UNCOUNTABLE FAMILIES OF PRIME $z$-IDEALS IN $C_0(\mathbb{R})$

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Abstract. Denote by $\mathfrak{c} = 2^{\aleph_0}$ the cardinal of continuum. We construct an intriguing family $(P_\alpha : \alpha \in \mathfrak{c})$ of prime $z$-ideals in $C_0(\mathbb{R})$ with the following properties:

• If $f \in P_{i_0}$ for some $i_0 \in \mathfrak{c}$, then $f \in P_i$ for all but finitely many $i \in \mathfrak{c}$;
• $\bigcap_{i \neq i_0} P_i \not\subset P_{i_0}$ for each $i_0 \in \mathfrak{c}$.

We also construct a well-ordered increasing chain, as well as a well-ordered decreasing chain, of order type $\kappa$ of prime $z$-ideals in $C_0(\mathbb{R})$ for any ordinal $\kappa$ of cardinality $\mathfrak{c}$.

1. Introduction

Let $\Omega$ be a locally compact space. In [5], we introduced the notion of pseudo-finite family of prime ideals as follows.

Definition 1.1. An indexed family $(P_i)_{i \in S}$ of prime ideals in $C_0(\Omega)$ is pseudo-finite if $f \in P_i$ for all but finitely many $i \in S$ whenever $f \in \bigcup_{i \in S} P_i$.

A pseudo-finite family $(P_i : i \in S)$ of prime ideals in $C_0(\Omega)$ has many interesting properties, for example, when $S$ is infinite, the union $\bigcup_{i \in S} P_i$ is again a prime ideal and any infinite subfamily of $(P_i)$ gives rise to the same union.

A pseudo-finite family of prime ideals $(P_i : i \in S)$ is said to be non-redundant if for every proper subset $T$ of $S$, $\bigcap_{i \in T} P_i \neq \bigcap_{i \in S} P_i$. Non-redundancy is equivalent to either of the following ([5, Lemma 3.4]):

(a) $P_\alpha \not\subset P_\beta$ ($\alpha \neq \beta \in S$);
(b) $\bigcap_{\beta \neq \alpha} P_\beta \not\subset P_\alpha$ for each $\alpha \in S$.

Note that (a) is apparently weaker, whereas (b) is apparently stronger than the non-redundancy. Thus, in this case, $\bigcap_{i \in S} P_i$ cannot be written as the intersection of less than $|S|$ prime ideals and $|S| \leq |C_0(\Omega)|$. Furthermore, for every pseudo-finite family of prime ideals, the subfamily consisting of those ideals that are minimal in the family is non-redundant and pseudo-finite and has the same intersection as the original family.

The notion of pseudo-finiteness has a connection with automatic continuity theory. It is proved in [5] that, assuming the Continuum Hypothesis, for each pseudo-finite family $(P_i : i \in S)$ of prime ideals in $C_0(\Omega)$ such that $|C_0(\Omega)/\bigcap_{i \in S} P_i| = \mathfrak{c}$, there exists a homomorphism from $C_0(\Omega)$ into a Banach algebra whose continuity ideal is $\bigcap_{i \in S} P_i$. Recall that the continuity ideal is the largest ideal of $C_0(\Omega)$ on which the homomorphism is continuous, and the continuity ideal as well as the kernel are always intersections of prime ideals in $C_0(\Omega)$ (see [1] for more details).

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Suppose that $\Omega$ is metrizable and that $\partial^{(\infty)}\Omega^{P} \neq \emptyset$; see §2 for the definition. Examples of such spaces include many countable locally compact spaces and all uncountable locally compact Polish spaces. For such $\Omega$, it is known that there exists an infinite non-redundant pseudo-finite sequence of prime ideals in $C_{0}(\Omega)$ ([5]). Here, we are going to show that there exists even a non-redundant pseudo-finite family $(P_{i} : i \in c)$ of prime ideals in $C_{0}(\Omega)$ (Theorem 3.9). As a consequence, assuming the Continuum Hypothesis, there exists a homomorphism from $C_{0}(\Omega)$ into a Banach algebra whose continuity ideal cannot be written as intersection of countably many prime ideals.

Note that when $\Omega$ is metrizable and $\partial^{(\infty)}\Omega^{P} = \emptyset$, then every non-redundant pseudo-finite families of prime ideals in $C_{0}(\Omega)$ is finite and the continuity ideal of every homomorphism from $C_{0}(\Omega)$ into a Banach algebra is the intersection of finitely many prime ideals. ([5])

In §4, we shall construct, for every ordinal $\kappa$ of cardinality at most $c$, a well-ordered decreasing chain of order type $\kappa$ of prime $z$-filters on any uncountable locally compact Polish space. In particular, combining with [3, Theorem 12.8], there exists a well-ordered decreasing chain of order type $\kappa$ of prime $z$-filters beginning with any non-minimal prime $z$-filters on $R$ (this was shown for $\kappa \leq \omega_{1}$, where $\omega_{1}$ is the first uncountable ordinal, in Theorems 8.5, 13.2 and Remark 13.2 of [3].) We also show that there are various countable compact subspaces of $R$ on which there is a well-ordered decreasing chain of order type $\epsilon$ of prime $z$-filters ($\epsilon$ is identified with the smallest ordinal of cardinality $\epsilon$).

In [3], it was asked whether there exists an uncountable well-ordered increasing chain of prime $z$-filters on $R$. We shall construct in §5, for every ordinal $\kappa$ of cardinality at most $\epsilon$, a well-ordered increasing chain of order type $\kappa$ of prime $z$-filters on any uncountable locally compact Polish space. We also construct well-ordered increasing chains of order type $\epsilon$ of prime $z$-filters on various countable compact subspaces of $R$.

All three constructions have as a common ingredient the result due to Sierpinski that $N$ can be expressed as the union of $\epsilon$ ”almost disjoint” infinite subsets.

2. Preliminary definitions and notations

For details of the theory of the algebras of continuous functions, see [2].

Let $\Omega$ be a locally compact space; our convention is that the topological spaces are always Hausdorff. The one-point compactification of $\Omega$ is denoted by $\Omega^{P}$.

For each prime ideal $P$ in $C_{0}(\Omega)$, either there exists a unique point $p \in \Omega$ such that $f(p) = 0$ ($f \in P$), in which case, we say that $P$ is supported at the point $p$; or otherwise, we say that $P$ is supported at the (point at) infinity.

It is an important fact that, for each prime ideal $P$ in $C_{0}(\Omega)$, the set of prime ideals in $C_{0}(\Omega)$ which contain $P$ is a chain with respect to the inclusion relation.

For each function $f$ continuous on $\Omega$, the zero set of $f$ is denoted by $Z(f)$. Define $Z[\Omega] = \{Z(f) : f \in C(\Omega)\}$. For each closed subset $Z \subset \Omega$, we have $Z = Z(f)$ for some function $f \in C_{0}(\Omega)$ if and only if $\Omega \setminus Z$ is $\sigma$-compact. An ideal $I$ of $C_{0}(\Omega)$ is a $z$-ideal if $g \in I$ whenever $g \in C_{0}(\Omega)$, $Z(g) \supset Z(f)$ and $f \in I$.

