Solutions to Assignment #2

Ex 2.3.2

(a) $\lambda > 0$, $\phi = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$\phi(0) = 0 \Rightarrow C_1 = 0$

$\phi(\pi) = 0 \Rightarrow C_2 \sin\sqrt{\lambda}\pi = 0 \Rightarrow \sin\sqrt{\lambda}\pi = 0 \Rightarrow \sqrt{\lambda}\pi = n\pi$

$\Rightarrow \lambda = n^2$, $n = 1, 2, 3, \cdots$

$\lambda = 0$, not eigenvalue
$\lambda < 0$, not eigenvalues } see my Lecture notes.

For eigenvalues $\lambda = n^2$, corresponding eigenfunctions

$\phi(x) = C_2 \sin nx$ or just $\phi(x) = \sin nx$ (pick $C_2 = 1$)
$n = 1, 2, 3, \cdots$

b) only positive eigenvalues $\lambda = (n\pi)^2$, $n = 1, 2, 3, \cdots$

Eigenfunctions $\phi(x) = C_2 \sin n\pi x$ or just

$\phi(x) = \sin n\pi x$ (pick $C_2 = 1$)

$n = 1, 2, 3, \cdots$

c) Eigenvalues $\lambda = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, \cdots$

Eigenfunctions $\phi(x) = \cos\frac{n\pi x}{L}$, $n = 0, 1, 2, \cdots$

Details are in my lecture notes of section 2.4.

d) $\lambda > 0$, $\phi = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$\phi(0) = 0 \Rightarrow C_1 = 0$

$\frac{d\phi}{dx}(L) = 0 \Rightarrow C_2 \sqrt{\lambda} \cos\sqrt{\lambda}L = 0 \Rightarrow \cos\sqrt{\lambda}L = 0$

$$\Rightarrow \sqrt{\lambda}L = n\pi - \frac{\pi}{2}, \quad n=1,2,3,\cdots$$

$$\Rightarrow \lambda = \left(\frac{n\pi-\frac{\pi}{2}}{L}\right)^2, \quad n=1,2,3,\cdots$$

Eigenvalues

Eigenfunctions $\phi(x) = \sin\frac{n\pi-\frac{\pi}{2}}{L}x, \quad n=1,2,3,\cdots$

$\lambda=0$, $\phi = C_1 + C_2 x$

$$\left.\begin{aligned} \phi(0)=0 &\Rightarrow C_1=0 \\ \frac{d\phi}{dx}(L)=0 &\Rightarrow C_2=0 \end{aligned}\right\}$$ trivial solution, i.e., $\lambda=0$ is not an eigenvalue.

$\lambda<0$, $\phi = C_3\cosh\sqrt{-\lambda}x + C_4\sinh\sqrt{-\lambda}x$

$$\left.\begin{aligned} \phi(0)=0 &\Rightarrow C_3=0 \\ \frac{d\phi}{dx}(L)=0 &\Rightarrow C_4\underbrace{\sqrt{-\lambda}}_{>0}\underbrace{\cosh\sqrt{-\lambda}L}_{>0}=0 \Rightarrow C_4=0 \end{aligned}\right\}$$ trivial solution

so, No negative eigenvalues.

(e) $\lambda>0$, $\phi = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$

$$\frac{d\phi}{dx}(0)=0 \Rightarrow C_2\sqrt{\lambda}=0 \Rightarrow C_2=0$$

$$\phi(L)=0 \Rightarrow C_1\cos\sqrt{\lambda}L=0 \Rightarrow \cos\sqrt{\lambda}L=0 \Rightarrow \sqrt{\lambda}L = n\pi-\frac{\pi}{2}$$

$\Rightarrow$ Eigenvalues $\lambda = \left(\frac{n\pi-\frac{\pi}{2}}{L}\right)^2, \quad n=1,2,3,\cdots$

Eigenfunctions $\phi(x) = \cos\frac{n\pi-\frac{\pi}{2}}{L}x, \quad n=1,2,3,\cdots$

$\lambda=0$, $\phi = C_1 + C_2 x$

$$\frac{d\phi}{dx}(0)=0 \Rightarrow C_2=0$$

$\phi(L)=0 \Rightarrow \quad c_1=0$

Trivial solution, i.e. $\lambda=0$ is not an eigenvalue.

$\lambda<0, \quad \phi=c_3\cosh\sqrt{-\lambda}x+c_4\sinh\sqrt{-\lambda}x$

$\frac{d\phi}{dx}(0)=0 \Rightarrow \quad c_4\sqrt{-\lambda}=0 \Rightarrow c_4=0$

$\phi(L)=0 \Rightarrow \quad c_3\underbrace{\cosh\sqrt{-\lambda}L}_{>0}=0 \Rightarrow c_3=0$

Trivial solution, i.e. No negative eigenvalues.

