THE KERNELS AND CONTINUITY IDEALS OF HOMOMORPHISMS FROM $C_0(\Omega)$

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In memoriam: Graham Robert Allan, 1937-2007

Abstract. We give a description of the continuity ideals and the kernels of homomorphisms from the algebras of continuous functions on locally compact spaces into Banach algebras. We also construct families of prime ideals satisfying a certain intriguing property in the algebras of continuous functions.

1. Introduction

Let $\theta : A \to B$ be a homomorphism from a commutative Banach algebra $A$ into a Banach algebra $B$. The continuity ideal of $\theta$ is defined to be the ideal

$$I(\theta) = \{a \in A : \text{the map } b \mapsto \theta(ab), A \to B, \text{ is continuous}\};$$

this ideal contains every ideal $I$ in $A$ on which $\theta$ is continuous. In the case where $A = C_0(\Omega)$, for a locally compact space $\Omega$, then $\theta$ is continuous on $I(\theta)$.

This paper aims to characterize the ideals which are the kernels or the continuity ideals of homomorphisms from $C_0(\Omega)$ into Banach algebras. This is, in some sense, a last piece of the picture of homomorphisms from $C_0(\Omega)$ into Banach algebras.

The study of homomorphisms from $C_0(\Omega)$ started with the theorem of Kaplansky [12] that every algebra norm on $C_0(\Omega)$ majorises the uniform norm. This essentially provides a description of all continuous homomorphisms from $C_0(\Omega)$ into Banach algebras.

Then, Bade and Curtis [1] gave a detail structural decomposition of discontinuous homomorphisms from $C_0(\Omega)$ into Banach algebras. The following statement of their theorem also includes some improvements from [6] and [16] (see also [3], [17]); see §2 for notations.

Theorem 1.1. Let $\Omega$ be a locally compact space, and let $\theta$ be a discontinuous homomorphism from $C_0(\Omega)$ into a Banach algebra $B$. Set $B_0 = \overline{\theta(C_0(\Omega))}$.

(i) The continuity ideal $I(\theta)$ is the largest ideal of $C_0(\Omega)$ on which $\theta$ is continuous.

(ii) There exists a non-empty finite subset $\{p_1, \ldots, p_n\}$ of $\Omega$ such that

$$\bigcap_{i=1}^n J_{p_i} \subset I(\theta) \subset \bigcap_{i=1}^n M_{p_i}.$$
(iii) There exists a continuous homomorphism \( \mu : C_0(\Omega) \to B_0 \) such that
\[
B_0 = \mu(C_0(\Omega)) \oplus \text{rad } B_0, \quad \mu(\bigcap_{i=1}^{n} M_{p_i}) \cdot \text{rad } B_0 = \{0\},
\]
and \( \mu = \theta \) on a dense subalgebra of \( C_0(\Omega) \) containing \( I(\theta) \).

(iv) Set \( \nu = \theta - \mu \). Then \( \nu \) maps into \( \text{rad } B_0 \), and the restriction of \( \nu \) to \( \bigcap_{i=1}^{n} M_{p_i} \) is a homomorphism \( \nu' \) onto a dense subalgebra of \( \text{rad } B_0 \).

(v) There exist linear maps \( \nu_1, \ldots, \nu_n : C_0(\Omega) \to \text{rad } B_0 \) such that
(a) \( \nu = \nu_1 + \cdots + \nu_n \),
(b) each \( \nu'_i = \nu_i|_{M_{p_i}} \) (\( 1 \leq i \leq n \)) is a non-zero homomorphism, and
(c) \( \nu_i(C_0(\Omega)) \cdot \nu_j(C_0(\Omega)) = \{0\} \) for each \( 1 \leq i \neq j \leq n \).

(vi) The ideals \( \ker \theta \) and \( I(\theta) \) are always intersections of prime ideals; we have
\[
\ker \theta = I(\theta) \cap \ker \mu \quad \text{and} \quad I(\theta) = \ker \nu' = \bigcap_{i=1}^{n} \ker \nu'_i.
\]

For brevity, we define a radical homomorphism to be a homomorphism into a radical Banach algebra. The above result points out the important roles of prime ideals and of radical homomorphisms as building blocks for general (discontinuous) homomorphisms from \( C_0(\Omega) \).

