. Noting that (2.6) can only hold for finite n, we hence plot Γ_j^n only for finite n.

3. Normal form of the double Hopf bifurcation

We will deduce the explicit formulae of normal form concerning the non-resonant double Hopf bifurcation obtained in Theorem 2.1 to understand the dynamics around the double Hopf bifurcation point. Denote $\tau = \tau^H + \xi_1, \delta_{12} = \delta_{12}^H + \xi_2$. Then $\xi = (\xi_1, \xi_2) = (0,0)$ is the double Hopf bifurcation point of model (1.1). We regard ξ_1, ξ_2 as variables when calculating normal form. We further denote

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ \beta_{21} & \beta_{22} \end{pmatrix},\ \mathfrak{D}_1 = \begin{pmatrix} \delta_{11} & 0 \\ 0 & \delta_{22} \end{pmatrix},\ \mathfrak{D}_2^H = \begin{pmatrix} 0 & \delta_{12}^H N_2 \\ 0 & 0 \end{pmatrix},$$

and adopt the following real-valued Hilbert space:

$$\Upsilon = \left\{ \mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2)^{\mathrm{T}} \in \left(H^2(0, \imath\pi)\right)^2 :\ \frac{\partial \mathcal{U}_1}{\partial x}\Big|_{x=0,\imath\pi} = \frac{\partial \mathcal{U}_2}{\partial x}\Big|_{x=0,\imath\pi} = 0 \right\}.$$

Define $\mathcal{C} := \mathcal{C}([-\max\{1, \frac{\sigma}{\tau}\}, 0]; \Upsilon)$ as the space constituted by all continuous mappings from $[-\max\{1, \frac{\sigma}{\tau}\}, 0]$ to Υ.

We now make the following transformations to normalize the memory delay and translate E_2 to the origin:

$$u_1(x,t) = N(x, \tau t) - N_2,\ u_2(x,t) = P(x, \tau t) - P_2.$$

We also denote $\mathcal{U}(x,t)$ and $\mathcal{U}_t(\rho) = \mathcal{U}(x, t+\rho)$ as $\mathcal{U}(t)$ and $\mathcal{U}_t \in \mathcal{C}$, respectively. We can then rewrite model (1.1) as below

$$\frac{\mathrm{d}\mathcal{U}(t)}{\mathrm{d}t} = \delta(\xi)(\mathcal{U}_t)_{xx} + \mathcal{L}(\xi)(\mathcal{U}_t) + \mathcal{F}(\mathcal{U}_t, \xi). \tag{3.1}$$

The $\phi = (\phi^{(1)}, \phi^{(2)})^{\mathrm{T}} \in \mathcal{C}$, $\delta(\xi)(\cdot)_{xx}$, $\mathcal{L}(\xi)(\cdot) : \mathcal{C} \to \Upsilon$, $\mathcal{F}(\cdot,\cdot) : \mathcal{C} \times \mathbb{R}^2 \to \Upsilon$ in Eq. (3.1) are respectively formulated by

$$\delta(\xi)(\phi)_{xx} = \delta_0(\phi)_{xx} + \mathcal{F}^{\delta}(\phi, \xi),\ \mathcal{L}(\xi)(\phi) = (\tau^H + \xi_1)\left(A\phi(0) + B\phi(-\tilde{\sigma})\right),$$

and

$$\mathcal{F}(\phi,\xi) =(\tau^H+\xi_1)\begin{pmatrix} f_1\left(\phi^{(1)}(0)+N_2,\phi^{(2)}(0)+P_2\right) \\ f_2\left(\phi^{(1)}(0)+N_2,\phi^{(2)}(0)+P_2,\phi^{(1)}(-\tilde{\sigma})+N_2,\phi^{(2)}(-\tilde{\sigma})+P_2\right)\end{pmatrix} - \mathcal{L}(\xi)(\phi), \tag{3.2}$$

where

$$\begin{aligned}\tilde{\sigma} &=\frac{\sigma}{\tau^H+\xi_1},\\ \delta_0(\phi)_{xx} &=\tau^H\mathfrak{D}_1(\phi)_{xx}(0)+\tau^H\mathfrak{D}_1^H(\phi)_{xx}(-1),\\ \mathcal{F}^\delta(\phi,\xi) &=\tau^H\begin{pmatrix}\delta_{12}^H\left(\phi_x^{(1)}(0)\phi_x^{(2)}(-1)+\phi^{(1)}(0)\phi_{xx}^{(2)}(-1)\right)\\ 0\end{pmatrix}\\ &\quad+\xi_1\begin{pmatrix}\delta_{11}\phi_{xx}^{(1)}(0)+\delta_{12}^H N_2\phi_{xx}^{(2)}(-1)\\ \delta_{22}\phi_{xx}^{(2)}(0)\end{pmatrix}+\tau^H\xi_2\begin{pmatrix}N_2\phi_{xx}^{(2)}(-1)\\0\end{pmatrix}\\ &\quad+(\delta_{12}^H\xi_1+\tau^H\xi_2)\begin{pmatrix}\phi_x^{(1)}(0)\phi_x^{(2)}(-1)+\phi^{(1)}(0)\phi_{xx}^{(2)}(-1)\\0\end{pmatrix}\\ &\quad+\xi_1\xi_2\begin{pmatrix}N_2\phi_{xx}^{(2)}(-1)\\0\end{pmatrix}+\xi_1\xi_2\begin{pmatrix}\phi_x^{(1)}(0)\phi_x^{(2)}(-1)+\phi^{(1)}(0)\phi_{xx}^{(2)}(-1)\\0\end{pmatrix}.\end{aligned} \tag{3.3}$$

Separating the linear term from Eq. (3.1), we have

$$\frac{\mathrm{d}\mathcal{U}(t)}{\mathrm{d}t}=\delta_0(\mathcal{U}_t)_{xx}+\mathcal{L}_0(\mathcal{U}_t)+\tilde{\mathcal{F}}(\mathcal{U}_t,\xi), \tag{3.4}$$

where $\mathcal{L}_0(\phi)=\tau^H(A\phi(0)+B\phi(-\frac{\sigma}{\tau^H}))$ and

$$\tilde{\mathcal{F}}(\mathcal{U}_t,\xi)=\mathcal{L}(\xi)(\phi)-\mathcal{L}_0(\phi)+\mathcal{F}(\phi,\xi)+\mathcal{F}^\delta(\phi,\xi). \tag{3.5}$$

We thus obtain the corresponding characteristic equation to system

$$\frac{\mathrm{d}\mathcal{U}(t)}{\mathrm{d}t}=\delta_0(\mathcal{U}_t)_{xx}+\mathcal{L}_0(\mathcal{U}_t) \tag{3.6}$$

is

$$\prod_{n=0}\det(\Delta_n(\lambda))=0, \tag{3.7}$$

where

$$\Delta_n(\lambda)=\lambda E_2+\tau^H\left(\frac{n}{\iota}\right)^2\mathcal{D}_1+\tau^H\left(\frac{n}{\iota}\right)^2\mathrm{e}^{-\lambda}\mathcal{D}_2^H-\tau^H\left(A+B\mathrm{e}^{-\frac{\lambda\sigma}{\tau^H}}\right).$$

It follows from Theorem 2.1 that Eq. (3.7) admits two pairs of purely imaginary roots $\pm i\varpi_{n_1}^H,\pm i\varpi_{n_2}^H$ where $\varpi_{n_1}^H=\varpi_{n_1}\tau^H$ and $\varpi_{n_2}^H=\varpi_{n_2}\tau^H$.

It is easy to check that $s=\left(\frac{n}{\iota}\right)^2$, $n\in\mathbb{N}_0$ are the eigenvalues of one-dimensional Laplace operator with homogeneous Neumann boundary conditions. The corresponding eigenfunctions with respect to $s=\left(\frac{n}{\iota}\right)^2$, $n\in\mathbb{N}_0$ are as below:

$$\varrho_n(x)=\frac{\cos(\frac{nx}{\iota})}{\|\cos\left(\frac{nx}{\iota}\right)\|_{2,2}}=\begin{cases}\frac{1}{\sqrt{\iota\pi}}, & \text{if } n=0,\\ \frac{\sqrt{2}}{\sqrt{\iota\pi}}\cos(\frac{nx}{\iota}), & \text{if } n\neq 0,\end{cases} \tag{3.8}$$

The symbol $\|\cdot\|_{2,2}$ in Eq. (3.8) refers to the norm induced by the following inner product

$$[\mathcal{U},\mathcal{V}]=\int_0^{\iota\pi}\mathcal{U}^{\mathrm{T}}\mathcal{V}\mathrm{d}x,\ \text{for } \mathcal{U},\mathcal{V}\in Y.$$

Denote $\varrho_n^{(i)}=\varrho_n(x)e_i,\ i=1,2$ where $e_1=(1,0)^{\mathrm{T}}$ and $e_2=(0,1)^{\mathrm{T}}$, and $\mathcal{S}_n=\mathrm{span}\{[u(\cdot),\varrho_n^{(i)}]\varrho_n^{(i)}\mid u\in C,\ i=1,2\}$. It is acknowledged that

$$\mathcal{L}_0(\mathcal{S}_n)\subset\mathrm{span}\{\varrho_n^{(1)},\varrho_n^{(2)}\}.$$

We then let $w_t(\zeta)\in\mathcal{C}_2=C([-\max\{-1,-\frac{\sigma}{\tau}\},0],\mathbb{R}^2)$ and $w_t^{\mathrm{T}}(\zeta)(\varrho_n^{(1)},\varrho_n^{(2)})^{\mathrm{T}}\in\mathcal{S}_n$. The linear system (3.6) can be then rewritten as the following equation on the space $\mathcal{S}_n$:

$$\dot{w}(t)=\mathcal{L}_0^\delta(w_t(\zeta))+\mathcal{L}_0(w_t(\zeta)), \tag{3.9}$$

where

$$\mathcal{L}_0^\delta(w_t(\zeta))=\tau^H\begin{pmatrix}-\delta_{11}(\frac{n}{\iota})^2 & 0\\ 0 & -\delta_{22}(\frac{n}{\iota})^2\end{pmatrix}w_t(0)-\tau^H\begin{pmatrix}0 & \delta_{12}^H N_2(\frac{n}{\iota})^2\\ 0 & 0\end{pmatrix}w_t(-1).$$

It is obvious that systems (3.6) and (3.9) have the same characteristic equation as is shown in Eq. (3.7). Let $C_2^* = C([0, \max\{1, \frac{\sigma}{\tau}\}]; \mathbb{R}^{2*})$, and choose the adjoint bilinear form on $C_2^* \times C_2$ as below:

$$\langle \mathcal{Q}(s), \mathcal{P}(\zeta)\rangle = \mathcal{Q}(0)\mathcal{P}(0) - \int_{-\max\{\frac{\sigma}{\tau},1\}}^{0}\int_{\rho=0}^{\zeta} \mathcal{Q}(\rho-\zeta)\mathrm{d}\Theta_n(\zeta)\mathcal{P}(\rho)\mathrm{d}\rho, \text{ for } \mathcal{Q}\in C_2^*, \mathcal{P}\in C_2,$$

where $\Theta_n(\zeta) \in BV([-\max\{1, \frac{\sigma}{\tau}\}, 0]; \mathbb{R}^{2\times 2})$. For $\mathcal{P}(\zeta) \in C_2$, we have

$$\mathcal{L}_0^{\delta}(\mathcal{P}(\zeta)) + \mathcal{L}_0(\mathcal{P}(\zeta)) = \int_{-\max\{1,\frac{\sigma}{\tau}\}}^{0} \mathrm{d}\Theta_n(\zeta)\mathcal{P}(\zeta).$$

Define $\Omega = \{i\varpi_{n_1}^H, -i\varpi_{n_1}^H, i\varpi_{n_2}^H, -i\varpi_{n_1}^H\}$. Let Ψ_{n_i} be the generalized eigenspace of (3.9) and $\Psi_{n_i}^*$ be the associated adjoint space. We then derive from [31] that C_2 can be decomposed as $C_2 = \Psi_{n_i} \oplus \Phi_{n_i},\ i = 1, 2$, where $\Phi_{n_i} = \{\psi \in C_2 : <\phi, \psi>= 0, \forall \phi \in \Psi_{n_i}^*\}$. We can respectively adopt the bases $P_{n_i}(\zeta)$ and $Q_{n_i}(\zeta)$ of Ψ_{n_i} and $\Psi_{n_i}^*$ as below:

$$P_{n_i}(\zeta) = (p_{n_i}(\zeta), \bar{p}_{n_i}(\zeta)), Q_{n_i}(s) = (q_{n_i}^{\mathrm{T}}(s), \bar{q}_{n_i}^{\mathrm{T}}(s))^{\mathrm{T}},$$

meeting $< Q_{n_i}, P_{n_i} >= E_2$. Doing some calculations, we have

$$p_{n_i}(\zeta) = \begin{pmatrix} p_{n_i}^{(1)}(\zeta) \\ p_{n_i}^{(2)}(\zeta) \end{pmatrix} = p_{n_i}(0)\mathrm{e}^{i\varpi_{n_i}^H \zeta}, \quad q_{n_i}(s) = \begin{pmatrix} q_{n_i}^{(1)}(s) \\ q_{n_i}^{(2)}(s) \end{pmatrix} = q_{n_i}(0)\mathrm{e}^{-i\varpi_{n_i}^H s},$$

and

$$\begin{aligned} p_{n_i}(0) &= \begin{pmatrix} 1 \\ \dfrac{-\beta_{21}\mathrm{e}^{-i\varpi_{n_i}\sigma}}{-\delta_{22}(\frac{n_i}{\iota})^2 + \alpha_{22} + \beta_{22}\mathrm{e}^{-i\varpi_{n_i}\sigma} - i\varpi_{n_i}} \end{pmatrix}, \\ q_{n_i}(0) &= \eta_i \begin{pmatrix} 1 \\ \dfrac{\delta_{11}(\frac{n_i}{\iota})^2 - \alpha_{11} + i\varpi_{n_i}}{\beta_{21}\mathrm{e}^{-i\varpi_{n_i}\sigma}} \end{pmatrix}, \end{aligned}$$

with

$$\eta_i = \frac{k_2}{k_2 - k_1 + \tau^H \delta_{12} N_2 (\frac{n_i}{\iota})^2 \beta_{21}\mathrm{e}^{-i\varpi_{n_i}(\sigma+\tau^H)} + k_1 k_2 \sigma - k_1 \sigma \beta_{22}\mathrm{e}^{-i\varpi_{n_i}\sigma}},$$

and $k_1 = \delta_{11}(\frac{n_i}{\iota})^2 - \alpha_{11} + i\varpi_{n_i}$, $k_2 = -\delta_{22}(\frac{n_i}{\iota})^2 + \alpha_{22} + \beta_{22}\mathrm{e}^{-i\varpi_{n_i}\sigma} - i\varpi_{n_i}$.