Ex 2.3.3

$$u(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2t}, \text{ where}$$

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

(a) ~~$B_n=\frac{2}{L}\int_0^L 6\sin\frac{9\pi x}{L}\sin$~~

It's special, we don't need the general formula with infinite series.

$$u(x,t)=6\sin\frac{9\pi x}{L}e^{-k(9\pi/L)^2t}$$

(b) $u(x,t)=3\sin\frac{\pi x}{L}e^{-k(\pi/L)^2t}-\sin\frac{3\pi x}{L}e^{-k(3\pi/L)^2t}$

(c) $B_n=\frac{2}{L}\int_0^L 2\cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}dx$ for the infinite series

(d) $B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^{L/2}\sin\frac{n\pi x}{L}dx+\frac{2}{L}\int_{L/2}^L 2\sin\frac{n\pi x}{L}dx$

$$=\frac{2}{L}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_0^{L/2}+\frac{2}{L}\left(\frac{-2L}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_{L/2}^L$$

$$=-\frac{2}{n\pi}\left(\cos\frac{n\pi}{2}-1\right)-\frac{4}{n\pi}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)=\frac{2}{n\pi}\cos\frac{n\pi}{2}-\frac{4}{n\pi}\cos n\pi+\frac{2}{n\pi}$$

Ex 2.4.1

(a) $u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}$, where

$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_{L/2}^L dx = \frac{1}{2}$

$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{L/2}^L \cos\frac{n\pi x}{L}\,dx = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$, $n \geq 1$.

(b)

$u(x,t) = 6 + 4\cos\frac{3\pi x}{L} e^{-(3\pi/L)^2 kt}$

(c) $u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}$, where

$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L -2\sin\frac{\pi x}{L}\,dx = -\frac{4}{\pi}$

$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L -2\sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx$

(d)

$u(x,t) = -3\cos\frac{8\pi x}{L} e^{-(8\pi/L)^2 kt}$.

Ex 2.5.1 (a) [illegible].

$u(x,y) = h(x)\,\phi(y)$

$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = -\lambda$ (two homog. B.C.'s for x, So we want $h(x)$ to oscillate.)

B.C.'s $\frac{dh}{dx}(0) = 0$, $\frac{dh}{dx}(L) = 0$, $\phi(0) = 0$

$\begin{cases} \frac{d^2h}{dx^2} = -\lambda h \\ \frac{dh}{dx}(0) = 0 \\ \frac{dh}{dx}(L) = 0 \end{cases}$ and $\begin{cases} \frac{d^2\phi}{dy^2} = \lambda\phi \\ \phi(0) = 0 \end{cases}$

From the first equation, we should be able to find eigenvalues and eigenfunctions.

Actually, we have known the answer

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \qquad n = 0, 1, 2, 3, \ldots$$
$$h(x) = \cos\frac{n\pi x}{L}$$

For the second equation, $\dfrac{d^2\phi}{dy^2} = \left(\dfrac{n\pi}{L}\right)^2 \phi$

$$\phi = a_1 \cosh\frac{n\pi y}{L} + a_2 \sinh\frac{n\pi y}{L}, \quad \text{for } n \geq 1$$

$$\phi(0) = 0 \Rightarrow a_1 = 0 \Rightarrow \phi = a_2 \sinh\frac{n\pi y}{L}, \quad \text{for } n \geq 1$$

For $n = 0$, $\dfrac{d^2\phi}{dy^2} = 0 \Rightarrow \phi = a_1 + a_2 y$

$$\phi(0) = 0 \Rightarrow a_1 = 0 \Rightarrow \phi = a_2 y$$

Product solutions: $u(x,y) = \begin{cases} A_0 y, & n = 0 \\ A_n \cos\frac{n\pi x}{L} \sinh\frac{n\pi y}{L}, & n \geq 1 \end{cases}$

Determine $u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \sinh\frac{n\pi y}{L}$ to satisfy the nonhomogeneous B.C. $u(x,H) = f(x)$

i.e.
$$f(x) = A_0 H + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \sinh\frac{n\pi H}{L}$$
$$\Leftrightarrow \quad f(x) = A_0 H + \sum_{n=1}^{\infty} \left(A_n \sinh\frac{n\pi H}{L}\right) \cos\frac{n\pi x}{L}$$

$$\int_0^L f(x) \cos\frac{m\pi x}{L}\,dx = A_0 H \int_0^L \cos\frac{m\pi x}{L}\,dx + \sum_{n=1}^{\infty} \left(A_n \sinh\frac{n\pi H}{L}\right) \int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\,dx$$

$m = 0$, $\int_0^L f(x)\,dx = A_0 H L + 0 \Rightarrow A_0 = \dfrac{1}{HL}\int_0^L f(x)\,dx$

$m \geq 1$, $\int_0^L f(x) \cos\frac{m\pi x}{L}\,dx = 0 + A_m \sinh\frac{m\pi H}{L} \cdot \frac{L}{2}$

$$\Rightarrow A_m = \frac{2}{L \sinh\frac{m\pi H}{L}} \int_0^L f(x) \cos\frac{m\pi x}{L}\,dx .$$

Ex 2.5.7 (a) with $f(\theta) = \theta(\frac{\pi}{3} - \theta)$.

Solution: Assuming that nontrivial product solutions have the form

$$u(r,\theta) = R(r)\Theta(\theta).$$

The rest is in the page 8 and page 9. ~~next one and a half typed pages~~.