A $z$-filter $\mathcal{F}$ on $\Omega$ is a non-empty proper subset of $Z[\Omega]$ that is closed under finite intersection and supersets. Each $z$-filter $\mathcal{F}$ associates with the ideal $Z^{-1}[\mathcal{F}] = \{f \in C(\Omega) : Z(f) \in \mathcal{F}\}$ of $C(\Omega)$. 
A z-filter \( \mathcal{P} \) is a prime z-filter if \( Z_1 \cup Z_2 \notin \mathcal{P} \) whenever \( Z_1, Z_2 \in \mathbb{Z}[\Omega] \setminus \mathcal{P} \).

Let \( \mathcal{P} \) be a prime z-filter on \( \Omega \). Then we say that \( \mathcal{P} \) is supported at a point \( p \in \Omega \), if \( p \in Z \) for each \( Z \in \mathcal{P} \); if there exists no such \( p \), we say that \( \mathcal{P} \) is supported at the (point at) infinity. The support point of each prime z-filter \( \mathcal{P} \) coincides with the support point of the prime \( \mathcal{P} \setminus \mathcal{P}^{-1}[\mathcal{P}] \).

Let \( \Omega \) be a compact space. Define \( \partial^{(1)} \Omega = \partial \Omega \) to be the set of all limit points of \( \Omega \). Since \( \Omega \) is compact, \( \partial \Omega \) is non-empty unless \( \Omega \) is finite. We then define inductively a non-increasing sequence \( (\partial^{(n)} \Omega : n \in \mathbb{N}) \) of compact subsets of \( \Omega \) by setting \( \partial^{(n+1)} \Omega = \partial (\partial^{(n)} \Omega) \) for each \( n \in \mathbb{N} \). Set \( \partial^{(\infty)} \Omega = \bigcap_{n=1}^{\infty} \partial^{(n)} \Omega \). By the compactness, either \( \partial^{(\infty)} \Omega \) is non-empty or \( \partial^{(l)} \Omega \) is empty for some \( l \in \mathbb{N} \).

A Polish space is a separable completely metrizable space. Every separable metrizable locally compact space is a Polish space.

3. Pseudo-finite families of prime ideals and prime z-filters

Let \( \Omega \) be a locally compact space. First, we shall make a connection between pseudo-finite families of prime ideals and pseudo-finite families of prime \( \mathcal{P} \)-ideals in \( \mathcal{C}_0(\Omega) \). For each closed subset \( E \) of \( \Omega \), we define the \( \mathcal{P} \)-ideal

\[
K_E = \{ f \in \mathcal{C}_0(\Omega) : E \subseteq \mathbb{Z}(f) \}
\]

The following is [4, 3.3 and 3.4], we shall give here a combined proof.

**Lemma 3.1.** [4] Let \( I \) be an ideal in \( \mathcal{C}_0(\Omega) \). Then

\[
I^2 = \bigcup \{ K_E : E \text{ is closed in } \Omega \text{ and } K_E \subseteq I \}
\]

is the largest \( \mathcal{P} \)-ideal contained in \( I \). If \( I \) is a prime ideal then so is \( I^2 \).

**Proof.** It is easy to see that \( I^2 \) contains every \( \mathcal{P} \)-ideal contained in \( I \). Since the sum of two \( \mathcal{P} \)-ideals is again a \( \mathcal{P} \)-ideal, we see that

\[
I^2 = \sum \{ K_E : E \text{ is closed in } \Omega \text{ and } K_E \subseteq I \}
\]

where the sum is algebraic. Thus \( I^2 \) is the largest \( \mathcal{P} \)-ideal contained in \( I \).

Now, suppose that \( I \) is prime. Let \( f_1, f_2 \in \mathcal{C}_0(\Omega) \setminus I^2 \). Then there exists \( g_1, g_2 \in \mathcal{C}_0(\Omega) \setminus I \) such that \( \mathbb{Z}(g_1) \supset \mathbb{Z}(f_1) \). Then \( \mathbb{Z}(g_1 g_2) \supset \mathbb{Z}(f_1 f_2) \). The primeness of \( I \) implies that \( g_1 g_2 \notin I \). So \( f_1 g_2 \notin I^2 \). \( \square \)

The following strengthens the implication (a)⇒(c) of [5, Lemma 8.4].

**Proposition 3.2.** Let \( \Omega \) be a locally compact space. Let \( \{ P_i : i \in S \} \) be an infinite non-redundant pseudo-finite family of prime ideals in \( \mathcal{C}_0(\Omega) \). Then \( P = \bigcup_{i \in S} P_i \) is a prime \( \mathcal{P} \)-ideal, and \( \{ P^*_i : i \in S \} \) is a non-redundant pseudo-finite family of prime \( \mathcal{P} \)-ideals whose union is \( P \) such that \( P^*_i \subseteq P \) (i \( \in S \)).

**Proof.** We shall need another theorem of [4] which say that the sum of two non-comparable prime ideals in \( \mathcal{C}_0(\Omega) \) is indeed a prime \( \mathcal{P} \)-ideal ([4, 3.2]).

We know that \( P \) must be a prime ideal. Assume toward a contradiction that \( P \) is not a prime \( \mathcal{P} \)-ideal. Choose \( \alpha_1 \neq \alpha_2 \in S \) arbitrary. Then \( P_{\alpha_1} + P_{\alpha_2} \) is a prime \( \mathcal{P} \)-ideal. Suppose that we already have distinct indices \( \alpha_1, \ldots, \alpha_n \in S \) such that \( \sum_{i=1}^n P_{\alpha_i} \) is a prime \( \mathcal{P} \)-ideal. Then \( P \neq \sum_{i=1}^n P_{\alpha_i} \), and so we can find \( \alpha_{n+1} \in S \)
such that \( P_{\alpha_{n+1}} \not\in \sum_{i=1}^{n} P_{\alpha_i} \). The induction can be continued. However, this gives a contradiction since then

\[
P = \bigcup_{n=1}^{\infty} P_{\alpha_n} = \bigcup_{n=1}^{\infty} \sum_{i=1}^{n} P_{\alpha_i}
\]
is a \( z \)-ideal. Hence, \( P \) is a prime \( z \)-ideal.

We claim that \( (P_i^\alpha : i \in S) \) is a pseudo-finite family with union \( P \). Indeed, assume toward a contradiction that there exists \( f \in P \) and distinct \( \alpha_n \in S \) (\( n \in \mathbb{N} \)) such that \( f \not\in P_{\alpha_n}^\alpha \) (\( n \in \mathbb{N} \)). For each \( n \), we can then find \( f_n \not\in P_{\alpha_n} \) such that \( \mathcal{Z}(f_n) \supset \mathcal{Z}(f) \); we can further assume that \( 0 \leq f_n \leq 2^{-n} \). Define \( f_* = \sum_{n=1}^{\infty} f_n \).

Then we see that \( f_n \leq f_* \) so \( f_* \not\in P_{\alpha_n} \) (\( n \in \mathbb{N} \)), and that \( \mathcal{Z}(f_n) \supset \mathcal{Z}(f) \) so \( f_* \in P \).