We can then decompose the phase space C as

$$C = \mathrm{Im}\pi \oplus \ker \pi. \tag{3.10}$$

$\pi : C \to \mathrm{Im}\pi$ in Eq. (3.10) is the projection map formulated by

$$\begin{aligned} \pi(\varphi) =& P_{n_1}(\zeta) \left\langle Q_{n_1}(\zeta), \begin{pmatrix} [\varphi(\cdot), \varrho_{n_1}^{(1)}] \\ [\varphi(\cdot), \varrho_{n_1}^{(2)}] \end{pmatrix} \right\rangle_{n_1} \varrho_{n_1}(x) \\ &+ P_{n_2}(\zeta) \left\langle Q_{n_2}(\zeta), \begin{pmatrix} [\varphi(\cdot), \varrho_{n_2}^{(1)}] \\ [\varphi(\cdot), \varrho_{n_2}^{(2)}] \end{pmatrix} \right\rangle_{n_2} \varrho_{n_2}(x). \end{aligned}$$

We further denote $C_0^1 = \{\varphi \in C : \dot{\varphi} \in C, \varphi(0) \in \mathrm{hom}(\delta(\cdot)_{xx})\}$, $w = (w_1(t), w_2(t), w_3(t), w_4(t))^{\mathrm{T}}$ and also set

$$w_x = \left(w_1(t)\varrho_{n_1}(x),\ w_2(t)\varrho_{n_1}(x),\ w_3(t)\varrho_{n_2}(x),\ w_4(t)\varrho_{n_2}(x)\right)^{\mathrm{T}},$$

and

$$P(\zeta) = \left(p_{n_1}(\zeta), \bar{p}_{n_1}(\zeta), p_{n_2}(\zeta), \bar{p}_{n_2}(\zeta)\right).$$

It follows from [36,40] that for $\phi(\zeta) \in C_0^1$, we can decompose $\phi(\zeta)$ as below

$$\phi(\zeta) = P(\zeta)w_x + z,\ z = (z^{(1)}, z^{(2)})^{\mathrm{T}} \in C_0^1 \cap \ker \pi := Q^1.$$

Following from [36], we denote

$$Y_0(\zeta) = \begin{cases} 0, & \zeta \in [-\max\{1, \frac{\sigma}{\tau}\}, 0), \\ 1, & \zeta = 0. \end{cases}$$

and

$$\begin{pmatrix} [\tilde{\mathcal{F}}, \varrho_\kappa^{(1)}] \\ [\tilde{\mathcal{F}}, \varrho_\kappa^{(2)}] \end{pmatrix}_{\kappa=n_1}^{\kappa=n_2} = \mathrm{col}\left(\begin{pmatrix} [\tilde{\mathcal{F}}, \varrho_{n_1}^{(1)}] \\ [\tilde{\mathcal{F}}, \varrho_{n_1}^{(2)}] \end{pmatrix}, \begin{pmatrix} [\tilde{\mathcal{F}}, \varrho_{n_2}^{(1)}] \\ [\tilde{\mathcal{F}}, \varrho_{n_2}^{(2)}] \end{pmatrix}\right).$$

We can then decompose system (3.4) as the following system on $\mathbb{R}^4 \times \ker \pi$:

$$\begin{cases} \dot{w} = \Lambda w + Q(0) \begin{pmatrix} [\tilde{\mathcal{F}}(P(\zeta)w_x + z, \xi), \varrho_\kappa^{(1)}] \\ [\tilde{\mathcal{F}}(P(\zeta)w_x + z, \xi), \varrho_\kappa^{(2)}] \end{pmatrix}_{\kappa=n_i}^{\kappa=n_j}, \\ \dot{z} = \mathcal{A}_{Q^1} z + (I-\pi)Y_0(\zeta)\tilde{\mathcal{F}}(P(\zeta)w_x + z, \xi), \end{cases} \tag{3.11}$$

where

$$\begin{cases} Q(0) = \mathrm{diag}\{Q_{n_1}(0), Q_{n_2}(0)\}, \\ \Lambda = \mathrm{diag}\{i\varpi_{n_1}^H, -i\varpi_{n_1}^H, i\varpi_{n_2}^H, -i\varpi_{n_2}^H\}, \\ \mathcal{A}_{Q^1} z = \dot{z} + Y_0(\zeta)\left(\mathcal{L}_0^\delta(z) + \mathcal{L}_0(z) - \dot{z}(0)\right). \end{cases}$$

Next we expand $\mathcal{F}(\phi,\xi)$, $\mathcal{F}^\delta(\phi,\xi)$, $\tilde{\mathcal{F}}(\phi,\xi)$ as below:

$$\mathcal{L}(\xi)(\phi) = \sum_{\kappa\ge 1}\frac{1}{\kappa!}\mathcal{L}_\kappa(\xi)(\phi),\ \mathcal{F}(\phi,\xi) = \sum_{\kappa\ge 2}\frac{1}{\kappa!}\mathcal{F}_\kappa(\phi,\xi),\ \tilde{\mathcal{F}}(\phi,\xi) = \sum_{\kappa\ge 2}\frac{1}{\kappa!}\tilde{\mathcal{F}}_\kappa(\phi,\xi), \tag{3.12}$$

and

$$\mathcal{F}^\delta(\phi,\xi) = \frac{1}{2}\mathcal{F}_2^\delta(\phi,\xi) + \frac{1}{6}\mathcal{F}_3^\delta(\phi,\xi) + \frac{1}{24}\mathcal{F}_4^\delta(\phi,\xi).$$

We can derive from (3.3) that

$$\mathcal{F}_2^\delta(\phi,\xi) = \mathcal{F}_2^{\delta(0,0)}(\phi) + \xi_1\mathcal{F}_2^{\delta(1,0)}(\phi) + \xi_2\mathcal{F}_2^{\delta(0,1)}(\phi), \tag{3.13}$$

and

$$\mathcal{F}_3^\delta(\phi,\xi) = \xi_1\mathcal{F}_3^{\delta(1,0)}(\phi) + \xi_2\mathcal{F}_3^{\delta(0,1)}(\phi) + \xi_1\xi_2\mathcal{F}_3^{\delta(1,1)}(\phi), \tag{3.14}$$

with

$$\begin{cases} \mathcal{F}_2^{\delta(0,0)}(\phi) = 2\delta_{12}^H\tau^H \begin{pmatrix} \phi_x^{(1)}(0)\phi_x^{(2)}(-1) + \phi^{(1)}(0)\phi_{xx}^{(2)}(-1) \\ 0 \end{pmatrix}, \\ \mathcal{F}_2^{\delta(1,0)}(\phi) = 2\mathcal{D}_1\phi_{xx}(0) + 2\mathcal{D}_2^H\phi_{xx}(-1), \\ \mathcal{F}_2^{\delta(0,1)}(\phi) = \dfrac{2\tau^H}{\delta_{12}^H}\mathcal{D}_2^H\phi_{xx}(-1), \\ \mathcal{F}_3^{\delta(1,0)}(\phi) = 6\delta_{12}^H \begin{pmatrix} \phi_x^{(1)}(0)\phi_x^{(2)}(-1) + \phi^{(1)}(0)\phi_{xx}^{(2)}(-1) \\ 0 \end{pmatrix}, \\ \mathcal{F}_3^{\delta(0,1)}(\phi) = 6\tau^H \begin{pmatrix} \phi_x^{(1)}(0)\phi_x^{(2)}(-1) + \phi^{(1)}(0)\phi_{xx}^{(2)}(-1) \\ 0 \end{pmatrix}, \\ \mathcal{F}_3^{\delta(1,1)}(\phi) = 6\begin{pmatrix} N_2\phi_{xx}^{(2)}(-1) \\ 0 \end{pmatrix}, \end{cases} \tag{3.15}$$

and

$$\mathcal{F}_4^\delta(\phi) = 24\xi_1\xi_2 \begin{pmatrix} \phi_x^{(1)}(0)\phi_x^{(2)}(-1) + \phi^{(1)}(0)\phi_{xx}^{(2)}(-1) \\ 0 \end{pmatrix}.$$

It then follows from (3.5) that

$$\tilde{\mathcal{F}}_2(\phi,\xi) = 2\xi_1\left(A\phi(0) + B\phi\left(-\frac{\sigma}{\tau^H}\right) + \frac{\sigma}{\tau^H}B\phi'\left(-\frac{\sigma}{\tau^H}\right)\right) + \mathcal{F}_2(\phi,\xi) + \mathcal{F}_2^\delta(\phi,\xi), \tag{3.16}$$

and

$$\tilde{\mathcal{F}}_3(\phi,\xi) = \mathcal{L}_3(\xi)(\phi) + \mathcal{F}_3(\phi,\xi) + \mathcal{F}_3^\delta(\phi,\xi). \tag{3.17}$$

We thus rewrite (3.11) as

$$\begin{cases} \dot{w} = \Lambda w + \sum_{\kappa\ge 2}\frac{1}{\kappa!}g_\kappa^1(w,z,\xi), \\ \dot{z} = \mathcal{A}_{Q^1} z + \sum_{\kappa\ge 2}\frac{1}{\kappa!}g_\kappa^2(w,z,\xi), \end{cases}$$

with

$$g_\kappa^1(w,z,\xi) = Q(0)\begin{pmatrix} [\tilde{\mathcal{F}}_\kappa(P(\zeta)w_x + z,\xi), \varrho_\iota^{(1)}] \\ [\tilde{\mathcal{F}}_\kappa(P(\zeta)w_x + z,\xi), \varrho_\iota^{(2)}] \end{pmatrix}_{\iota=n_2}^{\iota=n_1}, \tag{3.18}$$

and

$$g^2_\kappa(w,z,\xi)=(I-\pi)Y_0(\zeta)\tilde{\mathcal{F}}_\kappa(P(\zeta)w_x+z,\xi).$$

Following the notations in [36], we define the following space

$$\mathcal{V}^6_\kappa(\mathbb{X})=\left\{\sum_{|(m,\ell)|=\kappa}c_{(m,\ell)}w^m\xi^\ell\ :\ (m,\ell)\in\mathbb{N}^6_0,\ c_{(m,\ell)}\in\mathbb{X}\right\},$$

where $\mathbb{X}$ is a normed space. We also define the operator $\mathcal{M}_\kappa=(\mathcal{M}^1_\kappa,\mathcal{M}^2_\kappa),\ \kappa\geq 2$ by

$$\begin{aligned}&\mathcal{M}^1_\kappa\ :\ \mathcal{V}^6_\kappa(\mathbb{C}^4)\to\mathcal{V}^6_\kappa(\mathbb{C}^4),\\&(\mathcal{M}^1_\kappa\varrho)(w,\xi)=D_w\varrho(w,\xi)\Lambda w-\Lambda\varrho(w,\xi),\\&\mathcal{M}^2_\kappa\ :\ \mathcal{V}^6_\kappa(Q^1)\subset\mathcal{V}^6_\kappa(\ker\pi)\to\mathcal{V}^6_\kappa(\ker\pi),\\&(\mathcal{M}^2_\kappa\hbar)(w,\xi)=D_w\hbar(w,\xi)\Lambda w-\mathcal{A}_{Q^1}\hbar(w,\xi).\end{aligned}$$

We can easily check that

$$\begin{aligned}\mathcal{M}^1_\kappa(w^m\xi^\ell e_\jmath)&=D_w(w^m\xi^\ell e_\jmath)\Lambda w-\Lambda(w^m\xi^\ell e_\jmath)\\&=\begin{cases}\left(iw_{n_1}m_1-iw_{n_1}m_2+iw_{n_2}m_3-iw_{n_2}m_4+(-1)^\jmath iw_{n_1}\right)w^m\xi^\ell e_\jmath,\\\left(iw_{n_1}m_1-iw_{n_1}m_2+iw_{n_2}m_3-iw_{n_2}m_4+(-1)^\jmath iw_{n_2}\right)w^m\xi^\ell e_\jmath,\end{cases}\end{aligned}\tag{3.19}$$

where $\jmath=1,2,3,4,\ w^m=w_1^{m_1}w_2^{m_2}w_3^{m_3}w_4^{m_4},\ \xi=\xi_1^{\ell_1}\xi_2^{\ell_2},\ m_1+m_2+m_3+m_4+\ell_1+\ell_2=\kappa$. Therefore, for the non-resonant case, we have

$$\ker(\mathcal{M}^1_2)=\mathrm{span}\{\xi_jw_1e_1,\ \xi_jw_2e_2,\ \xi_jw_3e_3,\ \xi_jw_4e_4,\ j=1,2\},$$

$$\ker(\mathcal{M}^1_3)=\left\{\begin{array}{l}\xi_1\xi_2w_1e_1,\ \xi_j^2w_1e_1,\ w_1^2w_2e_1,\ w_1w_3w_4e_1,\ \xi_1\xi_2w_2e_2,\ \xi_j^2w_2e_2,\\ w_1w_2^2e_2,\ w_2w_3w_4e_2,\ \xi_1\xi_2w_3e_3,\ \xi_j^2w_3e_3,\ w_3^2w_4e_3,\ w_1w_2w_3e_3,\\ \xi_1\xi_2w_4e_4,\ \xi_j^2w_4e_4,\ w_3w_4^2e_4,\ w_1w_2w_3e_4,\ j=1,2\end{array}\right\},$$

and

$$\mathcal{S}=\mathrm{span}\{w_1^2w_2e_1,w_1w_3w_4e_1,w_1w_2^2e_2,w_2w_3w_4e_2,w_3^2w_4e_3,w_1w_2w_3e_3,w_3w_4^2e_4,w_1w_2w_4e_4\}.$$

It also follows from [36] that the local center manifold for system (3.4) can be written as below

$$\dot{w}=\Lambda w+\sum_{\kappa\geq 2}\frac{1}{\kappa!}f^1_\kappa(w,0,\xi),\tag{3.20}$$

by making the following recursive transformation of variables:

$$(w,z,\xi)=(\tilde{w},\tilde{z},\xi)+\frac{1}{\kappa}\Big(V^1_\kappa(\tilde{w},\xi),V^2_\kappa(\tilde{w},\xi)\Big),\tag{3.21}$$

where $V_\kappa=\Big(V^1_\kappa(\tilde{w},\xi),V^2_\kappa(\tilde{w},\xi)\Big)\in\mathcal{V}^6_\kappa(\mathbb{C}^4)\times\mathcal{V}^6_\kappa(Q^1)$. We recall from [38,40] that

$$f^1_2(w,0,\xi)=\mathrm{Proj}_{\ker(\mathcal{M}^1_2)}g^1_2(w,0,\xi),$$

and

$$f^1_3(w,0,\xi)=\mathrm{Proj}_{\ker(\mathcal{M}^1_3)}\tilde{g}^1_3(w,0,\xi)=\mathrm{Proj}_{\mathcal{S}}\tilde{g}^1_3(w,0,0)+O(|\xi|^2|w|),\tag{3.22}$$

where $\tilde{g}^1_3(w,0,\xi)$ is the cubic polynomial of (w,ξ) by performing transformation (3.21). Besides, $\tilde{g}^1_3(w,0,0)$ can be judged by , $\ker(\mathcal{M}^1_2)$, $\ker(\mathcal{M}^1_3)$ and $\mathcal{S}$. In the sequel of this paper, for the convenience of notation, we denote

$$\mathcal{B}(\beta w_1^{m_1}w_2^{m_2}w_3^{m_3}w_4^{m_4}\xi_1^{\ell_1}\xi_2^{\ell_2})=\begin{pmatrix}\beta w_1^{m_1}w_2^{m_2}w_3^{m_3}w_4^{m_4}\xi_1^{\ell_1}\xi_2^{\ell_2}\\\bar{\beta}w_1^{m_1}w_2^{m_2}w_3^{m_3}w_4^{m_4}\xi_1^{\ell_1}\xi_2^{\ell_2}\end{pmatrix},\ \beta\in\mathbb{C}.$$

3.1. Calculation of $f^1_2(w,0,\xi)$

We can derive from (3.18) that

$$g^1_2(w,0,\xi)=Q(0)\begin{pmatrix}[\tilde{\mathcal{F}}_2(P(\zeta)w_x,\xi),\varrho^{(1)}_\iota]\\ [\tilde{\mathcal{F}}_2(P(\zeta)w_x,\xi),\varrho^{(2)}_\iota]\end{pmatrix}_{\iota=n_1}^{\iota=n_2}.\tag{3.23}$$

We see from (3.16) that

$$\tilde{\mathcal{F}}_2(P(\zeta)w_x,\xi) = 2\xi_1\left(A\left(P(0)w_x\right)+B\left(P\left(-\frac{\sigma}{\tau^H}\right)w_x\right)+\frac{\sigma}{\tau^H}B\left(P'\left(-\frac{\sigma}{\tau^H}\right)w_x\right)\right) + \mathcal{F}_2(P(\zeta)w_x,\xi)+\mathcal{F}_2^{\delta}(P(\zeta)w_x,\xi). \tag{3.24}$$