The formulas $\int_0^{\pi/3} \sin 3n\theta \sin 3m\theta = \frac{\pi}{6}\delta_{n,m}$, $\forall n,m \geq 1$ are from (2.3.32) of textbook with $L = \pi/3$. The function $\delta_{n,m} = \begin{cases} 0, & \text{if } n \neq m \\ 1, & \text{if } n = m. \end{cases}$

Ex 2.5.8 (a) with $f(\theta) = \sin\theta$ and $g(\theta) = 2\sin\theta\cos\theta = \sin 2\theta$.

Solution: One way is to divide this problem into two sub-problems (as we did for standard Laplace equation in class), each of which has three homogeneous B.C.'s and one nonhomogeneous B.C..

The simplest way is to use the general solution of the Laplace equation in polar coordinaties. I explain how we obtain this general solution here.

Following my notes or Page 78–79 of the textbook, we have

eigenfunctions (for θ-dependent equation) $\sin n\theta$ and $\cos n\theta$, $(n = 1, 2, 3, \cdots)$

constant, $(n = 0)$

For r-dependent equation, $c_1 r^n + c_2 r^{-n}$, $(n = 1, 2, 3, \cdots)$

$\bar{c}_1 + \bar{c}_2 \ln r$, $(n = 0)$

(*Note that c_2 and $\bar{c}_2$ can be nonzero, since the condition $|u(0,\theta)| < \infty$ no longer exists; actually $r = 0$ is outside ~~the region~~ the region we consider, $a \leq r \leq b$.)

Hence, the general solution is the sum of all product solutions:

$$u(r,\theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty}\left[(A_n r^{-n} + B_n r^n)\cos n\theta + (C_n r^{-n} + D_n r^n)\sin n\theta\right].$$

Then we can follow the solution from Line 7 of page 9 to the end of page 10.

The function $\delta_{nm} = \begin{cases} 0, & \text{if } n \neq m \\ 1, & \text{if } n = m \end{cases}$.

The orthogonality formulas $\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta = 0, \forall n \geq 0, m \geq 1$, $\int_0^{2\pi} \sin(n\theta)\sin(m\theta)\,d\theta = \pi\delta_{nm}, \forall n,m \geq 1$, $\int_0^{2\pi} \cdots\cdots$, can be replaced by $\int_{-\pi}^{\pi} \cdots\cdots$ we learned in class as well as the textbook. Since 2π is a period of the inside functions ($\sin n\theta \cos m\theta$, $\sin n\theta \sin m\theta$, $\cos n\theta \cos m\theta$, $\cos n\theta$, $\sin n\theta$), $\int_0^{2\pi}$ is the same as $\int_{-\pi}^{\pi}$. Thus, you can use (2.4.40)–(2.4.42) with $L=\pi$ in textbook instead, and evaluate $\int_{-\pi}^{\pi} \cos n\theta \, d\theta$, $\int_{-\pi}^{\pi} \sin n\theta \, d\theta$ directly.

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Ex 2.5.7(a) (continued):

and appreciating that the homogeneous boundary conditions are with respect to θ leads to

$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda^2,$$

which implies that

$$\Theta''+\lambda^2\Theta=0 \text{ with } \Theta(0)=\Theta(\pi/3)=0,$$

$$r^2R''+rR'-\lambda^2R=0 \text{ with } |R(0)|<\infty.$$

The eigenfunctions will be given

$$\Theta_n(\theta)=A_n\sin(3n\theta) \text{ with } n=1,2,3,... \Longrightarrow \lambda=3n,$$

and it follows that the solution to the corresponding R function that is bounded at the origin is given by

$$R_n(r)=r^{3n},$$

where, without loss of generality, we have taken the free integration constant associated with the $R_n(r)$ functions to be "1" since it will be absorbed into A_n in any event. Thus, the general solution to Laplace's equation that satisfies the boundary conditions and is bounded at the origin is given by

$$u(r,\theta)=\sum_{n=1}A_nr^{3n}\sin(3n\theta).$$

Application of the boundary data along $r=a$ leads to

$$u(a,\theta)=\theta(\pi/3-\theta)=\sum_{n=1}^{\infty}A_na^{3n}\sin(3n\theta).$$

The orthogonality conditions are

$$\int_0^{\pi/3}\sin(3n\theta)\sin(3m\theta)=\frac{\pi}{6}\delta_{nm}\ \forall\ n,m\geq 1. \quad =\begin{cases}\pi/6, & n=m\\ 0, & n\neq m\end{cases}$$

Thus,

$$A_n=\frac{6}{\pi a^{3n}}\int_0^{\pi/3}\theta(\pi/3-\theta)\sin(3n\theta)\,d\theta$$

$$=-\frac{2}{\pi na^{3n}}\int_0^{\pi/3}\theta(\pi/3-\theta)\frac{d}{d\theta}\cos(3n\theta)\,d\theta$$

$$=-\frac{2}{\pi na^{3n}}\left[\theta(\pi/3-\theta)\cos(3n\theta)\right]_0^{\pi/3}+\frac{2}{\pi na^{3n}}\int_0^{\pi/3}(\pi/3-2\theta)\cos(3n\theta)\,d\theta$$

$$=\frac{2}{3\pi n^2a^{3n}}\int_0^{\pi/3}(\pi/3-2\theta)\frac{d}{d\theta}\sin(3n\theta)\,d\theta$$

$$=\frac{2}{3\pi n^2a^{3n}}\left[(\pi/3-2\theta)\sin(3n\theta)\right]_0^{\pi/3}+\frac{4}{3\pi n^2a^{3n}}\int_0^{\pi/3}\sin(3n\theta)\,d\theta$$