This is a contradiction to the pseudo-finiteness of \( (P_i : i \in S) \).

It remains to prove the non-redundancy of \( (P_i^\alpha : i \in S) \). So, assume that \( P_\alpha^\beta \subset P_\alpha^\gamma \) for some \( \alpha \neq \beta \in S \). Then \( P_\alpha^\beta \) is contained in both \( P_\alpha \) and \( P_\beta \), and so \( P_\alpha \) and \( P_\beta \) are in a chain. This contradicts the non-redundancy of \( (P_i : i \in S) \).

Conversely, it is obvious that if \( (Q_i : i \in S) \) is a pseudo-finite family of prime \( (z)\)-ideals and \( P_i \) is a prime ideal containing \( Q_i \), then \( (P_i : i \in S) \) is a pseudo-finite family of prime ideals.

We define a similar notion of pseudo-finite families of prime \( z \)-filters.

**Definition 3.3.** An indexed family \( (P_i)_{i \in S} \) of prime \( z \)-filters \( \Omega \) is pseudo-finite if \( Z \in P_i \) for all but finitely many \( i \in S \) whenever \( Z \in \bigcup_{i \in S} P_i \).

A pseudo-finite family of prime \( z \)-filters \( (P_i : i \in S) \) is said to be non-redundant if for every proper subset \( T \) of \( S \), \( \bigcap_{i \in T} P_i \neq \bigcap_{i \in S} P_i \). Similar to [5, Lemma 3.4] we have the following.

**Lemma 3.4.** Let \( (P_\alpha : \alpha \in S) \) be a pseudo-finite family of prime \( z \)-filters on \( \Omega \). Then the following are equivalent:

(a) \( (P_\alpha) \) is non-redundant;

(b) \( P_\alpha \not\subset P_\beta \) (\( \beta \neq \alpha \in S \));

(c) \( \bigcap_{\beta \neq \alpha} P_\beta \not\subset P_\alpha \) for each \( \alpha \in S \).

**Proof.** Obviously, (c) \( \Rightarrow \) (b).

We now prove (b) \( \Rightarrow \) (a). Fix \( \alpha \in S \). By condition (b), \( P_\beta \not\subset P_\alpha \) (\( \beta \neq \alpha \in S \)). Choose \( Z_0 \in P_\beta \setminus P_\alpha \) for some \( \beta \in S \setminus \{ \alpha \} \). Then, by the pseudo-finiteness, we have \( Z_0 \in P_\beta \) for all but finitely many \( \beta \in S \). Let \( \beta_1, \ldots, \beta_n \) be those indices \( \beta \in S \setminus \{ \alpha \} \) such that \( Z_0 \not\in P_\beta \). For each \( 1 \leq k \leq n \), choose \( Z_k \in P_{\beta_k} \setminus P_\alpha \), and set \( Z = \bigcup_{k=0}^{n} Z_k \). Then \( Z \in P_\beta \) (\( \beta \in S \setminus \{ \alpha \} \)), but \( Z \not\in P_\alpha \), by the primeness of \( P_\alpha \). Thus (c) holds.

The following definition and proposition are adapted from [5].

**Definition 3.5.** Let \( \Omega \) be a locally compact space, and let \( S \) be a non-empty index set. Let \( \mathcal{F} \) be a \( z \)-filter on \( \Omega \), and let \( (Z_\alpha : \alpha \in S) \) be a sequence of zero sets on \( \Omega \).

Then \( \mathcal{F} \) is extendible with respect to \( (Z_\alpha : \alpha \in S) \) if both the following conditions hold:

(a) \( Z_\alpha \not\in \mathcal{F} \), and \( Z_\alpha \cup Z_\beta \in \mathcal{F} \) (\( \alpha \neq \beta \in S \));

(b) for each \( Z \in \mathcal{Z}(\Omega) \), if \( Z \cup Z_\alpha \in \mathcal{F} \) for some \( \alpha \in S \), then \( Z \cup Z_\alpha \in \mathcal{F} \) for all except finitely many \( \alpha \in S \).
Proposition 3.6. Let $\Omega$ be a locally compact space. Suppose that there exist a $\beta$-filter $\mathcal{F}$ and a family $(Z_\alpha : \alpha \in S)$ in $\mathbb{Z}\Omega$ such that $\mathcal{F}$ is extendible with respect to $(Z_\alpha : \alpha \in S)$. Then there exists a pseudo-finite family of prime $\beta$-filters $(\mathcal{P}_\alpha : \alpha \in S)$ such that $Z_\alpha \in \bigcap_{\gamma \neq \alpha} \mathcal{P}_\gamma \setminus \mathcal{P}_\alpha$ for each $\alpha \in S$.

Proof. We see that the union of a chain of $\beta$-filters, each of which contains $\mathcal{F}$ and is extendible with respect to $(Z_\alpha)$, is also extendible with respect to $(Z_\alpha)$. Thus, by Zorn’s lemma, we can suppose that $\mathcal{F}$ is a maximal one among those $\beta$-filters.

For each $\alpha$, set $\mathcal{F}_\alpha = \{Z \in \mathbb{Z}\Omega : Z \cup Z_\alpha \in \mathcal{F}\}$, and set $\mathcal{P} = \bigcup_{\alpha \in S} \mathcal{F}_\alpha$. By the extensibility of $\mathcal{F}$, we see that whenever $Z \in \mathcal{P}$ then $Z \in \mathcal{F}_\alpha$ for all except finitely many $\alpha \in S$. Thus, in particular, the set $\mathcal{P}$ is actually a $\beta$-filter.

Claim 1: For each $Z_0 \in \mathbb{Z}\Omega \setminus \mathcal{P}$, we have $\{Z \in \mathbb{Z}\Omega : Z \cup Z_0 \in \mathcal{F}\} = \mathcal{F}$. Indeed, we see that $Z_\alpha \notin \mathcal{G} = \{Z \in \mathbb{Z}\Omega : Z \cup Z_0 \in \mathcal{F}\}$ ($\alpha \in S$); for otherwise, $Z_0$ would be in $\mathcal{P}$. It then follows easily that $\mathcal{G}$ is extendible with respect to $(Z_\alpha)$. This and the maximality of $\mathcal{F}$ imply the claim.

Claim 2: $\mathcal{P}$ is a prime $\beta$-filter. We have to prove that, whenever $Z_1, Z_2 \in \mathbb{Z}\Omega$ are such that $Z_1 \cup Z_2 \in \mathcal{P}$, but $Z_1 \notin \mathcal{P}$, then $Z_2 \notin \mathcal{P}$. Indeed, let $\alpha \in S$ be such that $Z_\alpha \cup Z_1 \cup Z_2 \in \mathcal{F}$. Then $Z_\alpha \cup Z_2 \in \mathcal{F}$, by the first claim, and so $Z_2 \in \mathcal{F}_\alpha$.