In conjunction with (3.8), (3.13)–(3.15), (3.23) and (3.24), for $n_1 \neq n_2$, we can deduce that

$$\begin{pmatrix} \left[2\xi_1\left(A\left(P(0)w_x\right)+B\left(P\left(-\frac{\sigma}{\tau^H}\right)w_x\right)+\frac{\sigma}{\tau^H}B\left(P'\left(-\frac{\sigma}{\tau^H}\right)w_x\right)\right),\varrho_\iota^{(1)}\right] \\ \left[2\xi_1\left(A\left(P(0)w_x\right)+B\left(P\left(-\frac{\sigma}{\tau^H}\right)w_x\right)+\frac{\sigma}{\tau^H}B\left(P'\left(-\frac{\sigma}{\tau^H}\right)w_x\right)\right),\varrho_\iota^{(2)}\right] \end{pmatrix} = \begin{cases} 2\xi_1\left(AP_{n_1}(0)+BP_{n_1}\left(-\frac{\sigma}{\tau^H}\right)+\frac{i\varpi_{n_1}^H\sigma}{\tau^H}BP_{n_1}\left(-\frac{\sigma}{\tau^H}\right)\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, & \iota = n_1, \\ 2\xi_1\left(AP_{n_2}(0)+BP_{n_2}\left(-\frac{\sigma}{\tau^H}\right)+\frac{i\varpi_{n_2}^H\sigma}{\tau^H}BP_{n_2}\left(-\frac{\sigma}{\tau^H}\right)\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, & \iota = n_2, \end{cases} \tag{3.25}$$

$$\begin{pmatrix} \left[\xi_1\mathcal{F}_2^{\delta(1,0)}(P(\zeta)w_x),\varrho_\iota^{(1)}\right] \\ \left[\xi_1\mathcal{F}_2^{\delta(1,0)}(P(\zeta)w_x),\varrho_\iota^{(2)}\right] \end{pmatrix} = \begin{cases} -2\xi_1\left(\frac{n_1}{\iota}\right)^2\left(\mathcal{D}_1P_{n_1}(0)+\mathcal{D}_2^HP_{n_1}(-1)\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, & \iota = n_1, \\ -2\xi_1\left(\frac{n_2}{\iota}\right)^2\left(\mathcal{D}_1P_{n_2}(0)+\mathcal{D}_2^HP_{n_2}(-1)\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, & \iota = n_2, \end{cases} \tag{3.26}$$

and

$$\begin{pmatrix} \left[\xi_2\mathcal{F}_2^{\delta(0,1)}(P(\zeta)w_x),\varrho_\iota^{(1)}\right] \\ \left[\xi_2\mathcal{F}_2^{\delta(0,1)}(P(\zeta)w_x),\varrho_\iota^{(2)}\right] \end{pmatrix} = \begin{cases} -2\xi_2\left(\frac{n_1}{\iota}\right)^2\frac{\tau^H}{\delta_{12}^H}\left(\mathcal{D}_2^HP_{n_1}(-1)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right), & \iota = n_1, \\ -2\xi_2\left(\frac{n_1}{\iota}\right)^2\frac{\tau^H}{\delta_{12}^H}\left(\mathcal{D}_2^HP_{n_2}(-1)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right), & \iota = n_2. \end{cases} \tag{3.27}$$

We can easily check from (3.2) that for all $\xi \in \mathbb{R}^2$,

$$\mathcal{F}_2(P(\zeta)w_x,\xi) = \mathcal{F}_2(P(\zeta)w_x,0). \tag{3.28}$$

In combination with Eqs. (3.25)–(3.27), we deduce that

$$f_2^1(w,0,\xi) = \mathrm{Proj}_{\ker(\mathcal{M}_2^1)}g_2^1(w,0,\xi) = \begin{pmatrix} \mathcal{B}\left((\mathfrak{B}_{11A}\xi_1+\mathfrak{B}_{21A}\xi_2)w_1\right) \\ \mathcal{B}\left((\mathfrak{B}_{13A}\xi_1+\mathfrak{B}_{23A}\xi_2)w_3\right) \end{pmatrix}, \tag{3.29}$$

where

$$\mathfrak{B}_{11A} = 2q_{n_1}^{\mathrm{T}}(0)\left(Ap_{n_1}(0)+Bp_{n_1}\left(-\frac{\sigma}{\tau^H}\right)\left(1+\frac{i\varpi_{n_1}^H\sigma}{\tau^H}\right)-\left(\frac{n_1}{\iota}\right)^2\left(\mathcal{D}_1p_{n_1}(0)+\mathcal{D}_2^Hp_{n_1}(-1)\right)\right),$$
$$\mathfrak{B}_{31A} = 2q_{n_2}^{\mathrm{T}}(0)\left(Ap_{n_2}(0)+Bp_{n_2}\left(-\frac{\sigma}{\tau^H}\right)\left(1+\frac{i\varpi_{n_2}^H\sigma}{\tau^H}\right)-\left(\frac{n_2}{\iota}\right)^2\left(\mathcal{D}_1p_{n_2}(0)+\mathcal{D}_2^Hp_{n_2}(-1)\right)\right),$$
$$\mathfrak{B}_{21A} = -2\left(\frac{n_1}{\iota}\right)^2\frac{\tau^H}{\delta_{12}^H}q_{n_1}^{\mathrm{T}}(0)\left(\mathcal{D}_2^Hp_{n_1}(-1)\right),$$
$$\mathfrak{B}_{23A} = -2\left(\frac{n_2}{\iota}\right)^2\frac{\tau^H}{\delta_{12}^H}q_{n_2}^{\mathrm{T}}(0)\left(\mathcal{D}_2^Hp_{n_2}(-1)\right).$$

For the scenario of $n_1 = n_2$, it is straightforward to check $f_2^1(w,0,\xi)$ has the same expression.

3.2. Calculation of $f_3^1(w,0,\xi)$

We devote ourselves to the calculation of $f_3^1(w,0,\xi)$ via Eq. (3.22). Denote

$$g_2^{(1,1)}(w,z,0) = Q(0)\begin{pmatrix} [\tilde{\mathcal{F}}_2(P(\zeta)w_x+z,0),\varrho_\iota^{(1)}] \\ [\tilde{\mathcal{F}}_2(P(\zeta)w_x+z,0),\varrho_\iota^{(2)}] \end{pmatrix}_{\iota=n_1}^{\iota=n_2}, \tag{3.30}$$

and

$$g_2^{(1,2)}(w,z,0) = Q(0)\begin{pmatrix} [\tilde{\mathcal{F}}_2^{\delta}(P(\zeta)w_x+z,0),\varrho_\iota^{(1)}] \\ [\tilde{\mathcal{F}}_2^{\delta}(P(\zeta)w_x+z,0),\varrho_\iota^{(2)}] \end{pmatrix}_{\iota=n_1}^{\iota=n_2}.$$

We can also deduce from (3.29) that $f_2^1(w,0,0) = (0,0,0,0)^{\mathrm{T}}$. We thus obtain the expression $\tilde{g}_3^1(w,0,0)$ as below:

$$\begin{aligned}\tilde{g}_3^1(w,0,0) =& g_3^1(w,0,0) + \frac{3}{2}\left[\left(D_w g_2^1(w,0,0)\right)V_2^1(w,0) + \left(D_z g_2^{(1,1)}(w,0,0)\right)V_2^2(w,0)(\zeta)\right.\\ &\left.+\left(D_{z,z_x,z_{xx}}g_2^{(1,2)}(w,0,0)\right)V_2^{(2,\delta)}(w,0)(\zeta)\right],\end{aligned}$$

where $g_2^1(w,0,0) = g_2^{(1,1)}(w,0,0) + g_2^{(1,2)}(w,0,0)$,

$$\begin{aligned}D_{z,z_x,z_{xx}}g_2^{(1,2)}(w,0,0) &= \left(D_z g_2^{(1,2)}(w,0,0), D_{z_x}g_2^{(1,2)}(w,0,0), D_{z_{xx}}g_2^{(1,2)}(w,0,0)\right),\\ V_2^1(w,0) &= (\mathcal{M}_2^1)^{-1}\mathrm{Proj}_{\mathrm{Im}(\mathcal{M}_2^1)}g_2^1(w,0,0),\\ V_2^2(w,0)(\zeta) &= (\mathcal{M}_2^2)^{-1}g_2^2(w,0,0),\end{aligned}$$

and

$$V_2^{(2,\delta)}(w,0)(\zeta) = \left(V_2^2(w,0)(\zeta), V_{2x}^2(w,0)(\zeta), V_{2xx}^2(w,0)(\zeta)\right)^{\mathrm{T}}.$$

To complete the calculation of $\mathrm{Proj}_{\mathcal{S}}\tilde{g}_3^1(w,0,0)$, we will divide four procedures.

Procedure 1: The calculation of $\mathrm{Proj}_{\mathcal{S}}g_3^1(w,0,0)$

It follows from (3.14) and (3.17) that $\tilde{\mathcal{F}}_3(P(\zeta)w_x,0) = \mathcal{F}_3(P(\zeta)w_x,0)$. We can thus denote

$$\begin{aligned}\tilde{\mathcal{F}}_3(P(\zeta)w_x,0) =& \mathcal{F}_3(P(\zeta)w_x,0)\\ =& \sum_{m_1+m_2+m_3+m_4=3} R_{m_1m_2m_3m_4}\varrho_{n_1}^{m_1+m_2}(x)\varrho_{n_2}^{m_3+m_4}(x)w_1^{m_1}w_2^{m_2}w_3^{m_3}w_4^{m_4}.\end{aligned} \tag{3.31}$$

In conjunction with (3.18) and (3.31), we can deduce that

$$g_3^1(w,0,0) = Q(0)\begin{pmatrix} \sum_{m_1+m_2+m_3+m_4=3} R_{m_1m_2m_3m_4}\int_0^{\iota\pi}\varrho_{n_1}^{m_1+m_2+1}(x)\varrho_{n_2}^{m_3+m_4}(x)\mathrm{d}x \\ \cdot w_1^{m_1}w_2^{m_2}w_3^{m_3}w_4^{m_4} \\ \sum_{m_1+m_2+m_3+m_4=3} R_{m_1m_2m_3m_4}\int_0^{\iota\pi}\varrho_{n_1}^{m_1+m_2}(x)\varrho_{n_2}^{m_3+m_4+1}(x)\mathrm{d}x \\ \cdot w_1^{m_1}w_2^{m_2}w_3^{m_3}w_4^{m_4}\end{pmatrix}.$$

This along with the observation that

$$\int_0^{\iota\pi}\varrho_{n_1}^2(x)\varrho_{n_2}^2(x)\mathrm{d}x = \begin{cases}\frac{3}{2\iota\pi}, & n_2 = n_1,\\ \frac{1}{\iota\pi}, & n_2 \neq n_1,\end{cases}$$

yields

$$\mathrm{Proj}_{\mathcal{S}}g_3^1(w,0,0) = \begin{pmatrix}\mathfrak{B}\left(C_{11}w_1^2w_2 + C_{12}w_1w_3w_4\right)\\ \mathfrak{B}\left(C_{31}w_3^2w_4 + C_{32}w_1w_2w_3\right)\end{pmatrix}, \tag{3.32}$$

where

$$\begin{aligned}C_{11} &= \frac{3}{2\iota\pi}q_{n_1}^{\mathrm{T}}(0)R_{2100},\\ C_{31} &= \frac{3}{2\iota\pi}q_{n_2}^{\mathrm{T}}(0)R_{0021},\\ C_{12} &= \begin{cases}\frac{3}{2\iota\pi}q_{n_1}^{\mathrm{T}}(0)R_{1011}, & n_2 = n_1,\\ \frac{1}{\iota\pi}q_{n_1}^{\mathrm{T}}(0)R_{1011}, & n_2 \neq n_1,\end{cases}\\ C_{32} &= \begin{cases}\frac{3}{2\iota\pi}q_{n_2}^{\mathrm{T}}(0)R_{1110}, & n_2 = n_1\\ \frac{1}{\iota\pi}q_{n_2}^{\mathrm{T}}(0)R_{1110}, & n_2 \neq n_1.\end{cases}\end{aligned}$$

Procedure 2: The calculation of $\mathrm{Proj}_{\mathcal{S}}\left(D_w g_2^1(w,0,0)\right)V_2^1(w,0)$

It follows from (3.16) that

$$\tilde{\mathcal{F}}_2(P(\zeta)w_x,0) = \mathcal{F}_2(P(\zeta)w_x,0) + \mathcal{F}_2^{\delta(0,0)}(P(\zeta)w_x).$$

We can derive from (3.28) that

$$\begin{aligned}\mathcal{F}_2(P(\zeta)w_x+z,\xi)=&\mathcal{F}_2(P(\zeta)w_x+z,0)\\ =&\sum_{m_1+m_2+m_3+m_4=2} R_{m_1m_2m_3m_4}\varrho_{n_1}^{m_1+m_2}(x)\varrho_{n_2}^{m_3+m_4}(x)w_1^{m_1}\\ &\cdot w_2^{m_2}w_3^{m_3}w_4^{m_4}+\mathcal{T}_2(P(\zeta)w_x,z)+O(|z|^2),\end{aligned} \tag{3.33}$$

where $\mathcal{T}_2(P(\zeta)w_x,z)$ is the second cross terms of $P(\zeta)w_x$ and z. We can also deduce from (3.13)

$$\begin{aligned}&\mathcal{F}_2^{\delta}(P(\zeta)w_x+z,\xi)=\mathcal{F}_2^{\delta(0,0)}(P(\zeta)w_x+z)\\ &=\sum_{m_1+m_2+m_3+m_4=2} R^{(\delta,1)}_{m_1m_2m_3m_4}\left(\frac{-n_1}{\iota}\eta_{n_1}(x)\right)^{m_1+m_2}\left(\frac{-n_2}{\iota}\eta_{n_2}(x)\right)^{m_3+m_4} w_1^{m_1}w_2^{m_2}w_3^{m_3}w_4^{m_4}\\ &-\left(\frac{n_1}{\iota}\right)^2R^{(\delta,2)}_{2000}\varrho^2_{n_1}(x)w_1^2-\left(\frac{n_1}{\iota}\right)^2R^{(\delta,2)}_{0200}\varrho^2_{n_1}(x)w_2^2-\left(\frac{n_2}{\iota}\right)^2R^{(\delta,2)}_{0020}\varrho^2_{n_2}(x)w_3^2-\left(\frac{n_2}{\iota}\right)^2R^{(\delta,2)}_{0002}\varrho^2_{n_2}(x)w_4^2\\ &-\varrho_{n_1}(x)\varrho_{n_2}(x)\left\{\left(\left(\frac{n_1}{\iota}\right)^2R^{(\delta,2)}_{1010}+\left(\frac{n_2}{\iota}\right)^2R^{(\delta,3)}_{1010}\right)w_1w_3+\left(\left(\frac{n_1}{\iota}\right)^2R^{(\delta,2)}_{0110}+\left(\frac{n_2}{\iota}\right)^2R^{(\delta,3)}_{0110}\right)w_1w_4\right.\\ &\left.+\left(\left(\frac{n_1}{\iota}\right)^2R^{(\delta,2)}_{0110}+\left(\frac{n_2}{\iota}\right)^2R^{(\delta,3)}_{0110}\right)w_2w_3+\left(\left(\frac{n_1}{\iota}\right)^2R^{(\delta,2)}_{0101}+\left(\frac{n_2}{\iota}\right)^2R^{(\delta,3)}_{0101}\right)w_2w_4\right\},\end{aligned} \tag{3.34}$$

where $\eta_{n_i}=\frac{\sqrt{2}}{\sqrt{\iota\pi}}\sin\left(\frac{n_ix}{\iota}\right), i=1,2$ and

$$\begin{aligned}
&R^{(\delta,1)}_{1010}=2\delta^H_{12}\tau^H\begin{pmatrix} p^{(1)}_{n_1}(0)p^{(2)}_{n_2}(-1)+p^{(1)}_{n_2}(0)p^{(2)}_{n_1}(-1)\\ 0\end{pmatrix},\\
&R^{(\delta,1)}_{1001}=2\delta^H_{12}\tau^H\begin{pmatrix} \bar{p}^{(1)}_{n_2}(0)p^{(2)}_{n_1}(-1)+p^{(1)}_{n_1}(0)\bar{p}^{(2)}_{n_2}(-1)\\ 0\end{pmatrix},\\
&R^{(\delta,1)}_{0110}=\overline{R^{(\delta,1)}_{1001}},\qquad R^{(\delta,1)}_{0101}=\overline{R^{(\delta,1)}_{1010}},\\
&R^{(\delta,1)}_{2000}=R^{(\delta,2)}_{2000}=2\delta^H_{12}\tau^H\begin{pmatrix} p^{(1)}_{n_1}(0)p^{(2)}_{n_1}(-1)\\ 0\end{pmatrix},\\
&R^{(\delta,1)}_{0020}=R^{(\delta,2)}_{0020}=2\delta^H_{12}\tau^H\begin{pmatrix} p^{(1)}_{n_2}(0)p^{(2)}_{n_2}(-1)\\ 0\end{pmatrix},\\
&R^{(\delta,1)}_{1100}=R^{(\delta,2)}_{1100}=2\delta^H_{12}\tau^H\begin{pmatrix} 2\Re\{\bar{p}^{(1)}_{n_1}(0)p^{(2)}_{n_1}(-1)\}\\ 0\end{pmatrix},\\
&R^{(\delta,2)}_{1010}=2\delta^H_{12}\tau^H\begin{pmatrix} p^{(1)}_{n_2}(0)p^{(2)}_{n_1}(-1)\\ 0\end{pmatrix},\\
&R^{(\delta,2)}_{1010}=2\delta^H_{12}\tau^H\begin{pmatrix} p^{(1)}_{n_2}(0)p^{(2)}_{n_1}(-1)\\ 0\end{pmatrix},\\
&R^{(\delta,3)}_{1010}=2\delta^H_{12}\tau^H\begin{pmatrix} p^{(1)}_{n_1}(0)p^{(2)}_{n_2}(-1)\\ 0\end{pmatrix},\\
&R^{(\delta,2)}_{1001}=2\delta^H_{12}\tau^H\begin{pmatrix} p^{(2)}_{n_1}(-1)\bar{p}^{(1)}_{n_2}(0)\\ 0\end{pmatrix},\\
&R^{(\delta,3)}_{1001}=2\delta^H_{12}\tau^H\begin{pmatrix} p^{(1)}_{n_1}(0)\bar{p}^{(2)}_{n_2}(-1)\\ 0\end{pmatrix},\\
&R^{(\delta,1)}_{0200}=R^{(\delta,2)}_{0200}=\overline{R^{(\delta,1)}_{2000}},\ R^{(\delta,2)}_{0110}=\overline{R^{(\delta,2)}_{1001}},\ R^{(\delta,3)}_{0110}=\overline{R^{(\delta,3)}_{1001}},\\
&R^{(\delta,1)}_{0002}=R^{(\delta,2)}_{0002}=\overline{R^{(\delta,1)}_{0020}},\ R^{(\delta,2)}_{0101}=\overline{R^{(\delta,2)}_{1010}},\ R^{(\delta,3)}_{0101}=\overline{R^{(\delta,3)}_{1010}}.
\end{aligned}$$