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$$= -\frac{4}{9\pi n^3 a^{3n}} \left[\cos(3n\theta)\right]_0^{\pi/3} = \frac{4\left[1-(-1)^n\right]}{9\pi n^3 a^{3n}},$$

where we have integrated by parts repeatedly. We see that if n is even then $A_n = 0$ and if n is odd, then

$$A_n = \frac{8}{9\pi n^3 a^{3n}}.$$

Therefore, the solution may be written in the form (map $n \to 2n+1$)

$$u(r,\theta) = \frac{8}{9\pi} \sum_{n=0}^{\infty} \frac{r^{3(2n+1)} \sin\left[3(2n+1)\theta\right]}{a^{3(2n+1)}(2n+1)^3}. \qquad \square$$

Ex 2.5.8(a) (Continued):

Question 2.5.8(a): The pde and boundary conditions are given by

$$\Delta u = 0,\ 0 < a < r < b,\ 0 \le \theta < 2\pi,$$

with

$$u(a,\theta) = \sin(\theta) \text{ and } u(b,\theta) = 2\sin(\theta)\cos(\theta) = \sin(2\theta).$$

The *general* solution to Laplace's equation in polar coordinates may be written in the form

$$u(r,\theta) = A_0 + B_0 \ln(r) + \sum_{n=1}^{\infty} \left[\left(\frac{A_n}{r^n} + B_n r^n\right)\cos(n\theta) + \left(\frac{C_n}{r^n} + D_n r^n\right)\sin(n\theta)\right].$$

The orthogonality conditions are

$$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, d\theta = \int_0^{2\pi} \sin(n\theta)\cos(m\theta)\, d\theta = 0 \ \forall\, n \ge 0 \text{ and } m \ge 1,$$

$$\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta = \int_0^{2\pi} \sin(n\theta)\sin(m\theta)\, d\theta = \pi\delta_{nm} \ \forall\, n, m \ge 1,$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = \int_0^{2\pi} \cos(n\theta)\cos(m\theta)\, d\theta = \pi\delta_{nm} \ \forall\, n, m \ge 1,$$

$$\int_{-\pi}^{\pi} \cos n\theta \, d\theta = \int_0^{2\pi} \cos(n\theta)\, d\theta = 2\pi\delta_{n0} \text{ and } \overset{\int_{-\pi}^{\pi}\sin n\theta\, d\theta}{=} \int_0^{2\pi} \sin(n\theta)\, d\theta = 0,\ \forall\, n \ge 0.$$

Application of the boundary condition along $r = a$ leads to

$$\sin(\theta) = A_0 + B_0 \ln(a) + \sum_{n=1}^{\infty} \left[\left(\frac{A_n}{a^n} + B_n a^n\right)\cos(n\theta) + \left(\frac{C_n}{a^n} + D_n a^n\right)\sin(n\theta)\right],$$

$$\delta_{n0} = \begin{cases} 0, & n \ne 0 \text{ (i.e., } n \ge 1 \text{ here)} \\ 1, & n = 0 \end{cases}$$

which implies that

$$A_0 + B_0 \ln(a) = 0,$$

$$\frac{A_n}{a^n} + B_n a^n = 0 \ \forall\, n \ge 1,$$

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$$\frac{C_1}{a} + D_1 a = 1,$$

$$\frac{C_n}{a^n} + D_n a^n = 0 \; \forall \; n \geq 2.$$

Application of the boundary condition along $r = b$ leads to

$$\sin(2\theta) = A_0 + B_0 \ln(b) + \sum_{n=1}^{\infty} \left[\left(\frac{A_n}{b^n} + B_n b^n \right) \cos(n\theta) + \left(\frac{C_n}{b^n} + D_n b^n \right) \sin(n\theta) \right],$$

which implies that

$$A_0 + B_0 \ln(b) = 0,$$

$$\frac{A_n}{b^n} + B_n b^n = 0 \; \forall \; n \geq 1,$$

$$\frac{C_2}{b^2} + D_2 b^2 = 1,$$

$$\frac{C_n}{b^n} + D_n b^n = 0 \; \forall \; n \neq 2.$$

Thus, we can immediately conclude that

$$A_n = B_n = 0 \; \forall \; n \geq 0,$$

$$C_n = D_n = 0 \; \forall \; n \neq 1 \text{ or } 2,$$

and that

$$\frac{C_1}{a} + D_1 a = 1 \text{ and } \frac{C_1}{b} + D_1 b = 0$$

$$\implies C_1 = \frac{ab^2}{b^2 - a^2} \text{ and } D_1 = \frac{a}{a^2 - b^2},$$