Now, for each $\alpha \in S$, define
$$D_\alpha = \{Z_\alpha \cup Z : Z \in \mathbb{Z}\Omega \setminus \mathcal{P}\}.$$ Then, by Claim 2, the set $D_\alpha$ is closed under finite union. Obviously, $D_\alpha \cap \mathcal{F}_\alpha = \emptyset$. Thus, there exists a prime $\beta$-filter $\mathcal{P}_\alpha$ containing $\mathcal{F}_\alpha$ such that $D_\alpha \cap \mathcal{P}_\alpha = \emptyset$.

We see that $\mathcal{F}_\alpha \subset \mathcal{P}_\alpha \subset \mathcal{P}$ and $Z_\alpha \notin \mathcal{P}_\alpha$ ($\alpha \in S$). The result then follows. $\Box$

We now define a “prototype” space $\Xi$. Denote by $\aleph_1$ the point adjacent to $\aleph_0$ to obtain its one-point compactification $\aleph_0^\ast$. The product space $(\aleph_0^\ast)^\mathbb{N}$ is a compact metrizable space. Define $\Xi$ to be the compact subset of $(\aleph_0^\ast)^\mathbb{N}$ consisting of all elements $(n_1, n_2, \ldots)$ with the property that there exists $k \in \aleph_0$ such that $n_i \geq k$ $(1 \leq i \leq k)$ and such that $n_i = \aleph_0$ $(i > k)$. The convention is that $\aleph_0 > n$ ($n \in \aleph_0$).

Lemma 3.7. [5, Lemma 9.2] Let $\Omega$ be a locally compact metrizable space. Suppose that there exists a point $p \in \partial(\aleph_0^\ast)(\Omega^\ast)$. Then there exists a homeomorphic embedding $\iota$ of all $\aleph_0$ onto a closed subset of $\Omega^\ast$ such that $\iota(\aleph_0, \aleph_0, \ldots) = p$.

A key to our construction is the result due to Sierpinski that there exists a family $(E_\alpha : \alpha \in \varepsilon)$ of infinite subsets of $\aleph_0$ satisfying the following properties:

(i) $\aleph_0 = \bigcup_{\alpha \in \varepsilon} E_\alpha$, and
(ii) $E_\alpha \cap E_\beta$ is finite for each $\alpha \neq \beta \in \varepsilon$.

We sketch the nice construction of such family as follows (cf. [7]): The set $\aleph_0$ is isomorphic to
$$C = \bigcup_{n=1}^{\aleph_0} \{f : \{1, \ldots, n\} \to \{1, 2\}\}.$$ For each $f : \aleph_0 \to \{1, 2\}$, define
$$C_f = \{\text{the restrictions of } f \text{ to } \{1, \ldots, n\} : (n \in \aleph_0)\}.$$ We see that $C = \bigcup_{f \in \aleph_0 \setminus \{1, 2\}} C_f$ and that $C_f \cap C_g$ is finite for each $f \neq g$. We can then map back from $C$ to $\aleph_0$. Inspecting the construction, we see that $(E_\alpha : \alpha \in \varepsilon)$ enjoys the following property:

(i') The cardinality of $\{\alpha \in \varepsilon : n \in E_\alpha\}$ is $\varepsilon$ for each $n \in \aleph_0$. 
Lemma 3.8. There exists a non-redundant pseudo-finite family \( \{Q_\alpha : \alpha \in \mathfrak{c}\} \) of prime \( z \)-filters on \( \Xi \) such that each \( z \)-filter is supported at the point \( (\infty, \infty, \ldots) \).

Proof. Let \( \{E_\alpha : \alpha \in \mathfrak{c}\} \) be the family of infinite subsets of \( \mathbb{N} \) as in the previous paragraph. For each \( \alpha \in \mathfrak{c} \), define
\[
N_\alpha = \{(j_1, j_2, \ldots) \in \Xi : j_n = \infty (n \in E_\alpha)\}.
\]
Let \( F \) to be the \( z \)-filter generated by all \( N_\alpha \cup N_\beta (\alpha, \beta \in \mathfrak{c}, \alpha \neq \beta) \). We claim that \( F \) is extendible with respect to \( (N_\alpha : \alpha \in \mathfrak{c}) \); the proof will then be completed by applying Lemma 3.6.

Obviously, \( N_\alpha \cup N_\beta \in F (\alpha \neq \beta) \). We claim that \( N_\alpha \notin F (\alpha \in \mathfrak{c}) \). Indeed, assume the contrary. Then there exist \( \gamma_1, \ldots, \gamma_m \in \mathfrak{c} \setminus \{\alpha\} \) such that \( N_\alpha \supset \bigcap_{i=1}^m N_{\gamma_i} \). Since \( E_\alpha \) is infinite whereas each \( E_\alpha \cap E_{\gamma_i} \) is finite, there exists \( l \in E_\alpha \setminus \bigcup_{i=1}^m E_{\gamma_i} \). We see that \( (j_i) \in \bigcap_{i=1}^m N_{\gamma_i} \setminus N_\alpha \) where \( j_i = l \) and \( j_i = \infty (i \neq l) \); a contradiction.

Finally, suppose that \( N \in \mathsf{Z}[\Xi] \) such that \( N \cup N_\alpha \in F \) for some \( \alpha \in \mathfrak{c} \). Then, there exist \( \gamma_1, \ldots, \gamma_m \in \mathfrak{c} \setminus \{\alpha\} \) such that
\[
N \cup N_\alpha \supset \bigcap_{i=1}^m N_{\gamma_i}, \quad \text{and so } N \supset \bigcap_{i=1}^m N_{\gamma_i} \setminus N_\alpha.
\]
As above, there exists \( l \in E_\alpha \setminus \bigcup_{i=1}^m E_{\gamma_i} \). We can then choose \( \gamma_{m+1}, \ldots, \gamma_n \in \mathfrak{c} \) such that
\[
\{1, \ldots, l\} \subset \bigcup_{i=1}^n E_{\gamma_i}.
\]
We claim that \( N \supset \bigcap_{i=1}^n N_{\gamma_i} \). Indeed, let \( (j_i) \in \bigcap_{i=1}^n N_{\gamma_i} \). Then \( j_1 = \cdots = j_l = \infty \). We see that there exists \( k > \infty \) such that \( j_i \geq k \) \((1 \leq i \leq k)\) and \( j_i = \infty \) \((i > k)\). For each \( r \in \mathbb{N} \), set \( j^{(r)}_i = j_i \) \((i \neq l)\) and set \( j^{(r)}_i = k + r \). Then, we see that \( (j^{(r)}_i) \in \bigcap_{i=1}^n N_{\gamma_i} \setminus N_\alpha \). Since \( \lim_r (j^{(r)}_i) = (j_i) \). Thus \( (j_i) \in N \). Hence, for each \( \beta \in \mathfrak{c} \setminus \{\gamma_1, \ldots, \gamma_n\} \), we have \( N \cup N_\beta \in F \).

Theorem 3.9. Let \( \Omega \) be a locally compact metrizable space. Suppose that \( p \in \partial(\infty)[\Omega^p] \). Then there exists a non-redundant pseudo-finite family \( \{P_\alpha : \alpha \in \mathfrak{c}\} \) of prime \( z \)-filters on \( \Omega \), each \( z \)-filter is supported at \( p \).