We can then perform simple calculations according to (3.33) and (3.34) to obtain

$$\begin{aligned}g^1_2(w,0,0)=&Q(0)\begin{pmatrix}[\tilde{\mathcal{F}}_2(P(\zeta)w_x,0),\varrho^{(1)}_\iota]\\ [\tilde{\mathcal{F}}_2(P(\zeta)w_x,0),\varrho^{(2)}_\iota]\end{pmatrix}^{\iota=n_2}_{\iota=n_1}\\ =&\begin{cases}\frac{1}{\sqrt{2\iota\pi}}Q(0)\begin{pmatrix}\widetilde{R}_{1010}w_1w_3+\widetilde{R}_{1001}w_1w_4+\widetilde{R}_{0110}w_2w_3+\widetilde{R}_{0101}w_2w_4\\ \widetilde{R}_{2000}w_1^2+\widetilde{R}_{0200}w_2^2+\widetilde{R}_{1100}w_1w_2\end{pmatrix}, & n_2=2n_1,\\ (0,0,0,0)^{\mathrm{T}}, & n_2\neq 2n_1,\end{cases}\end{aligned}$$

where

$$\begin{cases}\widetilde{R}_{m_1m_2m_3m_4} = R_{m_1m_2m_3m_4} + \frac{n_1n_2}{\imath^2} R^{(\delta,1)}_{m_1m_2m_3m_4} - \frac{n_1^2}{\imath^2} R^{(\delta,2)}_{m_1m_2m_3m_4} - \frac{n_2^2}{\imath^2} R^{(\delta,3)}_{m_1m_2m_3m_4},\\ m_1,m_2,m_3,m_4 = 0,1,\ m_1+m_2=1,\ m_3+m_4=1,\end{cases}$$

and

$$\begin{cases}\widetilde{R}_{m_1m_2m_3m_4} = R_{m_1m_2m_3m_4} - \frac{n_1^2}{\imath^2}\left(R^{(\delta,1)}_{m_1m_2m_3m_4} + R^{(\delta,2)}_{m_1m_2m_3m_4}\right),\\ m_1,m_2 = 0,1,2,\ m_1+m_2=2,\ m_3+m_4=0.\end{cases}$$

We thus have for $n_2 \neq 2n_1$, $V_2^1(w,0) = (0,0,0,0)^{\mathrm{T}}$, and for $n_1 = 2n_1$,

$$\begin{aligned} V_2^1(w,0) &= (\mathcal{M}_2^1)^{-1}\mathrm{Proj}_{\mathrm{Im}\mathcal{M}_2^1} g_2^1(w,0,0)\\ &= \frac{1}{i\sqrt{2\imath\pi}}\begin{pmatrix} q_{n_1}^{\mathrm{T}}(0)\left(\frac{1}{\varpi_{n_2}^H}\widetilde{R}_{1010}w_1w_3 - \frac{1}{\varpi_{n_2}^H}\widetilde{R}_{1001}w_1w_4 + \frac{1}{\varpi_{n_2}^H-2\varpi_{n_1}^H}\widetilde{R}_{0110}w_2w_3 - \frac{1}{\varpi_{n_2}^H+2\varpi_{n_1}^H}\widetilde{R}_{0101}w_2w_4\right)\\ \bar{q}_{n_1}^{T}(0)\left(\frac{1}{\varpi_{n_2}^H+2\varpi_{n_1}^H}\widetilde{R}_{1010}w_1w_3 - \frac{1}{\varpi_{n_2}^H-2\varpi_{n_1}^H}\widetilde{R}_{1001}w_1w_4 + \frac{1}{\varpi_{n_2}^H}\widetilde{R}_{0110}w_2w_3 - \frac{1}{\varpi_{n_2}^H}\widetilde{R}_{0101}w_2w_4\right)\\ q_{n_2}^{\mathrm{T}}(0)\left(-\frac{1}{\varpi_{n_2}^H-2\varpi_{n_1}^H}\widetilde{R}_{2000}w_1^2 - \frac{1}{\varpi_{n_2}^H+2\varpi_{n_1}^H}\widetilde{R}_{0200}w_2^2 - \frac{1}{\varpi_{n_2}^H}\widetilde{R}_{1100}w_1w_2\right)\\ \bar{q}_{n_2}^{\mathrm{T}}(0)\left(\frac{1}{\varpi_{n_2}^H+2\varpi_{n_1}^H}\widetilde{R}_{2000}w_1^2 + \frac{1}{\varpi_{n_2}^H-2\varpi_{n_1}^H}\widetilde{R}_{0200}w_2^2 + \frac{1}{\varpi_{n_2}^H}\widetilde{R}_{1100}w_1w_2\right)\end{pmatrix}.\end{aligned}$$

We readily check that

$$\mathrm{Proj}_{\mathcal{S}}\left(D_w g_2^1(w,0,0)\right)V_2^1(w,0) = \begin{pmatrix}\mathcal{B}\left(\mathcal{D}_{11}w_1^2w_2 + \mathcal{D}_{12}w_1w_3w_4\right)\\ \mathcal{B}\left(\mathcal{D}_{31}w_3^2w_4 + \mathcal{D}_{32}w_1w_2w_3\right)\end{pmatrix}, \tag{3.35}$$

where for $n_2 \neq 2n_1$, $\mathcal{D}_{11} = \mathcal{D}_{12} = \mathcal{D}_{31} = \mathcal{D}_{32} = 0$, and for $n_2 = 2n_1, \mathcal{D}_{31} = 0$, and

$$\begin{aligned}\mathcal{D}_{11} =& \frac{1}{2i\imath\pi}\Bigg(-\frac{1}{\varpi_{n_2}^H}\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1010}\right)\left(q_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{1100}\right) + \frac{1}{\varpi_{n_2}^H}\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1001}\right)\left(\bar{q}_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{1100}\right)\\ &-\frac{1}{\varpi_{n_2}^H-2\varpi_{n_1}^H}\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{0110}\right)\left(q_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{2000}\right)\\ &+\frac{1}{\varpi_{n_2}^H+2\varpi_{n_1}^H}\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{0101}\right)\left(\bar{q}_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{2000}\right)\Bigg),\\ \mathcal{D}_{12} =& \frac{1}{2i\imath\pi}\Bigg(-\frac{1}{\varpi_{n_2}^H}\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1010}\right)\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1001}\right) + \frac{1}{\varpi_{n_2}^H}\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1001}\right)\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1010}\right)\\ &-\frac{1}{\varpi_{n_2}^H-2\varpi_{n_1}^H}\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{0110}\right)\left(\bar{q}_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1001}\right)\\ &+\frac{1}{\varpi_{n_2}^H+2\varpi_{n_1}^H}\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{0101}\right)\left(\bar{q}_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1010}\right)\Bigg),\\ \mathcal{D}_{32} =& \frac{1}{2i\imath\pi}\Bigg(\frac{2}{\varpi_{n_2}^H-2\varpi_{n_1}^H}\left(q_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{2000}\right)\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{0110}\right)\\ &+\frac{1}{\varpi_{n_2}^H}\left(q_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{1100}\right)\left(q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1010}\right) + \frac{1}{\varpi_{n_2}^H}\left(q_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{1100}\right)\left(\bar{q}_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{0110}\right)\\ &+\frac{2}{\varpi_{n_2}^H+2\varpi_{n_1}^H}\left(q_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{0200}\right)\left(\bar{q}_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1010}\right)\Bigg).\end{aligned}$$

Procedure 3: The calculation of $\mathrm{Proj}_{\mathcal{S}}\Big(D_w g_2^{(1,1)}(w,0,0)\Big)V_2^2(w,0)(\zeta)$

Let

$$V_2^2(w,0)(\zeta) = \hbar(\zeta,w) = \sum_{n\in\mathbb{N}_0}\hbar_n(\zeta,w)\varrho_n(x), \tag{3.36}$$

where

$$\hbar_n(\zeta,w)\varrho_n(x) = \sum_{m_1+m_2+m_3+m_4=2}\hbar_{n,m_1m_2m_3m_4}(\zeta)w_1^{m_1}w_2^{m_2}w_3^{m_3}w_4^{m_4},$$

with

$$\hbar_{n,m_1m_2m_3m_4}(\zeta) = \left(\hbar^{(1)}_{n,m_1m_2m_3m_4}(\zeta), \hbar^{(2)}_{n,m_1m_2m_3m_4}(\zeta)\right)^{\mathrm{T}}.$$

We then utilize (3.36) to find

$$\begin{cases} V^2_{2x}(w,0)(\zeta) = \hbar_x(\zeta,w) = -\sum_{n\in\mathbb{N}_0}\left(\frac{n}{\iota}\right)\hbar_n(\zeta,w)\eta_n(x), \\ V^2_{2xx}(w,0)(\zeta) = \hbar_{xx}(\zeta,w) = -\sum_{n\in\mathbb{N}_0}\left(\frac{n}{\iota}\right)^2\hbar_n(\zeta,w)\varrho_n(x). \end{cases}$$

It then follows from (3.30) and (3.36) that

$$\begin{aligned} &\left(D_z g_2^{(1,1)}(w,0,0)\right)V_2^2(w,0) \\ &= Q(0)\begin{pmatrix} \left[D_z\mathcal{F}_2(P(\zeta)w_x+z,0)\mid_{z=0}\left(\sum_{n\in\mathbb{N}_0}\hbar_n(\zeta,w)\varrho_n(x)\right),\varrho_\iota^{(1)}\right] \\ \left[D_z\mathcal{F}_2(P(\zeta)w_x+z,0)\mid_{z=0}\left(\sum_{n\in\mathbb{N}_0}\hbar_n(\zeta,w)\varrho_n(x)\right),\varrho_\iota^{(2)}\right] \end{pmatrix}_{\iota=n_1}^{\iota=n_2}. \end{aligned}$$

It is straightforward to check that by noting (3.33)

$$D_z\mathcal{F}_2(P(\zeta)w_x+z,0)\mid_{z=0}\left(\sum_{n\in\mathbb{N}_0}\hbar_n(\zeta,w)\varrho_n(x)\right) = \mathcal{T}_2\left(P(\zeta)w_x,\sum_{n\in\mathbb{N}_0}\hbar_n(\zeta,w)\varrho_n(x)\right),$$

and

$$\begin{aligned} &\begin{pmatrix} \left[\mathcal{T}_2\left(P(\zeta)w_x,\sum_{n\in\mathbb{N}_0}\hbar_n(\zeta,w)\varrho_n(x)\right),\varrho_\iota^{(1)}\right] \\ \left[\mathcal{T}_2\left(P(\zeta)w_x,\sum_{n\in\mathbb{N}_0}\hbar_n(\zeta,w)\varrho_n(x)\right),\varrho_\iota^{(2)}\right] \end{pmatrix} \\ &= \sum_{n\in\mathbb{N}_0}\beta_{n_1,n,\vartheta}\left(\mathcal{T}_2(p_{n_1}(\zeta)w_1,\hbar_n(\zeta,w)) + \mathcal{T}_2(\overline{p}_{n_1}(\zeta)w_2,\hbar_n(\zeta,w))\right) \\ &\quad + \sum_{n\in\mathbb{N}_0}\beta_{n_2,n,\vartheta}\left(\mathcal{T}_2(p_{n_2}(\zeta)w_3,\hbar_n(\zeta,w)) + \mathcal{T}_2(\overline{p}_{n_2}(\zeta)w_4,\hbar_n(\zeta,w))\right), \end{aligned}$$

where $n = 0,1,2,\ldots,\ \vartheta = n_1, n_2$ and

$$\beta_{n_i,n,\vartheta} = \int_0^{\iota\pi}\varrho_{n_i}(x)\varrho_n(x)\varrho_\vartheta(x)\mathrm{d}x = \begin{cases} \frac{1}{\sqrt{\iota\pi}}, & n=0,\ \vartheta=n_i, \\ \frac{1}{\sqrt{2\iota\pi}}, & n=2n_i,\ \vartheta=n_i, \\ \frac{1}{\sqrt{2\iota\pi}}, & n=n_1+n_2,\ \vartheta=n_{i+(-1)^{i+1}}, \\ \frac{1}{\sqrt{2\iota\pi}}, & n=n_2-n_1,\ \vartheta=n_{i+(-1)^{i+1}}, n_1<n_2, \\ 0, & otherwise. \end{cases}$$

We thus have

$$\begin{aligned} &\left(D_w g_2^{(1,1)}(w,0,0)\right)V_2^2(w,0)(\zeta) \\ &= Q(0)\begin{pmatrix} \sum_{n\in 0,2n_1}\beta_{n_1,n,n_1}\left(\mathcal{T}_2(p_{n_1}(\zeta)w_1,\hbar_n(\zeta,w)) + \mathcal{T}_2(\overline{p}_{n_1}(\zeta)w_2,\hbar_n(\zeta,w))\right) \\ +\sum_{n\in n_1+n_2,n_2-n_1}\beta_{n_2,n,n_1}\left(\mathcal{T}_2(p_{n_2}(\zeta)w_3,\hbar_n(\zeta,w)) + \mathcal{T}_2(\overline{p}_{n_2}(\zeta)w_4,\hbar_n(\zeta,w))\right) \\ \sum_{n\in 0,2n_2}\beta_{n_2,n,n_2}\left(\mathcal{T}_2(p_{n_2}(\zeta)w_3,\hbar_n(\zeta,w)) + \mathcal{T}_2(\overline{p}_{n_2}(\zeta)w_4,\hbar_n(\zeta,w))\right) \\ +\sum_{n\in n_1+n_2,n_2-n_1}\beta_{n_1,n,n_2}\left(\mathcal{T}_2(p_{n_1}(\zeta)w_1,\hbar_n(\zeta,w)) + \mathcal{T}_2(\overline{p}_{n_1}(\zeta)w_2,\hbar_n(\zeta,w))\right) \end{pmatrix}. \end{aligned}$$