and that

$$\frac{C_2}{a^2} + D_2 a^2 = 0 \text{ and } \frac{C_2}{b^2} + D_2 b^2 = 1,$$

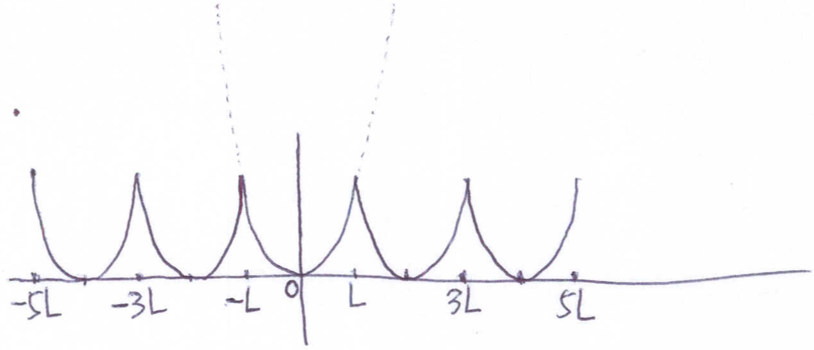
$$\implies C_2 = \frac{a^4 b^2}{a^4 - b^4} \text{ and } D_2 = \frac{b^2}{b^4 - a^4}.$$

Hence, the solution can be written in the form

$$u(r,\theta) = \left(\frac{C_1}{r} + D_1 r \right) \sin(\theta) + \left(\frac{C_2}{r^2} + D_2 r^2 \right) \sin(2\theta)$$

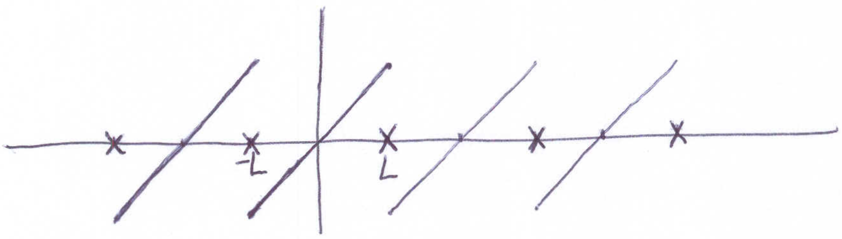
$$= \frac{(b^2 - r^2)\, a}{(b^2 - a^2)\, r} \sin(\theta) + \frac{(r^4 - a^4)\, b^2}{(b^4 - a^4)\, r^2} \sin(2\theta).$$

Ex 3.2.1 (b).



Ex 3.2.2.

(a)



$$a_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx = \frac{1}{2L}\cdot\frac{x^2}{2}\Big|_{-L}^{L} = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{L}\,dx = \frac{1}{n\pi}\cdot\int_{-L}^{L} x\,d\sin\frac{n\pi x}{L} = \frac{1}{n\pi}\left(x\sin\frac{n\pi x}{L}\Big|_{-L}^{L} - \int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx\right)$$

$$= \frac{1}{n\pi}(0-0) = 0$$

(Note: You can directly see that $a_0=0$, $a_n=0$ from the observation that both x and $x\cos\frac{n\pi x}{L}$ are odd functions.)

$$b_n = \frac{1}{L}\int_{-L}^{L} x\sin\frac{n\pi x}{L}\,dx = -\frac{1}{n\pi}\int_{-L}^{L} x\,d\cos\frac{n\pi x}{L}$$

$$= -\frac{1}{n\pi}\left(x\cos\frac{n\pi x}{L}\Big|_{-L}^{L} - \int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx\right)$$

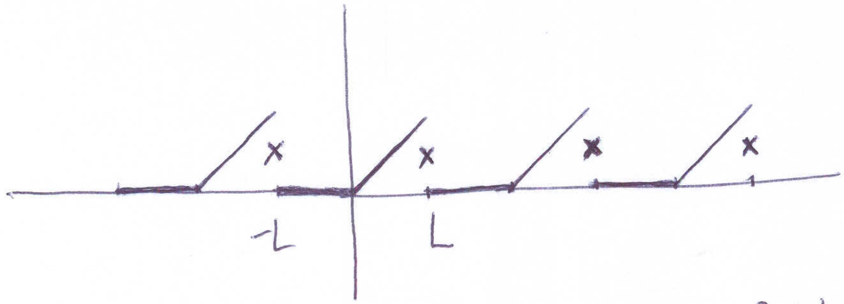
$$= -\frac{1}{n\pi}\left(L\cos n\pi - (-L)\cos(-n\pi) - \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L}\right)$$

$$= -\frac{1}{n\pi}\left(2L\cos n\pi - 0\right)$$

$$= \begin{cases} -\frac{2L}{n\pi}, & n \text{ even} \\ \frac{2L}{n\pi}, & n \text{ odd} \end{cases}$$

$$= \frac{2L}{n\pi}(-1)^{n+1}$$

(d)



