INCOMPARABLE PRIME IDEALS IN COMMUTATIVE RADICAL FRÉCHET ALGEBRAS

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Abstract. Let $R$ be a commutative radical Fréchet algebra having a non-nilpotent element $a$ with $a \in Ra$. Then $R$ contains a continuum of incomparable prime ideals.

In [3], J. Esterle proved the following result; it is a main ingredient in his proof that epimorphisms from $C_0(\Omega)$ onto Banach algebras are continuous.

Theorem (Esterle). Let $R$ be a commutative radical Banach algebra. Suppose that there exists a non-nilpotent element $a \in R$ with $a \in Ra$. Then the set of prime ideals in $R$, ordered by inclusion, does not form a chain.

Thus, each algebra $R$ as in the theorem contains at least two prime ideals which are incomparable. In [1], Bouloussa extended the result to commutative radical Fréchet algebra. In this paper, we shall extend this by producing a continuum of pairwise incomparable prime ideals. This will be proved as a consequence of a result on ideals in (not necessarily commutative) radical Fréchet algebras and the existence of a continuum of "almost disjoint" subsets of $\mathbb{N}$ due to Sierpinski.

1. Preliminaries

More details of the following can be found, for example, in [2].

A Fréchet algebra is a topological algebra $A$ whose topology is determined by a sequence of algebra seminorms $(p_n)$ such that

$$d(a, b) = \sum_{n=1}^{\infty} \min \left\{ \frac{p_n(a-b), 1}{2^n} \right\} \quad (a, b \in A)$$

is a complete metric.

Let $A$ be an algebra. Denote by $A^\#$ the conditional unitization of $A$: adjoining an identity in the case where $A$ is non-unital.

Let $S$ be a subset of an algebra $A$. For $n \in \mathbb{N}$, denote by $S^n$ the linear span of $\{a_1 \cdots a_n : a_i \in S\}$ (and $A^{(n)}$ the $n$-fold Cartesian product of $A$).

For clarity, we shall use boldface characters to denote tuples of elements; for example, we set

$$\mathbf{x} = (x_1, \ldots, x_m) \quad \text{or} \quad \mathbf{y} = (y_1, \ldots, y_n).$$

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2. A continuum of incomparable prime ideals

Lemma 1. Let $R$ be a radical Fréchet algebra. Let $m, n \in \mathbb{N}$, and let $(k_1, \ldots, k_n) \subset \mathbb{N}$. Let $I$ be a left ideal in $R$ satisfying that $I^{(m)} = R \cdot I^{(n)}$. Suppose that $p$ is a continuous algebra seminorm on $R$ such that $I^{2^{k-1}} \not\subset \ker p$, where $k = k_1 + \cdots + k_n$. Then, for each function $\phi : R \to \mathbb{R}^+$, the set of all $(x, y) \in I^{(m)} \times I^{(n)}$ satisfying

$$\sum_{i=1}^{m} \phi(vx_i) < p(v^{k_1}y_1 \cdots y_\alpha),$$

for some $v \in R^\#$, is dense in $I^{(m+n)}$.

Proof. Denote by $U$ the set under consideration. First, let $(a, b)$ be arbitrary in $I^{(m+n)}$ with $a \in R \cdot I^{(m)}$ and $p(b_{1}^{k_1} \cdots b_{n}^{k_n}) \neq 0$. Let $a' = (a_1', \ldots, a_m') \in I^{(m)}$ and $c \in R$ such that $c \cdot a' = a$. Since $p(b_{1}^{k_1} \cdots b_{n}^{k_n}) \neq 0$, we see that (cf. [1, Lemme 1.1]), for each $r \in \mathbb{N}$, there exists $\lambda_r \in \mathbb{C}$ such that

$$0 < |\lambda_r| < \frac{1}{r} \quad \text{and that} \quad p((\lambda_r + c)^{-1}b_{1}^{k_1} \cdots b_{n}^{k_n}) > \sum_{i=1}^{m} \phi(a'_i).$$

Set $v_r = \lambda_r + c \in R^\#$ and $x_r = v_r \cdot a' \in I^{(m)}$. Then

$$\sum_{i=1}^{m} \phi(v_r^{-1}x_{r,i}) = \sum_{i=1}^{m} \phi(a'_i) < p(v_r^{-1}b_{1}^{k_1} \cdots b_{n}^{k_n}),$$

so that $(x_r, b) \in U \quad (r \in \mathbb{N})$. We have $\lim x_r = a$, so $(a, b) \in U$.