Therefore, we obtain

$$\mathrm{Proj}_S\left(D_w g_2^{(1,1)}(w,0,0)V_2^2(w,0)\right) = \begin{pmatrix} \mathcal{B}(\mathcal{E}_{11}w_1^2w_2 + \mathcal{E}_{12}w_1w_2w_3) \\ \mathcal{B}(\mathcal{E}_{31}w_3^2w_4 + \mathcal{E}_{32}w_1w_2w_3) \end{pmatrix}, \tag{3.37}$$

where

$$\begin{aligned} \mathcal{E}_{11} =& \frac{1}{\sqrt{\iota\pi}}q_{n_1}^{\mathrm{T}}(0)\left(\mathcal{T}_2\left(p_{n_1}(\zeta),\hbar_{0,1100}(\zeta)\right) + \mathcal{T}_2\left(\overline{p}_{n_1}(\zeta),\hbar_{0,2000}(\zeta)\right)\right) \\ &+ \frac{1}{\sqrt{2\iota\pi}}q_{n_1}^{\mathrm{T}}(0)\left(\mathcal{T}_2\left(p_{n_1}(\zeta),\hbar_{2n_1,1100}(\zeta)\right) + \mathcal{T}_2\left(\overline{p}_{n_1}(\zeta),\hbar_{2n_1,2000}(\zeta)\right)\right), \\ \mathcal{E}_{12} =& \frac{1}{\sqrt{\iota\pi}}q_{n_1}^{\mathrm{T}}(0)\mathcal{T}_2\left(p_{n_1}(\zeta),\hbar_{0,0011}(\zeta)\right) + \frac{1}{\sqrt{2\iota\pi}}q_{n_1}^{\mathrm{T}}(0)\mathcal{T}_2\left(p_{n_1}(\zeta),\hbar_{2n_1,0011}(\zeta)\right) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\sqrt{2\iota\pi}} q_{n_1}^{\mathrm{T}}(0)\left(\mathcal{T}_2\left(p_{n_2}(\zeta), \hbar_{n_1+n_2,1001}(\zeta)\right) + \mathcal{T}_2\left(\bar{p}_{n_2}(\zeta), \hbar_{n_1+n_2,1010}(\zeta)\right)\right) \\
&+ \gamma_{n_1 n_2} q_{n_1}^{\mathrm{T}}(0)\left(\mathcal{T}_2\left(p_{n_2}(\zeta), \hbar_{n_2-n_1,1001}(\zeta)\right) + \mathcal{T}_2\left(\bar{p}_{n_2}(\zeta), \hbar_{n_2-n_1,1010}(\zeta)\right)\right), \\
\mathcal{E}_{31} =& \frac{1}{\sqrt{\iota\pi}} q_{n_2}^{\mathrm{T}}(0)\left(\mathcal{T}_2\left(p_{n_2}(\zeta), \hbar_{0,0011}(\zeta)\right) + \mathcal{T}_2\left(\bar{p}_{n_2}(\zeta), \hbar_{0,0020}(\zeta)\right)\right) \\
&+ \frac{1}{\sqrt{2\iota\pi}} q_{n_2}^{\mathrm{T}}(0)\left(\mathcal{T}_2\left(p_{n_2}(\zeta), \hbar_{2n_2,0011}(\zeta)\right) + \mathcal{T}_2\left(\bar{p}_{n_2}(\zeta), \hbar_{2n_2,0020}(\zeta)\right)\right), \\
\mathcal{E}_{32} =& \frac{1}{\sqrt{\iota\pi}} q_{n_2}^{\mathrm{T}}(0)\mathcal{T}_2\left(p_{n_2}(\zeta), \hbar_{0,1100}(\zeta)\right) + \frac{1}{\sqrt{2\iota\pi}} q_{n_2}^{\mathrm{T}}(0)\mathcal{T}_2\left(p_{n_2}(\zeta), \hbar_{2n_2,1100}(\zeta)\right) \\
&+ \frac{1}{\sqrt{2\iota\pi}} q_{n_2}^{\mathrm{T}}(0)\left(\mathcal{T}_2\left(p_{n_1}(\zeta), \hbar_{n_1+n_2,0110}(\zeta)\right) + \mathcal{T}_2\left(\bar{p}_{n_1}(\zeta), \hbar_{n_1+n_2,1010}(\zeta)\right)\right) \\
&+ \gamma_{n_1 n_2} q_{n_2}^{\mathrm{T}}(0)\left(\mathcal{T}_2\left(p_{n_1}(\zeta), \hbar_{n_2-n_1,0110}(\zeta)\right) + \mathcal{T}_2\left(\bar{p}_{n_1}(\zeta), \hbar_{n_2-n_1,1010}(\zeta)\right)\right),
\end{aligned}
$$

with

$$
\gamma_{n_1 n_2} = \begin{cases} \frac{1}{\sqrt{2\iota\pi}}, & n_1 < n_2, \\ \frac{1}{\sqrt{\iota\pi}}, & n_1 = n_2. \end{cases}
$$

Procedure 4: The calculation of $\mathrm{Proj}_{\mathcal{S}}\left(\left(D_{z,z_x,z_{xx}} g_2^{(1,2)}(w,0,0)\right)V_2^{(2,\delta)}(w,0)(\zeta)\right)$

We can perform procedure 4 similar to procedure 3. Therefore, we obtain

$$
\mathrm{Proj}_{\mathcal{S}}\left(\left(D_{z,z_x,z_{xx}} g_2^{(1,2)}(w,0,0)\right)V_2^{(2,\delta)}(w,0)(\zeta)\right) = \begin{pmatrix} \mathcal{B}(\mathcal{E}_{11}^d w_1^2 w_2 + \mathcal{E}_{12}^d w_1 w_2 w_3) \\ \mathcal{B}(\mathcal{E}_{31}^d w_3^2 w_4 + \mathcal{E}_{32}^d w_1 w_2 w_3) \end{pmatrix}, \tag{3.38}
$$

where

$$
\begin{aligned}
\mathcal{E}_{11}^d =& -\frac{1}{\sqrt{\iota\pi}}\left(\frac{n_1}{\iota}\right)^2 q_{n_1}^{\mathrm{T}}(0)\left(\mathcal{T}_2^{(\delta,1)}\left(p_{n_1}(\zeta), \hbar_{0,1100}(\zeta)\right) + \mathcal{T}_2^{(\delta,1)}\left(\bar{p}_{n_1}(\zeta), \hbar_{0,2000}(\zeta)\right)\right) \\
&+ \frac{1}{\sqrt{2\iota\pi}} q_{n_1}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{2n_1}^{(1,i)}\mathcal{T}_2^{(\delta,i)}\left(p_{n_1}(\zeta), \hbar_{2n_1,1100}(\zeta)\right) \\
&+ \frac{1}{\sqrt{2\iota\pi}} q_{n_1}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{2n_1}^{(1,i)}\mathcal{T}_2^{(\delta,i)}\left(\bar{p}_{n_1}(\zeta), \hbar_{2n_1,2000}(\zeta)\right), \\
\mathcal{E}_{31}^d =& -\frac{1}{\sqrt{\iota\pi}}\left(\frac{n_2}{\iota}\right)^2 q_{n_2}^{\mathrm{T}}(0)\left(\mathcal{T}_2^{(\delta,1)}\left(p_{n_2}(\zeta), \hbar_{0,0011}(\zeta)\right) + \mathcal{T}_2^{(\delta,1)}\left(\bar{p}_{n_2}(\zeta), \hbar_{0,0020}(\zeta)\right)\right) \\
&+ \frac{1}{\sqrt{2\iota\pi}} q_{n_2}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{2n_2}^{(2,i)}\mathcal{T}_2^{(\delta,i)}\left(p_{n_2}(\zeta), \hbar_{2n_2,0011}(\zeta)\right) \\
&+ \frac{1}{\sqrt{2\iota\pi}} q_{n_2}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{2n_2}^{(2,i)}\mathcal{T}_2^{(\delta,i)}\left(\bar{p}_{n_2}(\zeta), \hbar_{2n_2,0020}(\zeta)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{E}_{12}^d =& -\frac{1}{\sqrt{\iota\pi}}\left(\frac{n_1}{\iota}\right)^2 q_{n_1}^{\mathrm{T}}(0)\mathcal{T}_2^{(\delta,1)}\left(p_{n_1}(\zeta), \hbar_{0,0011}(\zeta)\right) \\
&+ \frac{1}{\sqrt{2\iota\pi}} q_{n_1}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{2n_1}^{(1,i)}\mathcal{T}_2^{(\delta,i)}\left(p_{n_1}(\zeta), \hbar_{2n_1,0011}(\zeta)\right) \\
&+ \frac{1}{\sqrt{2\iota\pi}} q_{n_1}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{n_1+n_2}^{(2,i)}\mathcal{T}_2^{(\delta,i)}\left(p_{n_2}(\zeta), \hbar_{n_1+n_2,1001}(\zeta)\right) \\
&+ \frac{1}{\sqrt{2\iota\pi}} q_{n_1}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{n_1+n_2}^{(2,i)}\mathcal{T}_2^{(\delta,i)}\left(\bar{p}_{n_2}(\zeta), \hbar_{n_1+n_2,1010}(\zeta)\right) \\
&+ \gamma_{n_1 n_2} q_{n_1}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{n_2-n_1}^{(2,i)}\mathcal{T}_2^{(\delta,i)}\left(p_{n_2}(\zeta), \hbar_{n_2-n_1,1001}(\zeta)\right) \\
&+ \gamma_{n_1 n_2} q_{n_1}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{n_2-n_1}^{(2,i)}\mathcal{T}_2^{(\delta,i)}\left(\bar{p}_{n_2}(\zeta), \hbar_{n_2-n_1,1010}(\zeta)\right),
\end{aligned}
$$

$$\begin{aligned}
\mathcal{E}_{32}^{d} =& -\frac{1}{\sqrt{\iota\pi}}\left(\frac{n_2}{\iota}\right)^2 q_{n_2}^{\mathrm{T}}(0)\mathcal{T}_2^{(\delta,1)}\left(p_{n_2}(\zeta),\hbar_{0,1100}(\zeta)\right)\\
&+\frac{1}{\sqrt{2\iota\pi}}q_{n_2}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{2n_2}^{(2,i)}\mathcal{T}_2^{(\delta,i)}\left(p_{n_2}(\zeta),\hbar_{2n_2,1100}(\zeta)\right)\\
&+\frac{1}{\sqrt{2\iota\pi}}q_{n_2}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{n_1+n_2}^{(1,i)}\mathcal{T}_2^{(\delta,i)}\left(p_{n_1}(\zeta),\hbar_{n_1+n_2,0110}(\zeta)\right)\\
&+\frac{1}{\sqrt{2\iota\pi}}q_{n_2}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{n_1+n_2}^{(1,i)}\mathcal{T}_2^{(\delta,i)}\left(\bar{p}_{n_1}(\zeta),\hbar_{n_1+n_2,1010}(\zeta)\right)\\
&+\gamma_{n_1n_2}q_{n_2}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{n_2-n_1}^{(1,i)}\mathcal{T}_2^{(\delta,i)}\left(p_{n_1}(\zeta),\hbar_{n_2-n_1,0110}(\zeta)\right)\\
&+\gamma_{n_1n_2}q_{n_2}^{\mathrm{T}}(0)\sum_{i=1,2,3}\beta_{n_2-n_1}^{(1,i)}\mathcal{T}_2^{(\delta,i)}\left(\bar{p}_{n_1}(\zeta),\hbar_{n_2-n_1,1010}(\zeta)\right),
\end{aligned}$$

with

$$\begin{cases}
\beta_\kappa^{(1,1)} = -\dfrac{n_1^2}{\iota^2},\ \beta_\kappa^{(1,3)} = -\dfrac{\kappa^2}{\iota^2},\ \kappa = 2n_1,\ n_1+n_2,\ n_2-n_1,\\
\beta_\kappa^{(1,2)} = \begin{cases}\frac{n_1\kappa}{\iota^2}, & \kappa = 2n_1,\ n_1+n_2,\\ -\frac{n_1\kappa}{\iota^2}, & \kappa = n_2-n_1,\end{cases}\\
\beta_\kappa^{(2,1)} = -\dfrac{n_2^2}{\iota^2},\ \beta_\kappa^{(2,2)} = \dfrac{n_2\kappa}{\iota^2},\ \beta_\kappa^{(2,3)} = -\dfrac{\kappa^2}{\iota^2}, \kappa = 2n_2,\ n_1+n_2,\ n_2-n_1,
\end{cases}$$

and

$$\begin{cases}
\mathcal{T}_2^{(\delta,1)}(a(\zeta),b(\zeta)) = 2\delta_{12}^H\tau^H\begin{pmatrix} a^{(2)}(-1)b^{(1)}(0)\\ 0\end{pmatrix},\\
\mathcal{T}_2^{(\delta,2)}(a(\zeta),b(\zeta)) = 2\delta_{12}^H\tau^H\begin{pmatrix} a^{(2)}(-1)b^{(1)}(0)\\ 0\end{pmatrix} + 2\delta_{12}^H\tau^H\begin{pmatrix} a^{(1)}(0)b^{(2)}(-1)\\ 0\end{pmatrix},\\
\mathcal{T}_2^{(\delta,3)}(a(\zeta),b(\zeta)) = 2\delta_{12}^H\tau^H\begin{pmatrix} a^{(1)}(0)b^{(2)}(-1)\\ 0\end{pmatrix},
\end{cases}$$

for $a(\zeta) = (a^{(1)}(\zeta),\ a^{(2)}(\zeta))^{\mathrm{T}},\ b(\zeta) = (b^{(1)}(\zeta),\ b^{(2)}(\zeta))^{\mathrm{T}}$.

We finally calculate $R_{m_1m_2m_3m_4}$, $\mathcal{T}_2(P(\zeta)w,z)$ and $\hbar_{n,m_1m_2m_3m_4}$ that are in C_{ij}, $\mathcal{D}_{ij}$, $\mathcal{E}_{ij}$ as well as $\mathcal{E}_{ij}^d$. It follows from (3.12) that

$$\mathfrak{F}_2(\phi,\zeta) = \mathfrak{F}_2(\phi,0) = f_{2000}\phi_1^2(0) + f_{1100}\phi_1(0)\phi_2(0) + f_{0011}\phi_1(-\hat{\sigma})\phi_2(-\hat{\sigma}) + f_{0002}\phi_2^2(-\hat{\sigma}),$$

and

$$\mathfrak{F}_3(\phi,0) = f_{0012}\phi_1(-\hat{\sigma})\phi_2^2(-\hat{\sigma}) + f_{0003}\phi_2^3(-\hat{\sigma}),$$

where $\hat{\sigma} = \frac{\sigma}{\tau^H}$ and

$$f_{2000} = \begin{pmatrix} -\dfrac{2r\tau^H}{K}\\ 0\end{pmatrix},\qquad f_{1100} = \begin{pmatrix} -2b\tau^H\\ 0\end{pmatrix},$$

$$f_{0011} = \begin{pmatrix} 0\\ \dfrac{2\tau^H\beta P_2 \mathrm{e}^{-d\sigma}(P_2+2h)}{(P_2+h)^2}\end{pmatrix},\quad f_{0002} = \begin{pmatrix} 0\\ \dfrac{2\tau^H N_2\beta \mathrm{e}^{-d\sigma}h^2}{(P_2+h)^3}\end{pmatrix},$$

$$f_{0012} = \begin{pmatrix} 0\\ \dfrac{6\tau^H\beta \mathrm{e}^{-d\sigma}h^2}{(P_2+h)^3}\end{pmatrix},\qquad f_{0003} = \begin{pmatrix} 0\\ \dfrac{-6\tau^H\beta \mathrm{e}^{-d\sigma}N_2h^2}{(P_2+h)^4}\end{pmatrix}.$$