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_0^L x\,dx = \frac{1}{2L}\cdot\frac{x^2}{2}\Big|_0^L = \frac{L}{4}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx = \frac{1}{n\pi}\int_0^L x\,d\sin\frac{n\pi x}{L}$$

$$= \frac{1}{n\pi}\left(x\sin\frac{n\pi x}{L}\Big|_0^L - \int_0^L \sin\frac{n\pi x}{L}\,dx\right)$$

$$= \frac{1}{n\pi}\left(0 + \frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L\right)$$

$$= \frac{1}{n\pi}\cdot\frac{L}{n\pi}(\cos n\pi - 1)$$

$$= \frac{L}{(n\pi)^2}(\cos n\pi - 1)$$

$$= \begin{cases} 0 & , n \text{ even} \\ \dfrac{-2L}{(n\pi)^2} & , n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx = -\frac{1}{n\pi}\int_0^L x\,d\cos\frac{n\pi x}{L}$$

$$= -\frac{1}{n\pi}\left(x\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L \cos\frac{n\pi x}{L}\,dx\right)$$

$$= -\frac{1}{n\pi}\left(L\cos n\pi - \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L\right)$$

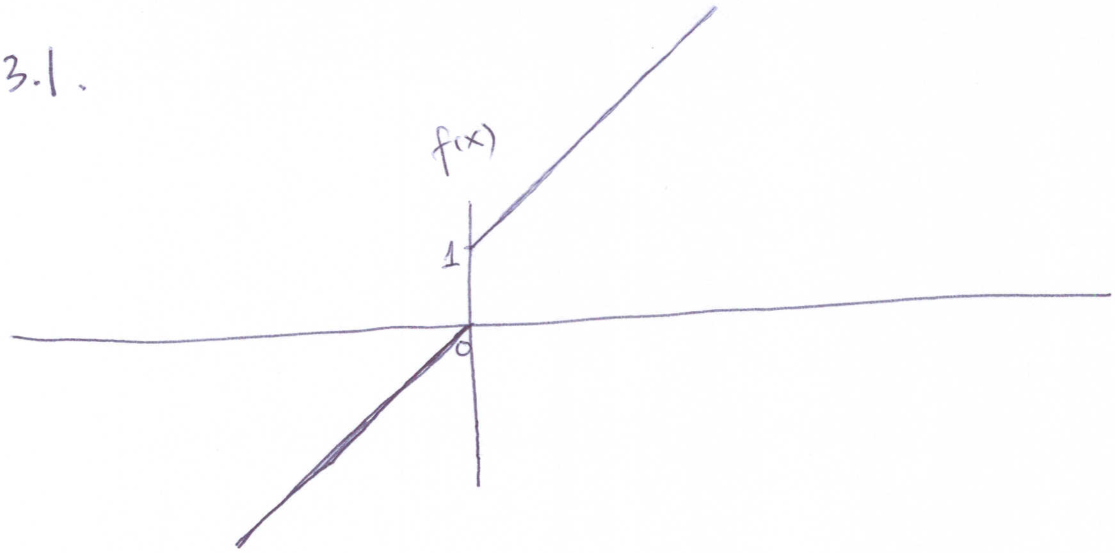
$$= -\frac{1}{n\pi}(L\cos n\pi - 0)$$

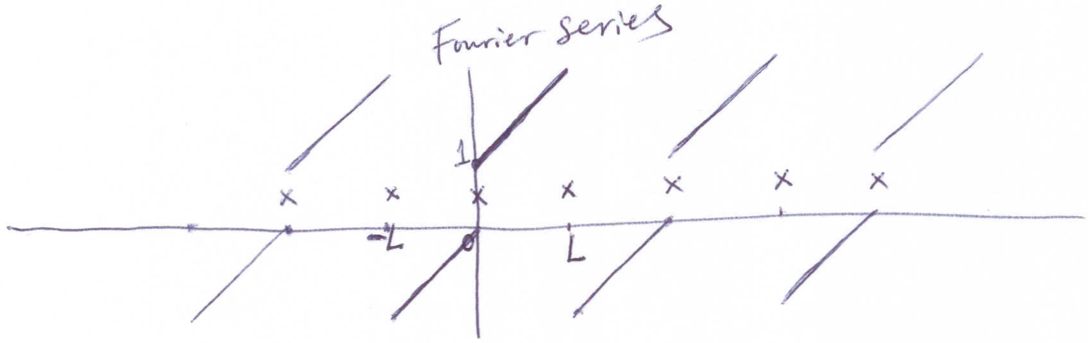
$$= -\frac{L}{n\pi}\cos n\pi$$

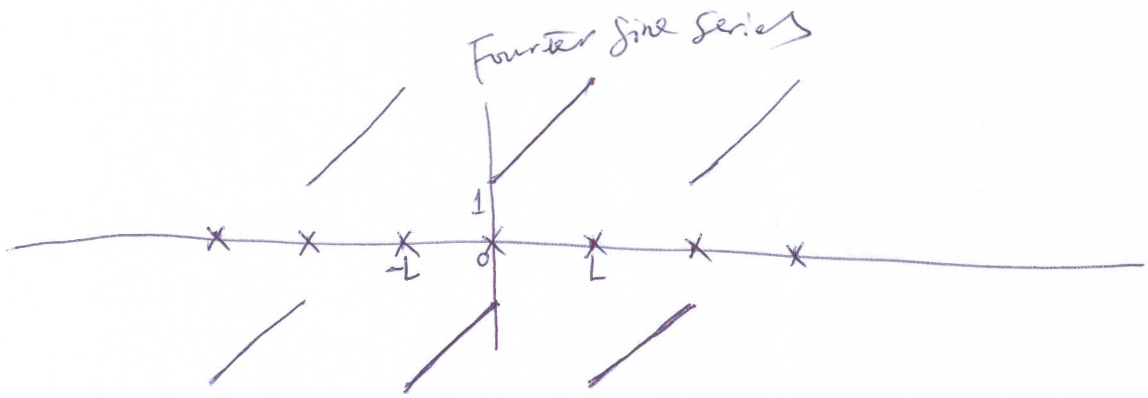
$$= \begin{cases} -\dfrac{L}{n\pi} & , n \text{ even} \\ \dfrac{L}{n\pi} & , n \text{ odd} \end{cases}$$