Moreover, by setting \( P_\alpha = C_0(\Omega) \cap \mathsf{Z}^{-1}[P_\alpha] \), we obtain a non-redundant pseudo-finite family of prime \( z \)-ideals in \( C_0(\Omega) \), each ideal is supported at \( p \), such that
\[
\left|C_0(\Omega) \left/ \bigcap_{\alpha \in \mathfrak{c}} P_\alpha \right. \right| = \mathfrak{c}.
\]

Proof. In this proof, we shall identify \( \Xi \) with a closed subset of \( \Omega^\mathfrak{c} \) such that \( (\infty, \infty, \ldots) \) is identified with \( p \); in the case where \( p \in \Omega \), we can further assume that \( \Xi \subset \Omega \) (cf. Lemmas 3.7).

Let \( \{Q_\alpha : \alpha \in \mathfrak{c}\} \) be the family of prime \( z \)-filters on \( \Xi \) as constructed in Lemma 3.8. For each \( \alpha \in \mathfrak{c} \), set
\[
P_\alpha = \{Z \in \mathsf{Z}[\Xi] : (Z \cup \{p\}) \cap \Xi \in Q_\alpha\}.
\]
Note that every closed subset of \( \Xi \) is in \( \mathsf{Z}[\Xi] \), so we can see that each \( P_\alpha \) is a prime \( z \)-filter on \( \Omega \). The pseudo-finiteness of \( \{P_\alpha : \alpha \in \mathfrak{c}\} \) and of \( \{P_\alpha : \alpha \in \mathfrak{c}\} \) then follows from that of \( \{Q_\alpha\} \). The cardinality condition follows from the fact that \( |\mathsf{C}(\Xi)| = \mathfrak{c} \).
By the non-redundancy of \((Q_\alpha : \alpha \in \mathfrak{c})\), for each \(\alpha \in \mathfrak{c}\), there exists
\[ N_\alpha \in \bigcap_{\beta \neq \alpha} Q_\beta \setminus Q_\alpha. \]

By the Uryson’s lemma, we can find \(Z_\alpha \in \mathbb{Z}[\Omega]\) such that \((Z_\alpha \cup \{p\}) \cap \Xi = \alpha\); we can even require \(\Omega \setminus Z_\alpha\) to be \(\sigma\)-compact so that \(Z_\alpha = \mathbb{Z}(f_\alpha)\) for some \(f_\alpha \in C_0(\Omega)\). Thus we see that
\[ Z_\alpha \in \bigcap_{\beta \in \mathfrak{c}, \beta \neq \alpha} P_\beta \setminus P_\alpha \quad \text{and} \quad f_\alpha \in \bigcap_{\beta \in \mathfrak{c}, \beta \neq \alpha} P_\beta \setminus P_\alpha. \]

Finally, we shall prove that each \(P_\alpha\) (and hence each \(P_\alpha\)) is supported at \(p\) (\(\alpha \in \mathfrak{c}\)). Indeed, in the case where \(p\) is the point at infinity of \(\Omega\), for each \(x \in \Omega\), there exists \(N \in \mathfrak{c}\) such that \(x \notin N\). We can then find \(Z \in \mathbb{Z}[\Omega]\) such that \(x \notin Z\) and that \((Z \cup \{p\}) \cap \Xi = N\). Thus \(Z \in P_\alpha\) and \(x \notin Z\). So \(P_\alpha\) is support at infinity. On the other hand, in the case where \(p \in \Omega\), let \(Z \in \mathfrak{c}\) be arbitrary. Then \(Z \cap \Xi\) is closed in \(\Xi\), and so it is in \(\mathbb{Z}[\Xi]\). Since \(\{p\} \in \mathbb{Z}[\Xi] \setminus Q_\alpha\), we deduce that \(Z \cap \Xi \in Q_\alpha\). Hence, \(p \in Z\), and thus \(P_\alpha\) is supported at \(p\). □

Combining the above with the result in [5], we have the following.

**Corollary 3.10.** Assuming the Continuum Hypothesis. Let \(\Omega\) be a locally compact metrizable space.

(i) Suppose that \(\partial^{(\infty)}(\Omega^\mathfrak{c}) \neq \emptyset\). Then there exists a homomorphism from \(C_0(\Omega)\) into a Banach algebra whose continuity ideal is not the intersection of any countable family of prime ideals.

(ii) Suppose that the infinity point belongs to \(\partial^{(\infty)}(\Omega^\mathfrak{c})\). Then there exists a homomorphism \(C_0(\Omega)\) into a radical Banach algebra whose kernel is not the intersection of any countable family of prime ideals. □

**Corollary 3.11.** Let \(p \in \mathbb{R}^\mathfrak{c}\). There exists a family \((P_\alpha : \alpha \in \mathfrak{c})\) of prime \(z\)-ideals in \(C_0(\mathbb{R})\), each ideal is supported at \(p\), with the following properties:

- If \(f \in P_\alpha\) for some \(\alpha_0 \in \mathfrak{c}\), then \(f \in P_\alpha\) for all but finitely many \(\alpha \in \mathfrak{c}\);
- \(\bigcap_{\alpha \neq \alpha_0} P_\alpha \subsetneq P_{\alpha_0}\) for each \(\alpha_0 \in \mathfrak{c}\). □

There are many countable compact metrizable spaces \(\Omega\) with \(\partial^{(\infty)}\Omega \neq \emptyset\). We note as a specific example the following countable compact subset of \([0, 1]\):

\[ \Delta = \{0\} \cup \left\{ \sum_{i=1}^{k} 2^{-n_i} : k, n_1, n_2, \ldots, n_k \in \mathbb{N} \text{ and } k \leq n_1 < \cdots < n_k \right\}. \]

**Corollary 3.12.** There exists a family \((P_\alpha : \alpha \in \mathfrak{c})\) of non-modular prime \(z\)-ideals in \(C_0(\Delta \setminus \{0\})\) with the following properties:

- If \(f \in P_{\alpha_0}\) for some \(\alpha_0 \in \mathfrak{c}\), then \(f \in P_\alpha\) for all but finitely many \(\alpha \in \mathfrak{c}\);
- \(\bigcap_{\alpha \neq \alpha_0} P_\alpha \subsetneq P_{\alpha_0}\) for each \(\alpha_0 \in \mathfrak{c}\). □

4. Well-ordered Decreasing Chains of Prime Ideals and Prime \(z\)-Filters

Let \(\Omega\) be a metrizable locally compact space. If \(\partial^{(n)}\Omega^\mathfrak{c} = \emptyset\) for some \(n \in \mathbb{N}\), then it can be seen that every chain of prime \(z\)-ideals in \(C_0(\Omega)\) or prime \(z\)-filters on \(\Omega\) has length at most \(n\). Hence, in this section we shall suppose that \(\partial^{(\infty)}\Omega^\mathfrak{c} \neq \emptyset\).
In the following, $\Xi$ is the compact subset of $(\mathbb{N}^p)^\mathbb{N}$ defined in the previous section. Also, our convention is that $\max\emptyset$ is smaller and $\min\emptyset$ is bigger than everything, and that $\bigcap_{\alpha \in \emptyset} N_\alpha$ is the whole space (i.e. $\Xi$ in the next lemma) and $\bigcup_{\alpha \in \emptyset} N_\alpha = \emptyset$.