We can acquire that

$$\begin{aligned}
R_{2000} =& f_{0002}\left(p_{n_1}^{(2)}(-\tilde{\sigma})\right)^2 + f_{0011}p_{n_1}^{(1)}(-\tilde{\sigma})p_{n_1}^{(2)}(-\tilde{\sigma}) + f_{1100}p_{n_1}^{(1)}(0)p_{n_1}^{(2)}(0) + f_{2000}\left(p_{n_1}^{(1)}(0)\right)^2,\\
R_{0200} =& f_{0002}\left(\bar{p}_{n_1}^{(2)}(-\tilde{\sigma})\right)^2 + f_{0011}\bar{p}_{n_1}^{(1)}(-\tilde{\sigma})\bar{p}_{n_1}^{(2)}(-\tilde{\sigma}) + f_{1100}\bar{p}_{n_1}^{(1)}(0)\bar{p}_{n_1}^{(2)}(0) + f_{2000}\left(\bar{p}_{n_1}^{(1)}(0)\right)^2,\\
R_{1100} =& 2f_{0002}p_{n_1}^{(2)}(-\tilde{\sigma})\bar{p}_{n_1}^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_1}^{(1)}(-\tilde{\sigma})\bar{p}_{n_1}^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_1}^{(2)}(-\tilde{\sigma})\bar{p}_{n_1}^{(1)}(-\tilde{\sigma})\\
&+ f_{1100}p_{n_1}^{(2)}(0)\bar{p}_{n_1}^{(1)}(0) + 2f_{2000}p_{n_1}^{(1)}(0)\bar{p}_{n_1}^{(1)}(0) + f_{1100}p_{n_1}^{(1)}(0)\bar{p}_{n_1}^{(2)}(0),
\end{aligned}$$

$$
\begin{aligned}
R_{1010} =&2f_{0002}p_{n_1}^{(2)}(-\tilde{\sigma})p_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_1}^{(1)}(-\tilde{\sigma})p_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_1}^{(2)}(-\tilde{\sigma})p_{n_2}^{(1)}(-\tilde{\sigma})\\
&+ f_{1100}p_{n_1}^{(1)}(0)p_{n_2}^{(2)}(0) + f_{1100}p_{n_1}^{(2)}(0)p_{n_2}^{(1)}(0) + 2f_{2000}p_{n_1}^{(1)}(0)p_{n_2}^{(1)}(0),\\
R_{0101} =&2f_{0002}\bar{p}_{n_1}^{(2)}(-\tilde{\sigma})\bar{p}_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0011}\bar{p}_{n_1}^{(1)}(-\tilde{\sigma})\bar{p}_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0011}\bar{p}_{n_1}^{(2)}(-\tilde{\sigma})\bar{p}_{n_2}^{(1)}(-\tilde{\sigma})\\
&+ f_{1100}\bar{p}_{n_1}^{(1)}(0)\bar{p}_{n_2}^{(2)}(0) + f_{1100}\bar{p}_{n_1}^{(2)}(0)\bar{p}_{n_2}^{(1)}(0) + 2f_{2000}\bar{p}_{n_1}^{(1)}(0)\bar{p}_{n_2}^{(1)}(0),\\
R_{1001} =&2f_{0002}p_{n_1}^{(2)}(-\tilde{\sigma})\overline{p}_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_1}^{(1)}(-\tilde{\sigma})\bar{p}_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_1}^{(2)}(-\tilde{\sigma})\bar{p}_{n_2}^{(1)}(-\tilde{\sigma})\\
&+ f_{1100}p_{n_1}^{(1)}(0)\bar{p}_{n_2}^{(2)}(0) + 2f_{2000}p_{n_1}^{(1)}(0)\bar{p}_{n_2}^{(1)}(0) + f_{1100}p_{n_1}^{(2)}(0)\bar{p}_{n_2}^{(1)}(0),\\
R_{0110} =&2f_{0002}p_{n_2}^{(2)}(-\tilde{\sigma})\overline{p}_{n_1}^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_2}^{(1)}(-\tilde{\sigma})\bar{p}_{n_1}^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_1}^{(1)}(-\tilde{\sigma})\\
&+ f_{1100}p_{n_2}^{(1)}(0)\bar{p}_{n_1}^{(2)}(0) + 2f_{2000}p_{n_2}^{(1)}(0)\bar{p}_{n_1}^{(1)}(0) + f_{1100}p_{n_2}^{(2)}(0)\bar{p}_{n_1}^{(1)}(0),\\
R_{0020} =&f_{0002}\left(p_{n_2}^{(2)}(-\tilde{\sigma})\right)^2 + f_{0011}p_{n_2}^{(1)}(-\tilde{\sigma})p_{n_2}^{(2)}(-\tilde{\sigma}) + f_{1100}p_{n_2}^{(1)}(0)p_{n_2}^{(2)}(0) + f_{2000}\left(p_{n_2}^{(1)}(0)\right)^2,\\
R_{0011} =&2f_{0002}p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_2}^{(1)}(-\tilde{\sigma})\bar{p}_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_2}^{(1)}(-\tilde{\sigma})\\
&+ f_{1100}p_{n_2}^{(2)}(0)\bar{p}_{n_2}^{(1)}(0) + 2f_{2000}p_{n_2}^{(1)}(0)\bar{p}_{n_2}^{(1)}(0) + f_{1100}p_{n_2}^{(1)}(0)\bar{p}_{n_2}^{(2)}(0),\\
R_{2100} =&\Big(3f_{0003}p_{n_1}^{(2)}(-\tilde{\sigma})\bar{p}_{n_1}^{(2)}(-\tilde{\sigma}) + 2f_{0012}p_{n_1}^{(1)}(-\tilde{\sigma})\bar{p}_{n_1}^{(2)}(-\tilde{\sigma}) + f_{0012}p_{n_1}^{(2)}(-\tilde{\sigma})\bar{p}_{n_1}^{(1)}(-\tilde{\sigma})\Big)p_{n_1}^{(2)}(-\tilde{\sigma}),\\
R_{0021} =&\Big(3f_{0003}p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_2}^{(2)}(-\tilde{\sigma}) + 2f_{0012}p_{n_2}^{(1)}(-\tilde{\sigma})\bar{p}_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0012}p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_2}^{(1)}(-\tilde{\sigma})\Big)p_{n_2}^{(2)}(-\tilde{\sigma}),\\
R_{1011} =&2\Big(\ 3f_{0003}p_{n_1}^{(2)}(-\tilde{\sigma})p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0012}p_{n_1}^{(1)}(-\tilde{\sigma})p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_2}^{(2)}(-\tilde{\sigma})\\
&+ f_{0012}p_{n_1}^{(2)}(-\tilde{\sigma})p_{n_2}^{(1)}(-\tilde{\sigma})\bar{p}_{n_2}^{(2)}(-\tilde{\sigma}) + f_{0012}p_{n_1}^{(2)}(-\tilde{\sigma})p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_2}^{(1)}(-\tilde{\sigma})\ \Big),\\
R_{1110} =&2\Big(\ 3f_{0003}p_{n_1}^{(2)}(-\tilde{\sigma})p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_1}^{(2)}(-\tilde{\sigma}) + f_{0012}p_{n_1}^{(1)}(-\tilde{\sigma})p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_1}^{(2)}(-\tilde{\sigma})\\
&+ f_{0012}p_{n_1}^{(2)}(-\tilde{\sigma})p_{n_2}^{(1)}(-\tilde{\sigma})\bar{p}_{n_1}^{(2)}(-\tilde{\sigma}) + f_{0012}p_{n_1}^{(2)}(-\tilde{\sigma})p_{n_2}^{(2)}(-\tilde{\sigma})\bar{p}_{n_1}^{(1)}(-\tilde{\sigma})\ \Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{T}_2(p_{n_1}(\zeta), z(\zeta)) =&2f_{0002}p_{n_1}^{(2)}(-\tilde{\sigma})z^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_1}^{(1)}(-\tilde{\sigma})z^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_1}^{(2)}(-\tilde{\sigma})z^{(1)}(-\tilde{\sigma})\\
&+\ f_{1100}p_{n_1}^{(1)}(0)z^{(2)}(0) + f_{1100}p_{n_1}^{(2)}(0)z^{(1)}(0) + 2f_{2000}p_{n_1}^{(1)}(0)z^{(1)}(0),\\
\mathcal{T}_2(\bar{p}_{n_1}(\zeta), z(\zeta)) =&2f_{0002}\bar{p}_{n_1}^{(2)}(-\tilde{\sigma})z^{(2)}(-\tilde{\sigma}) + f_{0011}\bar{p}_{n_1}^{(1)}(-\tilde{\sigma})z^{(2)}(-\tilde{\sigma}) + f_{0011}\bar{p}_{n_1}^{(2)}(-\tilde{\sigma})z^{(1)}(-\tilde{\sigma})\\
&+\ f_{1100}\bar{p}_{n_1}^{(1)}(0)z^{(2)}(0) + f_{1100}\bar{p}_{n_1}^{(2)}(0)z^{(1)}(0) + 2f_{2000}\bar{p}_{n_1}^{(1)}(0)z^{(1)}(0),\\
\mathcal{T}_2(p_{n_2}(\zeta), z(\zeta)) =&2f_{0002}p_{n_2}^{(2)}(-\tilde{\sigma})z^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_2}^{(1)}(-\tilde{\sigma})z^{(2)}(-\tilde{\sigma}) + f_{0011}p_{n_2}^{(2)}(-\tilde{\sigma})z^{(1)}(-\tilde{\sigma})\\
&+\ f_{1100}p_{n_2}^{(1)}(0)z^{(2)}(0) + f_{1100}p_{n_2}^{(2)}(0)z^{(1)}(0) + 2f_{2000}p_{n_2}^{(1)}(0)z^{(1)}(0),\\
\mathcal{T}_2(\bar{p}_{n_2}(\zeta), z(\zeta)) =&2f_{0002}\bar{p}_{n_2}^{(2)}(-\tilde{\sigma})z^{(2)}(-\tilde{\sigma}) + f_{0011}\bar{p}_{n_2}^{(1)}(-\tilde{\sigma})z^{(2)}(-\tilde{\sigma}) + f_{0011}\bar{p}_{n_2}^{(2)}(-\tilde{\sigma})z^{(1)}(-\tilde{\sigma})\\
&+\ f_{1100}\bar{p}_{n_2}^{(1)}(0)z^{(2)}(0) + f_{1100}\bar{p}_{n_2}^{(2)}(0)z^{(1)}(0) + 2f_{2000}\bar{p}_{n_2}^{(1)}(0)z^{(1)}(0)
\end{aligned}
$$

by noting that

$$
\begin{aligned}
\phi(\zeta) + z =&P(\zeta)w_x + z = p_{n_1}(\zeta)w_1\varrho_{n_1}(x) + \overline{p}_{n_1}(\zeta)w_2\varrho_{n_1}(x)\\
&+\ p_{n_2}(\zeta)w_3\varrho_{n_2}(x) + \overline{p}_{n_2}(\zeta)w_4\varrho_{n_2}(x) + z\\
=&\begin{pmatrix} p_{n_1}^{(1)}(\zeta)w_1\varrho_{n_1}(x) + \overline{p}_{n_1}^{(1)}(\zeta)w_2\varrho_{n_1}(x) + p_{n_2}^{(1)}(\zeta)w_3\varrho_{n_2}(x) + \overline{p}_{n_2}^{(1)}(\zeta)w_4\varrho_{n_2}(x) + z^{(1)}\\ p_{n_1}^{(2)}(\zeta)w_1\varrho_{n_1}(x) + \overline{p}_{n_1}^{(2)}(\zeta)w_2\varrho_{n_1}(x) + p_{n_2}^{(2)}(\zeta)w_3\varrho_{n_2}(x) + \overline{p}_{n_2}^{(2)}(\zeta)w_4\varrho_{n_2}(x) + z^{(2)} \end{pmatrix}.
\end{aligned}
$$

Next, we will present the formulas of $\hbar_{n,m_1m_2m_3m_4}$. According to [36], we do some algebra yielding

$$
\begin{aligned}
&\begin{pmatrix} \left[\mathcal{M}_2^2\hbar_n(\zeta,w)\varrho_n(x), \varrho_\iota^{(1)}\right]\\ \left[\mathcal{M}_2^2\hbar_n(\zeta,w)\varrho_n(x), \varrho_\iota^{(2)}\right] \end{pmatrix}\\
&= 2i\varpi_{n_1}^H\left(\hbar_{n,2000}(\zeta)w_1^2 - \hbar_{n,0200}(\zeta)w_2^2\right) + 2i\varpi_{n_2}^H\left(\hbar_{n,0020}(\zeta)w_3^2 - \hbar_{n,0002}(\zeta)w_4^2\right)\\
&\quad + i(\varpi_{n_1}^H + \varpi_{n_2}^H)\hbar_{n,1010}(\zeta)w_1w_3 + i(\varpi_{n_1}^H - \varpi_{n_2}^H)\hbar_{n,1001}(\zeta)w_1w_4
\end{aligned}
$$

$$
\begin{aligned}
&- i(\varpi_{n_1}^H - \varpi_{n_2}^H)\hbar_{n,0110}(\zeta)w_1w_3 - i(\varpi_{n_1}^H + \varpi_{n_2}^H)\hbar_{n,0101}(\zeta)w_2w_4 \\
&- \left(\dot{\hbar}_n(\zeta,w) + Y_0(\zeta)(\mathfrak{L}_0(\hbar_n(\zeta,w)) - \dot{\hbar}_n(0,w))\right),
\end{aligned}
$$

where

$$
\mathfrak{L}_0(\hbar_n(\zeta,w)) = -\tau^H \frac{n^2}{\imath^2}\left(D_1\hbar_n(0,w) + D_2^H\hbar_n(-1,w)\right) + \tau^H\left(A\hbar_n(0,w) + B\hbar_n(-\frac{\sigma}{\tau^H},w)\right).
$$

It follows from [40] that the explicit of $\hbar_n(\zeta,w)$ can be obtained for the cases $n_1 \neq n_2$ and $n_1 = n_2$.

Case 1: $n_1 \neq n_2$

$$
\begin{cases}
\hbar_{0,0200}(\zeta) = \frac{1}{\sqrt{\imath\pi}}\left(\Delta_0(2i\varpi_{n_1}^H)\right)^{-1} R_{2000}\mathrm{e}^{2i\varpi_{n_1}^H\zeta}, \\
\hbar_{0,0020}(\zeta) = \frac{1}{\sqrt{\imath\pi}}\left(\Delta_0(2i\varpi_{n_2}^H)\right)^{-1} R_{0020}\mathrm{e}^{2i\varpi_{n_2}^H\zeta}, \\
\hbar_{0,1100}(\zeta) = \frac{1}{\sqrt{\imath\pi}}\left(\Delta_0(0)\right)^{-1} R_{1100}, \\
\hbar_{0,0011}(\zeta) = \frac{1}{\sqrt{\imath\pi}}\left(\Delta_0(0)\right)^{-1} R_{0011},
\end{cases}
$$

$$
\begin{cases}
\hbar_{2n_2,0020}(\zeta) = \frac{1}{\sqrt{2\imath\pi}}\left(\Delta_{2n_2}(2i\varpi_{n_2}^H)\right)^{-1} \widetilde{R}_{0020}\mathrm{e}^{2i\varpi_{n_2}^H\zeta}, \\
\hbar_{2n_2,1100}(\zeta) = (0,0)^{\mathrm{T}}, \\
\hbar_{2n_2,0011}(\zeta) = \frac{1}{\sqrt{2\imath\pi}}\left(\Delta_{2n_2}(0)\right)^{-1} R_{0011},
\end{cases}
$$

$$
\begin{cases}
\hbar_{n_1+n_2,1010}(\zeta) = \dfrac{1}{\sqrt{2\imath\pi}}\left(\Delta_{n_1+n_2}(i(\varpi_{n_1}^H + \varpi_{n_2}^H))\right)^{-1} \widehat{R}_{1010}\mathrm{e}^{i(\varpi_{n_1}^H+\varpi_{n_2}^H)\zeta}, \\
\hbar_{n_1+n_2,1001}(\zeta) = \dfrac{1}{\sqrt{2\imath\pi}}\left(\Delta_{n_1+n_2}(i(\varpi_{n_1}^H - \varpi_{n_2}^H))\right)^{-1} \widehat{R}_{1001}\mathrm{e}^{i(\varpi_{n_1}^H-\varpi_{n_2}^H)\zeta}, \\
\hbar_{n_1+n_2,0110}(\zeta) = \dfrac{1}{\sqrt{2\imath\pi}}\left(\Delta_{n_1+n_2}(i(\varpi_{n_2}^H - \varpi_{n_1}^H))\right)^{-1} \widehat{R}_{0110}\mathrm{e}^{i(\varpi_{n_2}^H-\varpi_{n_1}^H)\zeta},
\end{cases}
$$

$$
\begin{cases}
\hbar_{2n_1,2000}(\zeta) = \dfrac{1}{\sqrt{2\imath\pi}}\left(\Delta_{2n_1}(2i\varpi_{n_1}^H)\right)^{-1} \widetilde{R}_{2000}\mathrm{e}^{2i\varpi_{n_1}^H\zeta}, \\
\hbar_{2n_1,1100}(\zeta) = \dfrac{1}{\sqrt{2\imath\pi}}\left(\Delta_{2n_1}(0)\right)^{-1} \widetilde{R}_{1100}, \qquad \text{for } n_2 \neq 2n_1, \\
\hbar_{2n_1,0011}(\zeta) = (0,0)^{\mathrm{T}},
\end{cases}
$$