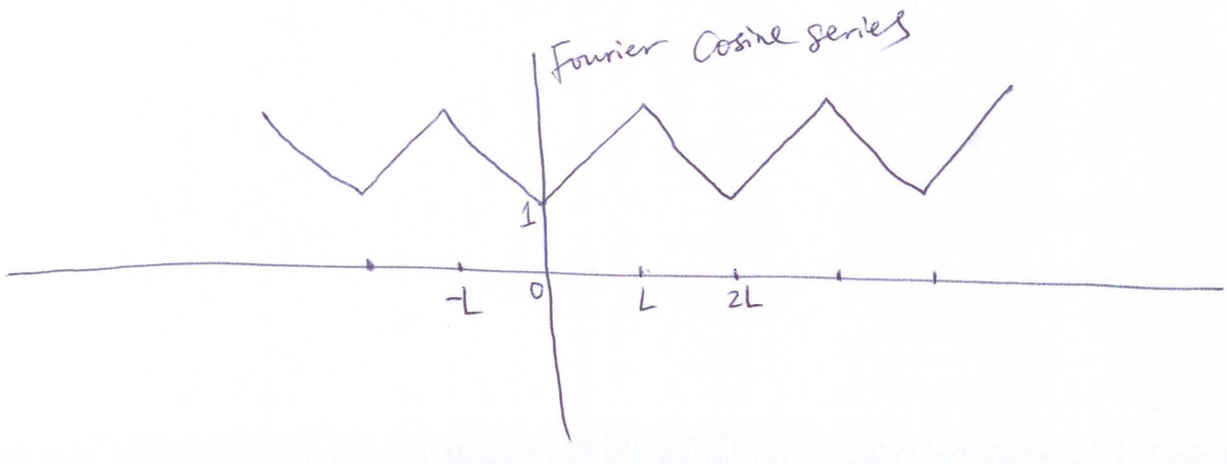
Ex 3.3.1.

(c)


f(x)
1
0


Fourier series
1
-L
0
L


Fourier sine series
1
-L
0
L

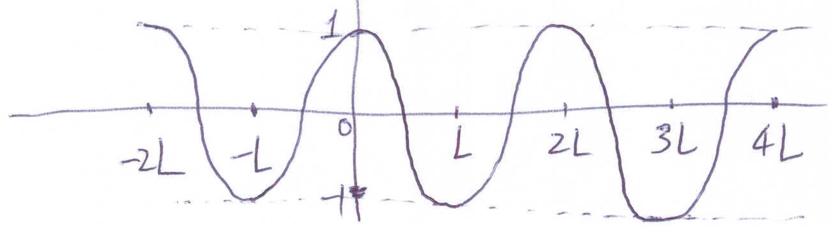

Fourier Cosine series
1
-L
0
L
2L

(d) See page 735 of textbook, that has much better figures than my handwriting.

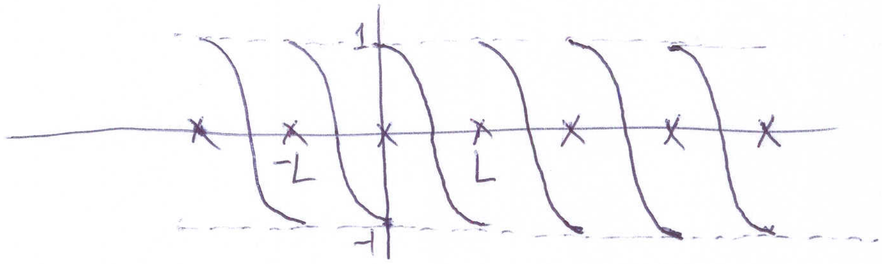
Ex 3.3.2.

(a)

$\cos\frac{\pi x}{L}$



Fourier sine series



$$\cos\frac{\pi x}{L} \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \quad \text{where}$$

$$B_n = \frac{2}{L}\int_0^L \cos\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L \frac{1}{2}\left[\sin\frac{(n+1)\pi x}{L} + \sin\frac{(n-1)\pi x}{L}\right]dx$$

$$= \frac{1}{L}\left[\frac{-L}{(n+1)\pi}\cos\frac{(n+1)\pi x}{L}\Big|_0^L + \frac{-L}{(n-1)\pi}\cos\frac{(n-1)\pi x}{L}\Big|_0^L\right]$$

$$= \frac{1}{L}\left[-\frac{L}{(n+1)\pi}\left(\cos(n+1)\pi - 1\right) - \frac{L}{(n-1)\pi}\left(\cos(n-1)\pi - 1\right)\right]$$

$$= \frac{1}{L}\left[-\frac{L}{(n+1)\pi}\left(-\cos n\pi - 1\right) - \frac{L}{(n-1)\pi}\left(-\cos n\pi - 1\right)\right]$$

$$= \frac{\cos n\pi}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right) + \frac{1}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right)$$