**Lemma 4.1.** There exists a family $(N_\alpha)_{\alpha \in \mathfrak{c}}$ of zero sets on $\Xi$ satisfying that, for every $\gamma \in \mathfrak{c}$ and disjoint finite subsets $F$ and $G$ of $\mathfrak{c}$, we can find a finite subset $H$ of $\mathfrak{c}$ with the properties that $\gamma \leq \min H$ and that

$$\bigcap_{\beta \in G} N_\beta \setminus \left( \bigcup_{\alpha \in F} N_\alpha \right) \supset \bigcap_{\beta \in G \cup H} N_\beta.$$  

*Proof*. Recall from the previous section that there exists a family $(E_\alpha : \alpha \in \kappa)$ of infinite subsets of $\mathbb{N}$ satisfying:

(a) $E_\alpha \cap E_\beta$ is finite for each $\alpha \neq \beta \in \kappa$, and

(b) the cardinality of $\{\alpha \in \mathfrak{c} : n \in E_\alpha\}$ is $\mathfrak{c}$ ($n \in \mathbb{N}$).

Similar to Lemma 3.8, we define, for each $\alpha \in \mathfrak{c}$,

$$N_\alpha = \{(j_1, j_2, \ldots) \in \Xi : j_n = \infty (n \in E_\alpha)\}.$$  

Let $G$ be a finite subset of $\kappa$ and let $\alpha, \gamma \in \mathfrak{c}$. Then, there exists $l \in E_\alpha \setminus \bigcup_{\beta \in G} E_\beta$. Since the cardinality of $\{\beta : \beta < \gamma\}$ is less than $\mathfrak{c}$, by (b) above, we can find a finite subset $H$ of $\mathfrak{c}$ such that $\gamma \leq \min H$ and that

$$\{1, \ldots, l\} \subset \bigcup_{\beta \in G \cup H} E_\beta.$$  

Then, similar to Lemma 3.8, we see that

$$\bigcap_{\beta \in G \setminus N_\alpha} \supset \bigcap_{\beta \in G \cup H} N_\beta.$$  

The general case follows by induction. \hfill \Box

**Theorem 4.2.** Let $\Omega$ be a metrizable locally compact space, and let $p \in \partial(\mathfrak{c}, \Omega)$. Then there exists a well-ordered decreasing chain $(Q_\alpha : \alpha \in \mathfrak{c})$ of prime z-filters on $\Omega$ each supported at $p$.

Furthermore, by setting $Q_\alpha = C_0(\Omega) \cap \mathbb{Z}^{-1}[Q_\alpha]$, we obtain a well-ordered decreasing chain $(Q_\alpha : \alpha \in \mathfrak{c})$ of prime z-ideals in $C_0(\Omega)$ each supported at $p$.

*Proof*. Similar to Theorem 3.9, we shall identify $\Xi$ with a closed subset of $\Omega^\mathfrak{c}$ such that $(\mathfrak{c}, \infty, \ldots)$ is identified with $p$; in the case where $p \in \Omega$, we can further assume that $\Xi \subset \Omega$ (cf. Lemmas 3.7).

Let $(N_\alpha : \alpha \in \mathfrak{c})$ be the family of zero sets on $\Xi$ as constructed in Lemma 4.1. For each $\alpha \in \mathfrak{c}$, choose $Z_\alpha = \mathbb{Z}(f_\alpha)$ for some $f_\alpha \in C_0(\Omega)$ such that $(Z_\alpha \cup \{p\}) \cap \Xi = N_\alpha$. Also, define

$$\mathcal{F}_\alpha = \left\{ Z \in \mathbb{Z}[\Omega] : Z \cup \{p\} \supset \bigcap_{i=1}^n N_{\beta_i} \text{ for some } \alpha \leq \beta_1, \ldots, \beta_n \in \mathfrak{c} \right\}.$$  

Then $(\mathcal{F}_\alpha)$ is a decreasing $\mathfrak{c}$-sequence of z-filters on $\Omega$; $Z_\alpha \in \mathcal{F}_\alpha$ but $Z_\alpha \notin \mathcal{F}_\beta$ ($\alpha < \beta \in \mathfrak{c}$).

Set

$$\mathcal{D}_\alpha = \{ Z \in \mathbb{Z}[\Omega] : (Z \cup \{p\}) \cap \Xi = \{p\} \}.$$  

Then $\mathcal{D}_\alpha$ is closed under taking finite union. Also, since $\mathcal{D}_\alpha \cap \mathcal{F}_0 = \emptyset$, there exists a prime z-filter $Q_0$ on $\Omega$ containing $\mathcal{F}_0$ such that $Q_0 \cap \mathcal{D}_\alpha = \emptyset$. Let $\gamma \in \mathfrak{c}$. Suppose
that we have already constructed a well-ordered decreasing chain \((Q_\alpha : \alpha < \gamma)\) of prime \(z\)-filters on \(\Omega\) such that \(\mathcal{F}_\alpha \subset Q_\alpha\) \((\alpha < \gamma)\). If \(\gamma\) is a limit ordinal, set \(Q_\gamma = \bigcap_{\alpha < \gamma} Q_\alpha\). Consider now the case where \(\gamma = \alpha + 1\) for some \(\alpha\). Set 
\[D_\gamma = \{Z \cup Z_\alpha : Z \in \mathbb{Z}[\Omega] \setminus Q_\alpha\}.
\]
Then \(D_\gamma\) is closed under taking finite union. Also, we have \(\mathcal{F}_\gamma \cap D_\gamma = \emptyset\); since otherwise, there exist \(Z \in \mathbb{Z}[\Omega] \setminus Q_\alpha\) and a finite subset \(G\) of \(\gamma\) such that \(\gamma \leq \min G\) and that 
\[Z \cup Z_\alpha \cup \{p\} \supseteq \bigcap_{\beta \in G} N_\beta\]
which implies that 
\[Z \cup \{p\} \supseteq \bigcap_{\beta \in G} N_\beta \setminus N_\alpha \supseteq \bigcap_{\beta \in H} N_\beta\]
for some finite subset \(H\) of \(\epsilon\) with \(\gamma \leq \min H\), by Lemma 4.1, or \(Z \in \mathcal{F}_\gamma\) \(\subset Q_\alpha\) a contradiction. Therefore, there exists a prime \(z\)-filter \(Q_\gamma\) such that \(\mathcal{F}_\gamma \subset Q_\gamma\) and \(Q_\gamma \cap D_\gamma = \emptyset\). We see that, in this case, \(Q_\gamma \not\subseteq Q_\alpha\) and \(Z_\alpha \not\in Q_\gamma\). Thus, in both cases, the construction can be continued inductively.