$$
\begin{cases}
\hbar_{2n_1,2000}(\zeta) = \dfrac{1}{\sqrt{2\imath\pi}}\left(\Delta_{2n_1}(i\varpi_{n_1}^H)\right)^{-1}\Big(\widetilde{R}_{2000} - M_1\Delta_{2n_1}(2i\varpi_{n_1}^H)p_{n_2}(0), \\
\qquad - M_2\Delta_{2n_1}(-i\varpi_{n_1}^H)\overline{p}_{n_2}(0)\Big)\mathrm{e}^{2i\varpi_{n_1}^H\zeta} + \dfrac{1}{\sqrt{2\imath\pi}}M_1p_{n_2}(\zeta) \\
\qquad + \dfrac{1}{\sqrt{2\imath\pi}}M_2\overline{p}_{n_2}(\zeta), \\
\hbar_{2n_1,1100}(\zeta) = \dfrac{1}{\sqrt{2\imath\pi}}\left(\Delta_{2n_1}(0)\right)^{-1}\Big(\widetilde{R}_{1100} - M_3\Delta_{2n_1}(i\varpi_{n_2}^H)p_{n_2}(0), \qquad \text{for } n_2 = 2n_1, \\
\qquad - M_4\Delta_{2n_1}(-i\varpi_{n_2}^H)\overline{p}_{n_2}(0)\Big) + \dfrac{1}{\sqrt{2\imath\pi}}M_3p_{n_2}(\zeta) \\
\qquad + \dfrac{1}{\sqrt{2\imath\pi}}M_4\overline{p}_{n_2}(\zeta), \\
\hbar_{2n_1,0011}(\zeta) = (0,0)^{\mathrm{T}},
\end{cases}
$$

$$\begin{cases} \hbar_{n_2-n_1,1010}(\zeta) = \frac{1}{\sqrt{2\iota\pi}} \left(\Delta_{n_2-n_1}(i(\varpi_{n_1}^H + \varpi_{n_2}^H))\right)^{-1} \widetilde{R}_{1010} \mathrm{e}^{i(\varpi_{n_1}^H+\varpi_{n_2}^H)\zeta}, \\ \hbar_{n_2-n_1,1001}(\zeta) = \frac{1}{\sqrt{2\iota\pi}} \left(\Delta_{n_2-n_1}(i(\varpi_{n_1}^H - \varpi_{n_2}^H))\right)^{-1} \widetilde{R}_{1001} \mathrm{e}^{i(\varpi_{n_1}^H-\varpi_{n_2}^H)\zeta}, \text{ for } n_2 \neq 2n_1, \\ \hbar_{n_2-n_1,0110}(\zeta) = \frac{1}{\sqrt{2\iota\pi}} \left(\Delta_{n_2-n_1}(i(\varpi_{n_2}^H - \varpi_{n_1}^H))\right)^{-1} \widetilde{R}_{0110} \mathrm{e}^{i(\varpi_{n_2}^H-\varpi_{n_1}^H)\zeta}, \end{cases}$$

$$\begin{cases} \hbar_{n_2-n_1,1010}(\zeta) = \frac{1}{\sqrt{2\iota\pi}} \left(\Delta_{n_2-n_1}(i(\varpi_{n_1}^H + \varpi_{n_2}^H))\right)^{-1} \Big(\widetilde{R}_{1010} - M_5\Delta_{n_2-n_1}(i\varpi_{n_1}^H)p_{n_1}(0) \\ \qquad - M_6\Delta_{n_2-n_1}(-i\varpi_{n_1}^H)\overline{p}_{n_1}(0) \Big) \mathrm{e}^{i(\varpi_{n_1}^H+\varpi_{n_2}^H)\zeta} + \frac{1}{\sqrt{\iota\pi}} M_5 p_{n_1}(\zeta) \\ \qquad + \frac{1}{\sqrt{\iota\pi}} M_6 \overline{p}_{n_1}(\zeta), \\ \hbar_{n_2-n_1,1001}(\zeta) = \frac{1}{\sqrt{2\iota\pi}} \left(\Delta_{n_2-n_1}(i(\varpi_{n_1}^H - \varpi_{n_2}^H))\right)^{-1} \Big(\widetilde{R}_{1001} - M_7\Delta_{n_2-n_1}(i\varpi_{n_1}^H)p_{n_1}(0) \\ \qquad - M_8\Delta_{n_2-n_1}(-i\varpi_{n_1}^H)\overline{p}_{n_1}(0) \Big) \mathrm{e}^{i(\varpi_{n_1}^H-\varpi_{n_2}^H)\zeta} + \frac{1}{\sqrt{\iota\pi}} M_7 p_{n_1}(\zeta) \\ \qquad + \frac{1}{\sqrt{\iota\pi}} M_8 \overline{p}_{n_1}(\zeta), \quad \text{for } n_2 = 2n_1, \\ \hbar_{n_2-n_1,0110}(\zeta) = \frac{1}{\sqrt{2\iota\pi}} \left(\Delta_{n_2-n_1}(i(\varpi_{n_2}^H - \varpi_{n_1}^H))\right)^{-1} \Big(\widetilde{R}_{0110} - M_9\Delta_{n_2-n_1}(i\varpi_{n_1}^H)p_{n_1}(0) \\ \qquad - M_{10}\Delta_{n_2-n_1}(-i\varpi_{n_1}^H)\overline{p}_{n_1}(0) \Big) \mathrm{e}^{i(\varpi_{n_2}^H-\varpi_{n_1}^H)\zeta} + \frac{1}{\sqrt{2\iota\pi}} M_9 p_{n_1}(\zeta) \\ \qquad + \frac{1}{\sqrt{2\iota\pi}} M_{10} \overline{p}_{n_1}(\zeta), \end{cases}$$

where

$$\begin{aligned} M_1 &= \frac{1}{i(\varpi_{n_2}^H - 2\varpi_{n_1}^H)} q_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{2000}, \ M_2 = -\frac{1}{i(\varpi_{n_2}^H + 2\varpi_{n_1}^H)} \overline{q}_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{2000}, \\ M_3 &= \frac{1}{i\varpi_{n_2}^H} q_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{1100}, \ M_4 = -\frac{1}{i\varpi_{n_2}^H} \overline{q}_{n_2}^{\mathrm{T}}(0)\widetilde{R}_{1100}, \\ M_5 &= -\frac{1}{i\varpi_{n_2}^H} q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1010}, \ M_6 = -\frac{1}{i(2\varpi_{n_1}^H + \varpi_{n_2}^H)} \overline{q}_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1010}, \\ M_7 &= \frac{1}{i\varpi_{n_2}^H} q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1001}, \ M_8 = -\frac{1}{i(2\varpi_{n_1}^H - \varpi_{n_2}^H)} \overline{q}_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{1001}, \\ M_9 &= \frac{1}{i(2\varpi_{n_1}^H - \varpi_{n_2}^H)} q_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{0110}, \ M_{10} = -\frac{1}{i\varpi_{n_2}^H} \overline{q}_{n_1}^{\mathrm{T}}(0)\widetilde{R}_{0110}, \end{aligned}$$

and

$$\begin{cases} \widetilde{R}_{m_1m_2m_3m_4} = R_{m_1m_2m_3m_4} - \frac{n_2^2}{\iota^2}\left(R^{(\delta,1)}_{m_1m_2m_3m_4} + R^{(\delta,2)}_{m_1m_2m_3m_4}\right), \\ m_3, m_4 = 0,1,2, \ m_3+m_4 = 4, \ m_1 = m_2 = 0, \end{cases}$$

$$\begin{cases} \widehat{R}_{m_1m_2m_3m_4} = R_{m_1m_2m_3m_4} - \frac{n_1n_2}{\iota^2} R^{(\delta,1)}_{m_1m_2m_3m_4} - \frac{n_1^2}{\iota^2} R^{(\delta,2)}_{m_1m_2m_3m_4} - \frac{n_2^2}{\iota^2} R^{(\delta,3)}_{m_1m_2m_3m_4}, \\ m_1, m_2, m_3, m_4 = 0,1, \ m_1+m_2 = 1, \ m_3+m_4 = 1. \end{cases}$$

Case 2: $n_1 = n_2$

In this case, we have

$$\begin{cases} \hbar_{0,0200}(\zeta) = \dfrac{1}{\sqrt{\iota\pi}} \left(\Delta_0(2i\varpi_{n_1}^H)\right)^{-1} R_{2000} e^{2i\varpi_{n_1}^H \zeta}, \\ \hbar_{0,0020}(\zeta) = \dfrac{1}{\sqrt{\iota\pi}} \left(\Delta_0(2i\varpi_{n_2}^H)\right)^{-1} R_{0020} e^{2i\varpi_{n_2}^H \zeta}, \\ \hbar_{0,1100}(\zeta) = \dfrac{1}{\sqrt{\iota\pi}} \left(\Delta_0(0)\right)^{-1} R_{1100}, \\ \hbar_{0,0011}(\zeta) = \dfrac{1}{\sqrt{\iota\pi}} \left(\Delta_0(0)\right)^{-1} R_{0011}, \\ \hbar_{0,1010}(\zeta) = \dfrac{1}{\sqrt{\iota\pi}} \left(\Delta_0(i(\varpi_{n_1}^H + \varpi_{n_2}^H))\right)^{-1} \check{R}_{1010} e^{i(\varpi_{n_1}^H + \varpi_{n_2}^H)\zeta}, \\ \hbar_{0,1001}(\zeta) = \dfrac{1}{\sqrt{\iota\pi}} \left(\Delta_0(i(\varpi_{n_1}^H - \varpi_{n_2}^H))\right)^{-1} \check{R}_{1001} e^{i(\varpi_{n_1}^H - \varpi_{n_2}^H)\zeta}, \\ \hbar_{0,0110}(\zeta) = \dfrac{1}{\sqrt{\iota\pi}} \left(\Delta_0(i(\varpi_{n_2}^H - \varpi_{n_1}^H))\right)^{-1} \check{R}_{0110} e^{i(\varpi_{n_2}^H - \varpi_{n_1}^H)\zeta}, \end{cases}$$

and

$$\begin{cases} \hbar_{2n_1,2000}(\zeta) = \dfrac{1}{\sqrt{2\iota\pi}} \left(\Delta_{2n_1}(2i\varpi_{n_1}^H)\right)^{-1} \widetilde{R}_{2000} e^{2i\varpi_{n_1}^H \zeta}, \\ \hbar_{2n_1,0020}(\zeta) = \dfrac{1}{\sqrt{2\iota\pi}} \left(\Delta_{2n_1}(2i\varpi_{n_2}^H)\right)^{-1} \widetilde{R}_{0020} e^{2i\varpi_{n_2}^H \zeta}, \\ \hbar_{2n_1,1100}(\zeta) = \dfrac{1}{\sqrt{2\iota\pi}} \left(\Delta_{2n_1}(0)\right)^{-1} \widetilde{R}_{1100}, \\ \hbar_{2n_1,0011}(\zeta) = \dfrac{1}{\sqrt{2\iota\pi}} \left(\Delta_{2n_1}(0)\right)^{-1} \widetilde{R}_{0011}, \\ \hbar_{2n_1,1010}(\zeta) = \dfrac{1}{\sqrt{2\iota\pi}} \left(\Delta_{2n_1}(i(\varpi_{n_1}^H + \varpi_{n_2}^H))\right)^{-1} \widehat{R}_{1010} e^{i(\varpi_{n_1}^H + \varpi_{n_2}^H)\zeta}, \\ \hbar_{2n_1,1001}(\zeta) = \dfrac{1}{\sqrt{2\iota\pi}} \left(\Delta_{2n_1}(i(\varpi_{n_1}^H - \varpi_{n_2}^H))\right)^{-1} \widehat{R}_{1001} e^{i(\varpi_{n_1}^H - \varpi_{n_2}^H)\zeta}, \\ \hbar_{2n_1,0110}(\zeta) = \dfrac{1}{\sqrt{2\iota\pi}} \left(\Delta_{2n_1}(i(\varpi_{n_2}^H - \varpi_{n_1}^H))\right)^{-1} \widehat{R}_{0110} e^{i(\varpi_{n_2}^H - \varpi_{n_1}^H)\zeta}, \end{cases}$$

where

$$\begin{cases} \check{R}_{m_1m_2m_3m_4} = R_{m_1m_2m_3m_4} + \dfrac{n_1^2}{\iota^2}\left(R^{(\delta,1)}_{m_1m_2m_3m_4} - R^{(\delta,2)}_{m_1m_2m_3m_4} - R^{(\delta,3)}_{m_1m_2m_3m_4}\right), \\ m_1, m_2, m_3, m_4 = 0, 1, \ m_1 + m_2 = 1, \ m_3 + m_4 = 1. \end{cases}$$

By noting that $2n_2 = n_1 + n_2 = 2n_1$ and $n_2 - n_1 = 0$, we hence have $\hbar_{2n_2,m_1m_2m_3m_4}(\zeta) = \hbar_{n_1+n_2,m_1m_2m_3m_4}(\zeta) = \hbar_{2n_1,m_1m_2m_3m_4}(\zeta)$ and $\hbar_{n_2-n_1,m_1m_2m_3m_4}(\zeta) = \hbar_{0,m_1m_2m_3m_4}(\zeta)$.

3.3. Normal form of double Hopf bifurcation truncated to third terms

In the final of this section, we will give the expression of normal form truncated to third terms. Define

$$\mathbb{B}_{11} = C_{11} + \frac{3}{2}\left(\mathcal{D}_{11} + \mathcal{E}_{11} + \mathcal{E}^d_{11}\right), \mathbb{B}_{12} = C_{12} + \frac{3}{2}\left(\mathcal{D}_{12} + \mathcal{E}_{12} + \mathcal{E}^d_{12}\right),$$
$$\mathbb{B}_{31} = C_{31} + \frac{3}{2}\left(\mathcal{D}_{31} + \mathcal{E}_{31} + \mathcal{E}^d_{31}\right), \mathbb{B}_{32} = C_{32} + \frac{3}{2}\left(\mathcal{D}_{32} + \mathcal{E}_{32} + \mathcal{E}^d_{32}\right),$$

In combination with (3.20), (3.29), (3.32), (3.35), (3.37) and (3.38), we obtain the following normal form truncated to third terms:

$$\dot{w} = \Lambda w + \frac{1}{2}\begin{pmatrix} \mathcal{B}\left((\mathfrak{B}_{11A}\xi_1 + \mathfrak{B}_{21A}\xi_2)w_1\right) \\ \mathcal{B}\left((\mathfrak{B}_{13A}\xi_1 + \mathfrak{B}_{23A}\xi_2)w_3\right) \end{pmatrix} + \frac{1}{6}\begin{pmatrix} \mathcal{B}(\mathbb{B}_{11}w_1^2w_2 + \mathbb{B}_{12}w_1w_3w_4) \\ \mathcal{B}(\mathbb{B}_{31}w_3^2w_4 + \mathbb{B}_{32}w_1w_2w_3) \end{pmatrix}. \tag{3.39}$$

Similarly to the arguments in [40], we obtain the amplitude equations for (3.39) as below:

$$\begin{cases} \frac{\mathrm{d}r_1}{\mathrm{d}t} = (c_1 + a_{11}r_1^2 + a_{12}r_2^2)r_1, \\ \frac{\mathrm{d}r_2}{\mathrm{d}t} = (c_2 + a_{21}r_1^2 + a_{22}r_2^2)r_2, \end{cases} \tag{3.40}$$

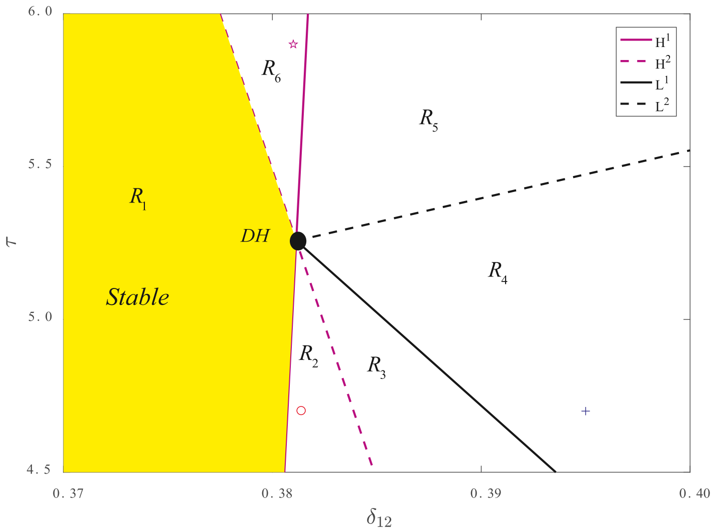

Fig. 2. Complete bifurcation region around the double Hopf bifurcation point for model (4.1).

where

$$c_1 = \frac{1}{2}\Re\left(\mathfrak{B}_{11A}\xi_1 + \mathfrak{B}_{21A}\xi_2\right),\ c_2 = \frac{1}{2}\Re\left(\mathfrak{B}_{13A}\xi_1 + \mathfrak{B}_{23A}\xi_2\right),$$

and

$$a_{11} = \frac{1}{6}\Re(\mathbb{B}_{11}),\ a_{12} = \frac{1}{6}\Re(\mathbb{B}_{12}),\ a_{21} = \frac{1}{6}\Re(\mathbb{B}_{31}),\ a_{22} = \frac{1}{6}\Re(\mathbb{B}_{32}).$$

Remark 3.1. It follows from [43] that there exist no periodic orbits bifurcating from the nontrivial equilibrium of the amplitude Eqs. (3.40) provided $a_{11}a_{22} > 0$. The dynamics near double Hopf bifurcation points can be completely determined by the amplitude Eqs. (3.40) in this case.