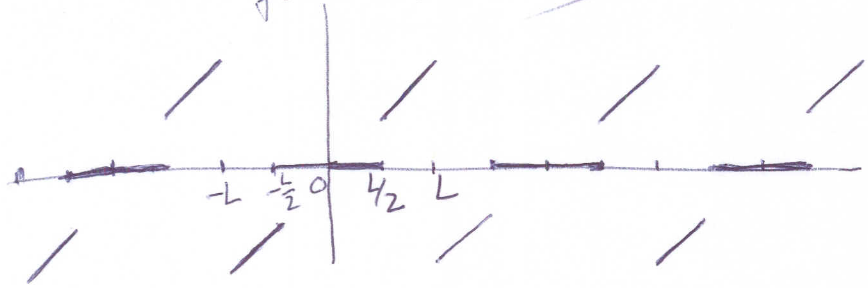
$$= \frac{2n}{\pi(n^2-1)}\left(\cos n\pi + 1\right)$$

$$= \begin{cases} \frac{4n}{\pi(n^2-1)}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

15

(c)

Fourier sine series



$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_{L/2}^{L} x \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\cdot\frac{-L}{n\pi}\int_{L/2}^{L} x\, d\cos\frac{n\pi x}{L}$$

$$= -\frac{2}{n\pi}\left(x\cos\frac{n\pi x}{L}\Big|_{L/2}^{L} - \int_{L/2}^{L} \cos\frac{n\pi x}{L}\,dx\right)$$

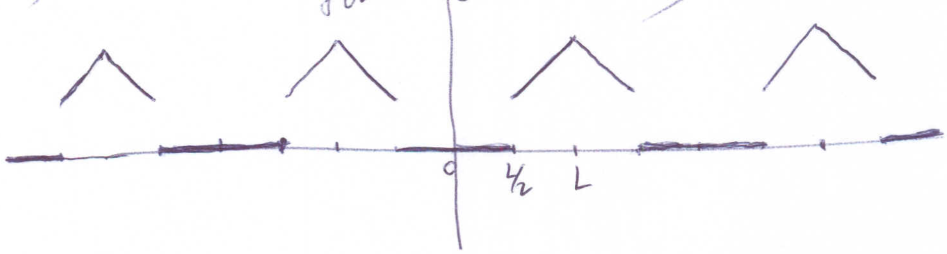
$$= -\frac{2}{n\pi}\left(L\cos n\pi - \frac{L}{2}\cos\frac{n\pi}{2} - \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{L/2}^{L}\right)$$

$$= -\frac{2}{n\pi}\left(L\cos n\pi - \frac{L}{2}\cos\frac{n\pi}{2} - 0 + \frac{L}{n\pi}\sin\frac{n\pi}{2}\right)$$

$$= -\frac{2}{n\pi}\left(L\cos n\pi - \frac{L}{2}\cos\frac{n\pi}{2} + \frac{L}{n\pi}\sin\frac{n\pi}{2}\right)$$

Ex 3.3.5 (c)

Fourier Cosine series



Fourier cosine coefficients are

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_{L/2}^{L} x\,dx = \frac{1}{L}\,\frac{x^2}{2}\Big|_{L/2}^{L} = \frac{1}{L}\left(\frac{L^2}{2} - \frac{L^2}{8}\right) = \frac{3}{8}L$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{L/2}^{L} x\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\cdot\frac{L}{n\pi}\int_{L/2}^{L} x\, d\sin\frac{n\pi x}{L}$$

$$= \frac{2}{n\pi}\left(x\sin\frac{n\pi x}{L}\Big|_{L/2}^{L} - \int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx\right) = \frac{2}{n\pi}\left(0 - \frac{L}{2}\sin\frac{n\pi}{2} + \frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{L/2}^{L}\right)$$

$$= \frac{2}{n\pi}\left(-\frac{L}{2}\sin\frac{n\pi}{2} + \frac{L}{n\pi}\cos n\pi - \frac{L}{n\pi}\cos\frac{n\pi}{2}\right)$$

Ex 3.3.10

Even part $f_e(x) = \frac{1}{2}[f(x)+f(-x)] = \begin{cases} \frac{1}{2}[x^2+e^{-(-x)}], & x<0 \\ \frac{1}{2}[e^{-x}+(-x)^2], & x>0 \end{cases}$

$$= \begin{cases} \frac{x^2+e^x}{2}, & x<0 \\ \frac{x^2+e^{-x}}{2}, & x>0 \end{cases}$$

Odd part $f_o(x) = \frac{1}{2}[f(x)-f(-x)] = \begin{cases} \frac{1}{2}[x^2-e^{-(-x)}], & x<0 \\ \frac{1}{2}[e^{-x}-(-x)^2], & x>0 \end{cases}$

$$= \begin{cases} \frac{x^2-e^x}{2}, & x<0 \\ \frac{-x^2+e^{-x}}{2}, & x>0 \end{cases}.$$