The set

$$\{ b \in I^{(n)} : p(b_{1}^{k_1} \cdots b_{n}^{k_n}) \neq 0 \}$$

must be dense in $I^{(n)}$. For otherwise, there exist open subsets $B_i$ of $I$ $(1 \leq i \leq n)$ such that $p(b_{1}^{k_1} \cdots b_{n}^{k_n}) = 0$ whenever $(b_1, \ldots, b_n) \in \prod_{i=1}^{n} B_i$. Then we see that $p(b^k) = 0$ whenever $b \in I$. It then follows from Nagata-Higman’s theorem (see, for example, [2, Theorem 1.3.33]) that $I^{2^{k-1}} \subset \ker p$; contradicting the hypothesis.

Hence $U$ is dense in $I^{(m+n)}$ as claimed. \qed

The following theorem extends [1, Theorem 1.2]; in the commutative case, it is possible to extend the proof in [1] to yield the same result.

Theorem 2. Let $R$ be a radical Fréchet algebra. Suppose that there exists a non-nilpotent element $a \in R$ such that $a \in \overline{Ra}$. Then there exists a sequence $(a_n) \in R$ such that $\overline{Ra_n} = \overline{Ra}$ $(n \in \mathbb{N})$, and that, for each Fréchet algebra $A$ containing $R$ as a topological subalgebra,

$$a_{j_1}^{k_1} \cdots a_{j_n}^{k_n} \notin a_i A + \cdots + a_m A$$

for every finite sequences $i = (i_1, \ldots, i_m)$, $j = (j_1, \ldots, j_n)$, and $k = (k_1, \ldots, k_n)$ in $\mathbb{N}$ such that $i$ and $j$ are disjoint.

Proof. Set $I = \overline{Ra}$. We see that $0 \neq a^m \in I^{(m)}$ and

$$I^{(m)} = \overline{R \cdot I^{(m)}} \quad (m \in \mathbb{N}).$$

For each $k \in \mathbb{N}$, fix a continuous algebra seminorm $q_k$ of $R$ such that

$$I^{2^{k-1}} \not\subset \ker q_k.$$
Without loss of generality, we can assume that \((q_k)\) is an increasing sequence of seminorms defining the topology of \(R\). Denote by \(\Omega\) the product space \(\mathbb{R}^\Omega\); its topology is defined by a complete metric.

Let \(d\) be any complete metric defining the topology of \(R\). For each \(n \in \mathbb{N}\), set \(V_n = \{x \in I : d(a, vx) < 1/n \text{ for some } v \in R\}\). Then \(V_n\) is an open subset of \(I\). We see that \(a \in V_n\), and \(V_n\) is closed under multiplication on the left by elements in \(R^\# \setminus R\). Hence, \(V_n\) is dense in \(\overline{Ra} = I\).

For each \(m, n \in \mathbb{N}\), set

\[ V_{n,m} = \{(x_r) \in \Omega : x_r \in V_n (1 \leq r \leq m)\}. \]

From the previous paragraph, we see that \(V_{n,m}\) is an open dense subset of \(\Omega\).

For each \(s = (i, j, k)\) in \(\mathbb{N} \times \mathbb{N}^m \times \mathbb{N}^n \times \mathbb{N}^n\), with \(i\) and \(j\) being disjoint, let \(U_s\) be the set of all \((x_r)\) in \(\Omega\) with the property that

\[ l^2 \sum_{t=1}^m q_t(vx_{1t}x_{2t}) < q_k(vx_{1t} \cdots x_{ktn}) \]

for some \(v \in R^\#\), where \(k = k_1 + \ldots + k_n\). By Lemma 1, this is a dense (open) subset of \(\Omega\).