4. Numerical example

In this section, we will present a numerical example to explicate the correctness of our obtained theoretical results. We choose the same parameters as provided in [30]. Then model (1.1) can be rewritten as

$$\begin{cases} \dfrac{\partial N}{\partial t} = 0.1N_{xx} + \delta_{12}\left(NP_x(x,t-\tau)\right)_x + 2N\left(1-\dfrac{N}{4}\right) - 1.2NP,\ 0<x<2\pi,\ t>0, \\ \dfrac{\partial P}{\partial t} = 0.1P_{xx} + \dfrac{0.96N_{3.1}P_{3.1}^2}{0.4+P_{3.1}}e^{-0.62} - 0.5P,\ 0<x<2\pi,\ t>0, \\ N_x(x,t) = P_x(x,t) = 0,\ x=0,2\pi,\ t\geq 0. \end{cases} \tag{4.1}$$

According to Theorem 2.1, we obtain the crossing curves in the (δ_{12},τ) plane as is shown in Fig. 1. We thus obtain the critical values for the double Hopf bifurcation point $n_1 = 4$, $n_2 = 5$, $\varpi_{n_1} \approx 0.3044$, $\varpi_{n_2} \approx 0.3059$, $\delta_{12}^H \approx 0.3811$, $\tau^H \approx 5.2576$.

According to procedures in Section 3, we can calculate $c_1 \approx -0.0009166\xi_1 + 1.2172312\xi_2$, $c_2 \approx 0.0064241\xi_1 + 1.3039521\xi_2$, $a_{11} \approx -0.0700132$, $a_{12} \approx -0.1021972$, $a_{21} \approx -0.0817567$, $a_{22} \approx -0.0732399$. We can thus classify the dynamics of model (4.1) into 6 categories in the (δ_{12},τ) plane as is displayed in Fig. 2 by the following curves:

$$\begin{aligned} &H1:\ \tau-\tau^H = 1327.9942(\delta_{12}-\delta_{12}^H), \\ &H2:\ \tau-\tau^H = -202.9793(\delta_{12}-\delta_{12}^H), \\ &L1:\ \tau-\tau^H = -60.9552(\delta_{12}-\delta_{12}^H),\ \delta_{12} > \delta_{12}^H, \\ &L2:\ \tau-\tau^H = 15.6715(\delta_{12}-\delta_{12}^H),\ \delta_{12} > \delta_{12}^H. \end{aligned} \tag{4.2}$$

The dynamical behaviors in each region of model (4.1) are displayed in Fig. 3. The points of O, A and B in Fig. 3 refer to constant steady state, spatially heterogeneous periodic solution with mode-4 and 5 respectively. We observe that when (δ_{12},τ) is in regions R_2 and R_6, model (4.1) can exhibit stable spatial periodic solutions with mode 4 and 5 respectively as is shown Fig. 4. When (δ_{12},τ) is in region R_4, the solution of model (4.1) can converge to spatially heterogeneous periodic solutions with mode 4 or 5 depending on the initial values as is illustrated in Fig. 5.

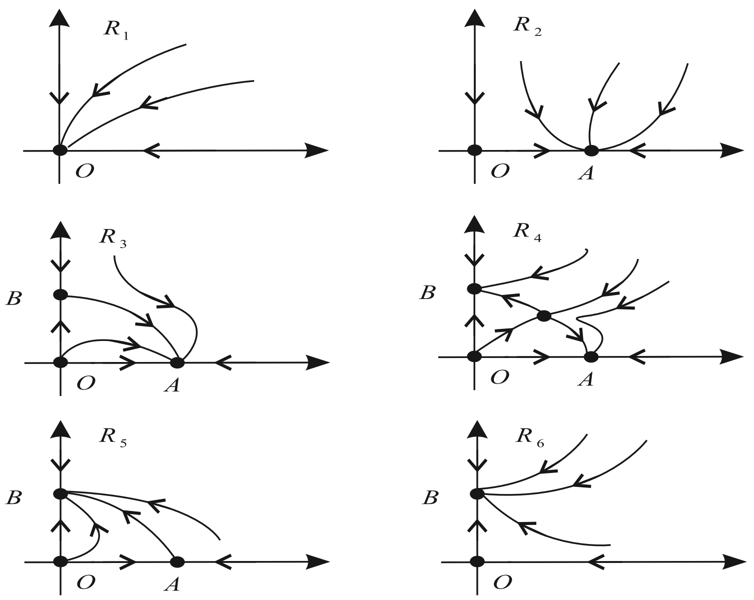

Fig. 3. Phase portraits in different regions of Fig. 2.

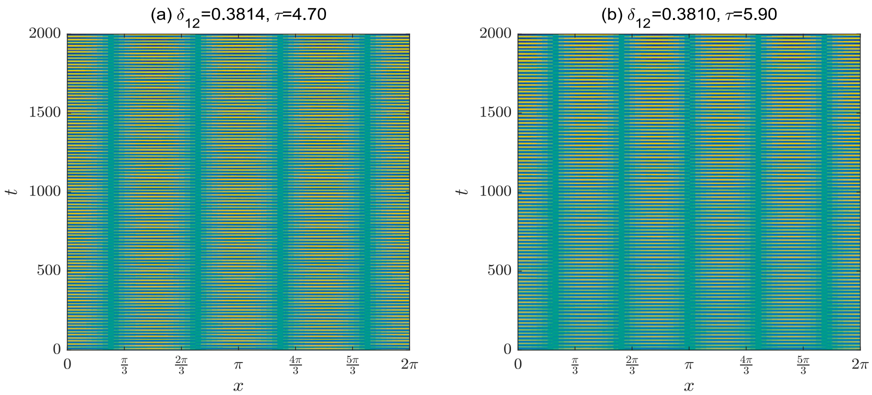

Fig. 4. The stable spatially inhomogeneous periodic solutions with mode 4 and 5 for $(\delta_{12}, \tau) \in R_2$ and $(\delta_{12}, \tau) \in R_6$ respectively.

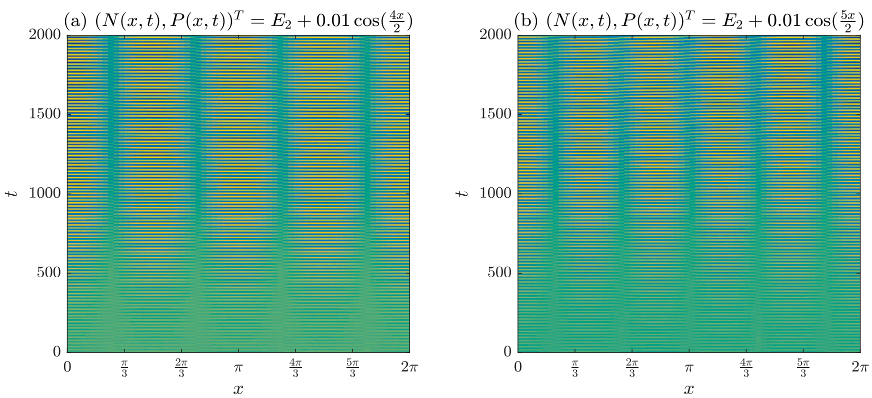

Fig. 5. The coexistence of stable spatially inhomogeneous periodic solutions with mode 4 and 5 for $(\delta_{12}, \tau) \in R_4$.

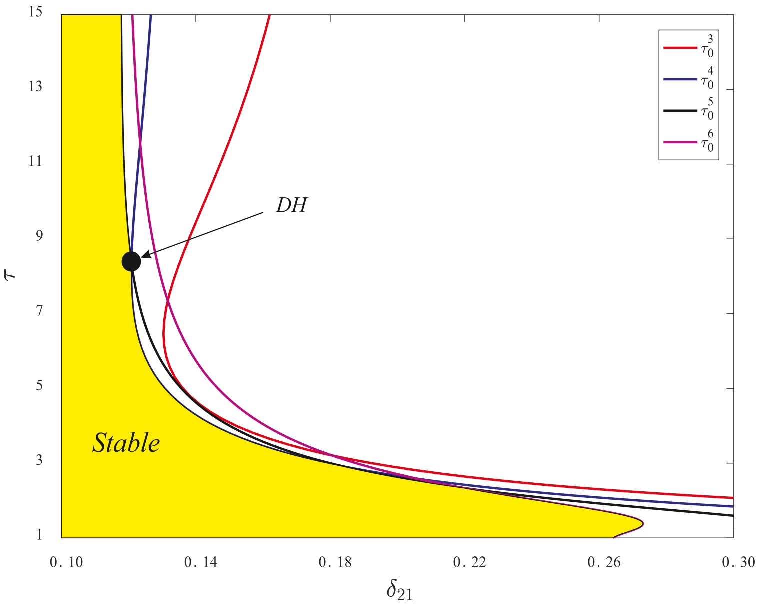

Fig. 6. Crossing curves and double Hopf bifurcation point of model (5.1).

5. Discussions

We investigated the double Hopf bifurcation in a spatial model with directed movement, i.e., model (1.1), with the aim to clarify the temporal–spatial distribution of predators and the prey under the joint influence of memory-driven movement, reproductive facilitation and maturation period via model (1.1). Some significant findings and main contributions in this paper are as follows.

- We presented an innovative technique to plot crossing curves in the (δ_{12}, τ) plane by treating them as parametric curves about frequency. We thus found the double Hopf bifurcation points.
- We derived the explicit formulae concerning normal form of the non-resonant double Hopf bifurcation triggered by spatial memory in memory-driven diffusive predator–prey models with double delays.
- Our research shows that it is the tactic diffusion term rather than the reaction term of the developed model that induces complicated dynamic behaviors around the double Hopf bifurcation point. This presents a completely different mechanism from classical models with only random diffusions [38].
- Spatial memory along with other biotic processes can induce complex spatial distribution of animals. For example, the transition between stable spatially inhomogeneous periodic solutions with higher modes and coexistence of them can be observed. These seem not to have ever been observed in the literature [26,28,40,44]. Also, these complex spatiotemporal distributions may be a signal of resilience for species [45].

As we state in the introduction section, we can also consider the scenario of spatial memory in predators. Assume that the prey population is viewed as "drunk" animals so that their memory or cognition can be ignored [27]. Then we consider the following model:

$$\begin{cases} \frac{\partial N}{\partial t} = \delta_{11} N_{xx} + rN\left(1 - \frac{N}{K}\right) - bNP, \ 0 < x < \iota\pi, \ t > 0, \\ \frac{\partial P}{\partial t} = \delta_{22} P_{xx} - \delta_{21}\left(PN_x(x, t-\tau)\right)_x + \frac{\beta N_\sigma P_\sigma^2}{h + P_\sigma} \mathrm{e}^{-d\sigma} - \mu P, \ 0 < x < \iota\pi, \ t > 0, \\ N_x(x,t) = P_x(x,t)\mid_{x=0,\iota\pi} = 0, \ t \ge 0, \end{cases} \tag{5.1}$$

where δ_{21} and τ represent respectively the memory-dependent diffusion and the averaged memory period coefficients of predators. We can analyze model (5.1) in a similar way as model (1.1). Here we only elucidate the main results of model (5.1) through numerical simulations.

Taking the same parameter values as in system (4.1), we can also plot the crossing curves in the (δ_{21}, τ) plane as is shown in Fig. 6. We thus obtain the critical values for the double Hopf bifurcation point $n_1 = 4$, $n_2 = 5$, $\varpi_{n_1} \approx 0.3045$, $\varpi_{n_2} \approx 0.3060$, $\delta_{21}^H \approx 0.1208$, $\tau^H \approx 8.3554$. Similarly, we can calculate $c_1 \approx -0.0014446\xi_1 + 6.1026204\xi_2$, $c_2 \approx 0.0102199\xi_1 + 6.5375531\xi_2$, $a_{11} \approx -0.1215701$, $a_{12} \approx -0.2278062$, $a_{21} \approx -0.1377265$, $a_{22} \approx -0.1377265$. We thus classify the dynamics of model (5.1) into 6 categories

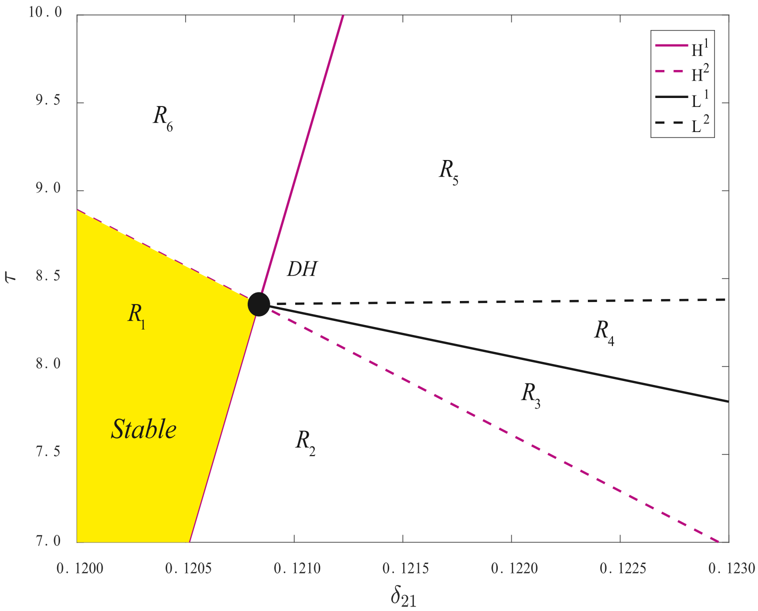

Fig. 7. Complete bifurcation region near the double Hopf bifurcation point.

in the (δ_{21}, τ) plane as is displayed in Fig. 7 by the following curves:

$$\begin{cases} H1 : \tau - \tau^H = 4224.4733(\delta_{21} - \delta_{21}^H), \\ H2 : \tau - \tau^H = -639.6865(\delta_{21} - \delta_{21}^H), \\ L1 : \tau - \tau^H = -256.7359(\delta_{21} - \delta_{21}^H),\ \delta_{12} > \delta_{12}^H, \\ L2 : \tau - \tau^H = 11.6484(\delta_{21} - \delta_{21}^H),\ \delta_{12} > \delta_{12}^H. \end{cases}$$

To sum up, the method developed in this paper by viewing some critical indices without explicit expression (such as frequency or equilibrium) as parameters can provide a new paradigm to detect some bifurcations. It can also be extensively used to reveal the complex effects of biotic process of animals on their distribution in space. There also exist some other problems that can be further considered. For instance, both the prey and predators can possess spatial memory. In this case, we can deduce from (2.6) that its left side is a high degree polynomial with respect to δ_{12}, and therefore it is difficult to obtain the associated crossing curves. Also, the derivation of normal form in this paper is under the assumption that the double Hopf bifurcation is non-resonant. When the double Hopf bifurcation is not a non-resonant one, then we cannot obtain $\ker(\mathcal{M}_2^1)$ and $\ker(\mathcal{M}_3^1)$ from (3.19). A natural question is to calculate normal form concerning resonant double Hopf bifurcation for diffusive predator–prey models with double delays and spatial memory in this case. We will devote ourselves to these problems in the future.

CRediT authorship contribution statement

Shuai Li: Conceptualization, Investigation, Writing – original draft. **Sanling Yuan:** Conceptualization, Funding acquisition, Investigation, Project administration, Supervision, Writing – review & editing. **Zhen Jin:** Conceptualization, Formal analysis, Investigation, Writing – review & editing. **Hao Wang:** Conceptualization, Funding acquisition, Investigation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

The first two authors are sponsored by Natural Science Foundation of Shanghai (Grant 23ZR1445100) and National Natural Science Foundation of China (Grant 12071293). The fourth author is partially sponsored by the Natural Sciences and Engineering Research Council of Canada (Discovery Grant RGPIN-2020-03911 and Accelerator Grant RGPAS-2020-00090).

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