Setting \(Q_\alpha = C_0(\Omega) \cap \mathbb{Z}^{-1}[Q_\alpha]\). Then \(f_* \not\in Q_0\) for \(f_* \in C_0(\Omega)\) such that \(Z(f) \in D_*\), and, for each \(\gamma = \alpha + 1 \in \epsilon\), we have \(f_\alpha \in Q_\alpha \setminus Q_\gamma\). It follows that the chain \((Q_\alpha : \alpha \in \epsilon)\) is decreasing.

The statement on support point follows from the fact that \(\bigcap_{\alpha \in \epsilon} N_\alpha = \{p\}\). \(\square\)

It was proved in [3, Theorem 13.2] that starting from any non-minimal prime \(z\)-filter containing a countable zero set on \(\mathbb{R}\) there exists a well-ordered decreasing full \(\omega_1\)-sequence of prime \(z\)-filters such that each prime \(z\)-filter contains a countable zero set. However, besides that \(\omega_1 < \epsilon\) in the absence of the Continuum Hypothesis, the union of those countable zero sets are not countable, and thus that \(\omega_1\)-sequence says nothing about uncountable chains of prime \(z\)-filters on countable spaces.

**Corollary 4.3.** Let \(\Delta\) be any countable compact subset of \(\mathbb{R}\) such that \(\partial^{(\infty)} \Delta \neq \emptyset\). There exists a well-ordered decreasing chain of order type \(\epsilon\) of prime \(z\)-filters on \(\mathbb{R}\) such that each prime \(z\)-filter contains \(\Delta\). \(\square\)

We now look for longer chains. We shall need to restrict to uncountable locally compact Polish spaces. Note that for any well-ordered decreasing chain of order type \(\kappa\) of prime \(z\)-filters on \(\Omega\) or prime ideals in \(C_0(\Omega)\), where \(\Omega\) is in addition \(\sigma\)-compact, \(\kappa\) must have cardinality at most \(\epsilon\).

**Lemma 4.4.** Let \(\kappa\) be an ordinal of cardinality \(\epsilon\). There exists a family \((N_\alpha)_{\alpha \in \kappa}\) of zero sets on \((\mathbb{N}^\omega)^\mathbb{R}\) such that for every disjoint finite subsets \(F\) and \(G\) of \(\kappa\), we have 
\[\bigcap_{\alpha \in F} N_\alpha \setminus \left( \bigcup_{\beta \in G} N_\beta \right) = \bigcap_{\alpha \in F} N_\alpha.
\]

*Proof.* Similar to (but simpler than) that of Lemma 4.1. \(\square\)

**Theorem 4.5.** Let \(\Omega\) be an uncountable locally compact Polish space. Let \(\kappa\) be an ordinal of cardinality \(\epsilon\). Then there exists a well-ordered decreasing chain \((Q_\alpha : \alpha \in \kappa)\) of prime \(z\)-filters on \(\Omega\).

Furthermore, by setting \(Q_\alpha = C_0(\Omega) \cap \mathbb{Z}^{-1}[Q_\alpha]\), we obtain a well-ordered decreasing chain \((Q_\alpha : \alpha \in \kappa)\) of prime \(z\)-ideals in \(C_0(\Omega)\).

*Proof.* Every uncountable Polish space contains a closed subsets homeomorphic to the Cantor space \(\{0, 1\}^\mathbb{N}\), which in turn contains a copy of \((\mathbb{N}^\omega)^\mathbb{R}\). Thus, we shall
There exists a well-ordered decreasing chain of order type $(i)$ follows from the theorem and [3, Theorem 12.8], and $(ii)$ follows from $(i)$.

The remaining of the proof is similar to that of Theorem 4.2, but applying Lemma 4.4 instead of Lemma 4.1.

**Corollary 4.6.** Let $\kappa$ be any ordinal of cardinality $c$. Then:

(i) There exists a well-ordered decreasing chain of order type $\kappa$ of prime $z$-filters on $\mathbb{R}$ starting from any non-minimal prime $z$-filter.

(ii) There exists a well-ordered decreasing chain of order type $\kappa$ of prime $z$-ideals in $C_0(\mathbb{R})$ starting from any non-minimal prime $z$-ideals.

**Proof.** (i) follows from the theorem and [3, Theorem 12.8], and (ii) follows from (i) and the fact that every zero set on $\mathbb{R}$ is the zero set of a function in $C_0(\mathbb{R})$.

5. **Well-ordered increasing chains of prime ideals and prime $z$-filters**

Let $\Omega$ be a metrizable locally compact space. Similar to the previous section we shall only consider the case where $\partial^{(\infty)}\Omega^\circ \neq \emptyset$. Recall that there are many countable compact space satisfying this condition. First we shall prove a general construction.

**Definition 5.1.** Let $\kappa$ be any ordinal, and let $(Z_\alpha : \alpha \in \kappa)$ be a family of zero sets on $\Omega$. A zero set $Z$ is said to have property (A) (with respect to the family $(Z_\alpha : \alpha \in \kappa)$) if for every (possibly empty) finite subset $F$ of $\kappa$ and every $\beta \in \kappa$ with $\max F < \beta$ then

$$Z \cap \bigcap_{\alpha \in F} Z_\alpha \not\subset Z_\beta.$$

A zero set $Z$ is said to have property (B) (with respect to the family $(Z_\alpha : \alpha \in \kappa)$) if whenever $Z = \bigcup_{i=1}^n Z_i$ for some $Z_1, \ldots, Z_n \in Z[\Omega]$ then there exists $1 \leq k \leq n$ such that $Z_k$ has property (A).

**Lemma 5.2.** Let $\kappa$ be any ordinal, and let $(Z_\alpha : \alpha \in \kappa)$ be a family of zero sets on $\Omega$. Suppose that $F$ is a $z$-filter on $\Omega$ such that every element of $F$ has property (B) with respect to $(Z_\alpha : \alpha \in \kappa)$. Then there exists a well-ordered increasing chain $(Q_\alpha : \alpha \in \kappa)$ of prime $z$-filters containing $F$ such that $Z_\alpha \not\in Q_\alpha$ but $Z_\alpha \in Q_\beta$ ($\alpha < \beta \in \kappa$).

**Proof.** Let $D$ be the collection of all zero sets not having property (B). Then obviously $D$ is closed under finite union and $(Z_\alpha : \alpha \in \kappa) \subset D$. Since $F \cap D = \emptyset$, there exists a prime $z$-filter $Q_0$ such that $F \subset Q_0$ and $Q_0 \cap D = \emptyset$. We then define $Q_\alpha$ to be the $z$-filter generated by $Q_0$ and $\{Z_\gamma : \gamma < \alpha\}$. It follows that $Q_\alpha$ is a prime $z$-filter, $Q_\alpha \subset Q_\beta$ and $Z_\alpha \in Q_\beta$ ($\alpha < \beta \in \kappa$). We need to show that $Z_\alpha \not\in Q_\alpha$ ($\alpha \in \kappa$) (and thus $(Q_\alpha)$ is increasing). Assume towards a contradiction that $Z_\alpha \in Q_\alpha$ for some $\alpha \in \kappa$. Then, there exist $N \in Q_0$ and a finite subset $F$ of $\{\gamma : \gamma < \alpha\}$ such that

$$Z_\alpha \supset N \cap \bigcap_{\gamma \in F} Z_\gamma.$$

This implies that $N \in D$ a contradiction.
**Lemma 5.3.** There exists a family \((N_\alpha)_{\alpha \in \mathfrak c}\) of zero sets on \(\Xi\) satisfying that \(\Xi\) has property (B) with respect to \((N_\alpha : \alpha \in \mathfrak c)\).