In 1970s, Dales [2] and Esterle [6], [7], [8] independently proved that, assuming the Continuum Hypothesis (CH), every ideal \( I \) which is the intersection of a finite number of non-modular prime ideals in \( C_0(\Omega) \) with \( \|C_0(\Omega)/I\| = \epsilon \) is the kernel of a radical homomorphism from \( C_0(\Omega) \). (For more details see [3].)

In fact, for some spaces \( \Omega \), the kernels of radical homomorphisms from \( C_0(\Omega) \) are always finite intersections of non-modular prime ideals ([6], [14]). However, in [14], we showed that for most metrisable non-compact locally compact spaces \( \Omega \), for example \( \mathbb{R} \), there exist radical homomorphisms from \( C_0(\Omega) \) whose kernels are not the intersection of any finite number of prime ideals.

In this paper, we shall show that the kernels of radical homomorphisms from \( C_0(\Omega) \) are always intersections of (relatively) compact families of non-modular prime ideals. More generally, we shall prove a similar result for continuity ideals of homomorphisms from Banach algebras into commutative Banach algebras (Corollary 4.10).

For the converse direction, we shall prove that, assuming the Continuum Hypothesis (CH), every ideal \( I \) which is the intersection of a relatively compact family \( \mathcal{P} \) of non-modular prime ideals such that \( \|C_0(\Omega)/I\| = \epsilon \) and such that each chain in the closure of \( \mathcal{P} \) is countable (thus, in particular, when \( \mathcal{P} \) itself is countable) is the kernel of a radical homomorphism from \( C_0(\Omega) \). A similar result holds for continuity ideals of homomorphisms from \( C_0(\Omega) \) into Banach algebras. (See §6.) We conjecture that the countability condition in these converses is redundant (cf. the last paragraphs of §6 and §7).

Remark. It was proved by Solovay and Woodin that the existence of discontinuous homomorphisms from \( C_0(\Omega) \) is not a theorem of ZFC alone (see [4] for more details). All results that require the Continuum Hypothesis will be marked with (CH).

In the final section, we shall construct non-trivial examples of (relatively) compact families of prime ideals in \( C_0(\Omega) \) for metrisable locally compact spaces \( \Omega \) with infinite limit level (Theorem 7.5). This, in particular, shows the complexity of the prime ideal structure of \( C_0(\Omega) \) even for countable compact subspaces \( \Omega \) of \( \mathbb{R} \).
2. Preliminary definitions and notations

Let \( A \) be a commutative algebra. The (conditional) unitisation \( A^\# \) of \( A \) is defined as the algebra \( A \) itself if \( A \) is unital, and as \( A \) with identity adjoined otherwise. The identity of \( A^\# \) is denoted by \( e_A \).

A prime ideal or semiprime ideal in \( A \) must be a proper ideal. However, we consider \( A \) itself as the intersection of the empty collection of prime ideals.

Let \( I \) be an ideal in \( A \). Define the prime radical \( \sqrt{I} \) of \( I \) to be the intersection of all prime ideals in \( A \) containing \( I \), so that
\[
\sqrt{I} = \{ a \in A : a^n \in I \text{ for some } n \in \mathbb{N} \}.
\]
For each element \( a \in A^\# \), define the quotient of \( I \) by \( a \) to be the ideal
\[
I/a = \{ b \in A : ab \in I \}.
\]
Clearly we have \( I \subset I/a \).

For the definition of universal algebras, see [3, Definition 5.7.8]. For example, the integral domain \( L^1(\mathbb{R}^+, \omega) \) is universal for each radical weight \( \omega \) bounded near the origin [3, Theorem 5.7.25]. Indeed, the class of universal commutative radical Banach algebras has been characterized in [10] (see also [3, Theorem 5.7.28]).

For a discussion of the theory of the algebras of continuous functions, see [3], [5] or [11]. Here, we just give some facts which are needed in our discussion.