By the Baire category’s theorem, there exists \((a_r)\) belonging to all \(U_s\) and \(V_{n,m}\) above. Since \((a_r) \in V_{n,m}\), it follows that \(a \in \overline{Ra_r}\), and so \(\overline{Ra} = \overline{Ra_r}\) \((r \in \mathbb{N})\).

Let \(a\) be a Fréchet algebra containing \(R\) as a topological subalgebra. Let \(m, n \in \mathbb{N}\), and let \(i = (i_1, \ldots, i_m)\), \(j = (j_1, \ldots, j_n)\), and \(k = (k_1, \ldots, k_n)\) be finite sequences in \(\mathbb{N}\) such that \(i\) and \(j\) are disjoint. It remains to prove that \(a_{j_1}^{k_1} \cdots a_{j_n}^{k_n} \notin a_{i_1}A + \ldots + a_{i_m}A\).

Indeed, assume toward a contradiction that \(a_{j_1}^{k_1} \cdots a_{j_n}^{k_n} = a_{i_1}c_1 + \ldots + a_{i_m}c_m\) for some \(c_r \in A\). Set \(k = k_1 + \ldots + k_n\). The previous paragraph shows that there exists an element \(v_l \in R^\#\) such that

\[ \sum_{t=1}^m q_t(\alpha_{it}) < \frac{1}{l} \text{ and } q_k(v_l a_{j_1}^{k_1} \cdots a_{j_n}^{k_n}) > l \text{ } (l \in \mathbb{N}). \]

We then see that \(\lim_{l \to \infty} v_la_{i_t} = 0\), so \(\lim_{l \to \infty} v_l \sum_{t=1}^m a_{i_t}c_t = 0\), but \((v_l a_{j_1}^{k_1} \cdots a_{j_n}^{k_n} : l \in \mathbb{N})\) can never converge in \(R\) (and hence, can never converge to 0 in \(A\)); a contradiction. \(\square\)

We now present the construction due to Sierpinski of a family \(\{E_\alpha : \alpha \in \mathfrak{c}\}\) of infinite subsets of \(\mathbb{N}\) satisfying the following properties (cf. [6]):

(i) \(\mathbb{N} = \bigcup_{\alpha \in \mathfrak{c}} E_\alpha\), and
(ii) \(E_\alpha \cap E_\beta\) is finite for each \(\alpha \neq \beta \in \mathfrak{c}\).

The set \(\mathbb{N}\) is isomorphic to \(\mathbb{N} = \bigcup_{n=1}^{\infty} \{f : \{1, \ldots, n\} \to \{1, 2\}\}\).

For each \(f : \mathbb{N} \to \{1, 2\}\), define

\[ C_f = \{\text{the restrictions of } f \text{ to } \{1, \ldots, n\} : (n \in \mathbb{N})\}. \]
We see that \( C = \bigcup_{f \in \mathbb{N} - \{1, 2\}} C_f \) and that \( C_f \cap C_g \) is finite for each \( f \neq g \). We can then map back from \( C \) to \( \mathbb{N} \).

**Corollary 3.** Let \( R \) be a commutative radical Fréchet algebra. Suppose that there exists a non-nilpotent element \( a \in R \) such that \( a \in \text{Ra} \). Then there exists a family of prime ideals \( (P_\alpha : \alpha \in \mathcal{C}) \) in \( R \) such that \( P_\alpha \notin P_\beta \) (\( \alpha \neq \beta \in \mathcal{C} \)).

**Proof.** Let \( (a_n) \) be a sequence in \( R \) specified in the theorem. We then see that, for each \( E \subset \mathbb{N} \), there exists a prime ideal \( Q_E \) in \( R \) such that \( a_i \in Q_E \) (\( i \in E \)) but \( a_j \notin Q_E \) (\( j \notin E \)). Set \( P_\alpha = Q_{E_\alpha} \) (\( \alpha \in \mathcal{C} \)) where \( (E_\alpha : \alpha \in \mathcal{C}) \) is the Sierpinski’s family of subsets of \( \mathbb{N} \) constructed in the previous paragraph. Then \( (P_\alpha) \) is the desired collection of prime ideals in \( R \). \( \square \)

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**References**