**Proof.** Let \((E_\alpha : \alpha \in \mathfrak c)\) and \((N_\alpha : \alpha \in \mathfrak c)\) be defined as in Lemma 4.1.

We shall prove a little stronger statement. Assume towards a contradiction that there exists \(Z_1, \ldots, Z_n \in \mathbb Z[\Xi]\) such that \(\Xi = \bigcup_{i=1}^n Z_i\) and that, for each \(1 \leq i \leq n\), there exist finite subsets \(F_i\) and \(G_i\) of \(\kappa\) with \(\max F_i < \min G_i\) such that

\[
Z_i \cap \bigcap_{\alpha \in F_i} N_\alpha \subset \bigcup_{\beta \in G_i} N_\beta.
\]

Without loss of generality, we can suppose that

\[
\max F_1 \leq \max F_2 \leq \ldots \leq F_n.
\]

Fix \(\gamma \in \mathfrak c\) such that \(\gamma > \max G_i\) \((1 \leq i \leq n)\). We shall prove by induction that there exist finite subsets \(H_i\) of \(\mathfrak c\) with \(\gamma \leq \min H_i\) such that

\[
\bigcap_{i=1}^{k-1} \bigcap_{\alpha \in F_i \cup H_i} N_\alpha \subset \bigcup_{i=k}^n Z_i \quad (1 \leq k \leq n + 1).
\]

This is obviously true when \(k = 1\) since both sides are \(\Xi\). Suppose that the above is true for some \(k < n\). Then we see that

\[
\bigcap_{i=1}^k \bigcap_{\alpha \in F_i \cup H_i} N_\alpha \cap \bigcap_{i=k+1}^n \bigcup_{\alpha \in F_k \cup H_k} N_\alpha \\
\subset \bigcup_{\beta \in G_k} N_\beta \subset \bigcup_{i=k+1}^n Z_i.
\]

Because

\[
G_k \cap \left( \bigcup_{i=1}^k F_i \cup \bigcup_{i=1}^k H_i \right) = \emptyset,
\]

by Lemma 4.1, there exists a finite subset \(H_k\) of \(\mathfrak c\) such that \(\min H_k > \gamma\) and that

\[
\bigcap_{i=1}^k \bigcap_{\alpha \in F_i \cup H_i} N_\alpha \subset \bigcap_{i=k+1}^{k-1} \bigcap_{\alpha \in F_k \cup H_k} N_\alpha \\
\subset \bigcup_{\alpha \in F_k \cup H_k} N_\alpha \subset \bigcup_{i=k+1}^n Z_i.
\]

Thus, the induction can be continued, and so, for \(k = n + 1\), we have

\[
\bigcap_{i=1}^n N_\alpha \subset \emptyset;
\]

this is a contradiction. \(\square\)

**Theorem 5.4.** Let \(\Omega\) be a metrizable locally compact space, and let \(p \in \partial^{(\infty)}\Omega^0\). Then there exists a well-ordered increasing chain \((Q_\alpha : \alpha \in \mathfrak c)\) of prime \(z\)-filters on \(\Omega\) each supported at \(p\).

Furthermore, by setting \(Q_\alpha = C_0(\Omega) \cap \mathbb Z^{-1}[Q_\alpha]\), we obtain a well-ordered increasing chain \((Q_\alpha : \alpha \in \mathfrak c)\) of prime \(z\)-ideals in \(C_0(\Omega)\) each supported at \(p\).

**Proof.** Similar to Theorem 3.9, we shall identify \(\Xi\) with a closed subset of \(\Omega^0\) such that \((\infty, \infty, \ldots)\) is identified with \(p\); in the case where \(p \in \Omega\), we can further assume that \(\Xi \subset \Omega\) (cf. Lemmas 3.7).

Let \((N_\alpha : \alpha \in \mathfrak c)\) be the family of zero sets on \(\Xi\) as constructed in Lemma 5.3. For each \(\alpha \in \mathfrak c\), choose \(Z_\alpha = \mathbb Z(f_\alpha)\) for some \(f_\alpha \in C_0(\Omega)\) such that \((Z_\alpha \cup \{p\}) \cap \Xi = N_\alpha\). It follows that \(\Omega\) has property (B) with respect to \((Z_\alpha : \alpha \in \mathfrak c)\). Thus, by Lemma
5.2 where $\mathcal{F} = \{\Omega\}$, there exists a well-ordered increasing chain $(Q_\alpha : \alpha \in \kappa)$ of prime $z$-filters such that $Z_\alpha \notin Q_\alpha$ but $Z_\alpha \in Q_\beta$ ($\alpha < \beta \in \kappa$).

The rest is similar to Theorem 4.2. □

**Corollary 5.5.** Let $\Delta$ be any countable compact subset of $\mathbb{R}$ such that $\partial^{(\infty)} \Delta \neq \emptyset$. There exists a well-ordered increasing chain of order type $\kappa$ of prime $z$-filters on $\mathbb{R}$ such that each prime $z$-filter contains $\Delta$. □

For longer chains, as in §4, we need to restrict to uncountable locally compact Polish spaces. Again, in the case where $\Omega$ is in addition $\sigma$-compact, it will restrict the ordinal $\kappa$ under consideration to have cardinality at most $\kappa$.

**Lemma 5.6.** Let $\kappa$ be an ordinal of cardinality $\kappa$. There exists a family $(N_\alpha)_{\alpha \in \kappa}$ of zero sets on $(\mathbb{N}^\omega)^\mathbb{R}$ such that $(\mathbb{N}^\omega)^\mathbb{R}$ has property (B) with respect to $(N_\alpha : \alpha \in \kappa)$.

**Proof.** Similar to Lemma 5.3, here we apply Lemma 4.4 instead of Lemma 4.1. □

**Theorem 5.7.** Let $\Omega$ be an uncountable locally compact Polish space. Let $\kappa$ be an ordinal of cardinality $\kappa$. Then there exists a well-ordered increasing chain $(Q_\alpha : \alpha \in \kappa)$ of prime $z$-filters on $\Omega$.

Furthermore, by setting $Q_\alpha = C_0(\Omega) \cap Z^{-1}[Q_\alpha]$, we obtain a well-ordered increasing chain $(Q_\alpha : \alpha \in \kappa)$ of prime $z$-ideals in $C_0(\Omega)$.

**Proof.** Similar to previous proofs. □

**Corollary 5.8.** Let $\kappa$ be any ordinal of cardinality $\kappa$. Then there exists a well-ordered increasing chain of order type $\kappa$ of prime $z$-filters on $\mathbb{R}$. □

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**References**