Let \( \Omega \) be a locally compact space; the convention is that locally compact spaces and compact spaces are Hausdorff. The one-point compactification of \( \Omega \) is denoted by \( \Omega^\# \). Denote by \( C_c(\Omega) \) the algebra of compactly supported continuous functions on \( \Omega \). For each \( p \in \Omega \), define
\[
J_p = \{ f \in C_0(\Omega) : f \text{ is zero on a neighbourhood of } p \},
\]
\[
M_p = \{ f \in C_0(\Omega) : f(p) = 0 \}.
\]
For \( p \) being the point (at infinity) adjoined to \( \Omega \) to obtain \( \Omega^\# \), we also set
\[
J_p = C_c(\Omega) \quad \text{and} \quad M_p = C_0(\Omega).
\]
For each prime ideal \( P \) in \( C_0(\Omega) \), there always exists a unique point \( p \in \Omega^\# \) such that \( J_p \subset P \subset M_p \), we say that \( P \) is supported at the point \( p \). It can be seen that \( P \) is modular if and only if its support point belongs to \( \Omega \).

We shall use frequently the important fact that, for each prime ideal \( P \) in \( C_0(\Omega) \), the set of prime ideals containing \( P \) is a chain with respect to the inclusion relation.

For each function \( f \) continuous on \( \Omega \), the zero set of \( f \) is denoted by \( Z(f) \). The set of zero sets of continuous functions on \( \Omega \) is denoted by \( Z[\Omega] \).

A z-filter \( \mathcal{F} \) on \( \Omega \) is a non-empty proper subset of \( Z[\Omega] \) satisfying:

(i) \( Z_1 \cap Z_2 \) belongs to \( \mathcal{F} \) whenever both \( Z_1 \) and \( Z_2 \) belong to \( \mathcal{F} \),
(ii) if \( Z_1 \in \mathcal{F} \), \( Z_2 \in Z[\Omega] \) and \( Z_1 \subset Z_2 \), then \( Z_2 \) also belongs to \( \mathcal{F} \).

Each z-filter \( \mathcal{F} \) corresponds to an ideal
\[
\{ f \in C(\Omega) : Z(f) \in \mathcal{F} \},
\]
denoted by \( Z^{-1}[\mathcal{F}] \).

Let \( \kappa \) be an infinite cardinal and let \( \mathcal{U} \) be a free ultrafilter on \( \kappa \). Define \( M_\mathcal{U} \) to be the maximal ideal of \( C^\kappa \) consisting of all elements \( f \in C^\kappa \) such that \( \{ \sigma \in \kappa : f(\sigma) = 0 \} \in \mathcal{U} \).
The field \( C^\kappa / M_\kappa \) is called an ultrapower, and is denoted by \( C^\kappa / U \). An element \( f + M_\kappa \in C^\kappa / U \) is an infinitesimal if
\[
\{ \sigma \in \kappa : |f(\sigma)| < \varepsilon \} \in U \quad \text{for each } \varepsilon > 0.
\]
The subalgebra of infinitesimals of \( C^\kappa / U \) is denoted by \( (C^\kappa / U)^\circ \).

For a well-ordered set \( \Lambda \), denote by \( o(\Lambda) \) the ordinal order isomorphic to \( \Lambda \).

3. RELATIVELY COMPACT FAMILIES OF PRIME IDEALS

In this section, let \( A \) be a commutative algebra.

**Definition 3.1** (cf. [14] Definition 3.1). An indexed family \( (P_i)_{i \in S} \) of prime ideals in \( A \) is pseudo-finite if \( a \in P_i \) for all but finitely many \( i \in S \) whenever \( a \in \bigcup_{i \in S} P_i \).

For a pseudo-finite family \( (P_i : i \in S) \) of prime ideals where \( S \) is infinite, it is obvious that \( \bigcup_{i \in T} P_i = \bigcup_{i \in S} P_i \) is either a prime ideal in \( A \) or the whole \( A \) for each infinite subset \( T \) of \( S \).

**Definition 3.2.** A family \( \mathcal{C} \) of prime ideals in \( A \) is relatively compact if every sequence of prime ideals in \( \mathcal{C} \) contains a pseudo-finite subsequence. The family \( \mathcal{C} \) is compact if it is relatively compact and contains the union of each of its pseudo-finite sequences.

Obviously, the union of finitely many pseudo-finite families is relatively compact.

In the rest of this section, we shall indeed relate our compactness to the usual meaning of this terminology.

Denote by \( \Pi \) the set of prime ideals in \( A \). For \( a_1, a_2, \ldots, a_m \), and \( b \) in \( A \), define
\[
U_{a_1, \ldots, a_m}^b = \{ P \in \Pi : a_i \in P \quad (1 \leq i \leq m) \quad \text{and} \quad b \notin P \}.
\]
Then the collection of all such sets \( U_{a_1, \ldots, a_m}^b \) is a base for a topology \( \tau \). Indeed, by the primeness, we have
\[
U_{a_1, \ldots, a_m, c_1, \ldots, c_n}^{b, d} = U_{a_1, \ldots, a_m}^b \cap U_{c_1, \ldots, c_n}^d.
\]
It is also easy to see that \( \tau \) is Hausdorff. We claim that \( U_0^a \) is \( \tau \)-compact \((u \in A)\).

Indeed, we see that \( \{ U_0^a, U_0^b : a \in A \} \) is a subbasis for the relative \( \tau \)-topology on \( U_0^a \), so by Alexander’s lemma, we need only to show that each cover of \( U_0^a \) by sets in this subbasis has a finite subcover. Let \( E, F \) be subsets of \( A \) such that
\[
U_0^a = \bigcup_{a \in E} U_v^a \cup \bigcup_{b \in F} U_0^b.
\]
Set \( S = \{ u^m a_1 \cdots a_n : m, n \in \mathbb{N}, a_1, \ldots, a_n \in E \} \), and let \( I \) be the ideal generated by \( F \). Assume toward a contradiction that \( S \cap I = \emptyset \). Then since \( S \) is closed under multiplication, there exists a prime ideal \( P \) such that \( P \supseteq I \) and \( P \cap S = \emptyset \); this implies that \( P \in U_0^a \) but \( P \notin \bigcup_{a \in E} U_v^a \cup \bigcup_{b \in F} U_0^b \), a contradiction. Thus, \( S \cap I \neq \emptyset \), so there exist \( k, m, n \in \mathbb{N}, a_1, \ldots, a_n \in E \) and \( b_1, \ldots, b_n \in F \), and \( c_1, \ldots, c_n \in A \) such that \( u^k a_1 \cdots a_m = b_1 c_1 + \cdots + b_n c_n \). We can then deduce that
\[
U_0^u = \bigcup_{i=1}^m U_v^{a_i} \cup \bigcup_{j=1}^n U_0^{b_j}.
\]
Thus \( \tau \) is locally compact.
The one point compactification of \((\Pi, \tau)\) can be considered as the set \(\Pi \cup \{A\}\); a basis of neighbourhood for \(A\) is given by

\[
U_{b_1, \ldots, b_n} = \{A\} \cup \{P \in \Pi : b_i \in P \ (1 \leq i \leq n)\}.
\]

**Proposition 3.3.** Let \(A\) be a commutative algebra. Denote by \(\Pi\) the set of prime ideals in \(A\). Define a topology \(\tau\) as above. Then \((\Pi, \tau)\) is a totally disconnected locally compact space, and every [relatively] compact family of prime ideals in \(A\) is a [relatively] sequentially \(\tau\)-compact subset of \(\Pi \cup \{A\}\).

**Proof.** It remains to prove the last assertion. We claim that a pseudo-finite sequence \((P_n)\) of prime ideals in \(A\) is \(\tau\)-convergent in \(\Pi \cup \{A\}\). In fact, set \(P = \bigcup_{n=1}^{\infty} P_n\). Then either \(P \in \Pi\) or \(P = A\), and, in both cases, \((P_n)\) \(\tau\)-converges to \(P\). \(\square\)

**Remark.** If \(A\) is either unital or \(\mathcal{C}_0(\Omega)\) for some locally compact space \(\Omega\), then, for each pseudo-finite sequence \((P_n)\) of prime ideals in \(A\), the union \(\bigcup_{n=1}^{\infty} P_n\) is in fact a prime ideal in \(A\). Thus, in the above proposition, we can replace \(\Pi \cup \{A\}\) by \(\Pi\).

**Remark.** Let us instead consider a topology \(\sigma\) on \(\Pi \cup \{A\}\) generated by

\[
U_{a_1, \ldots, a_m}^Q = \{P \in \Pi \cup \{A\} : P \subset Q \text{ and } a_i \in P \ (1 \leq i \leq m)\},
\]

where \(Q\) is either a semiprime ideal in \(A\) or \(A\) itself, and \(a_1, \ldots, a_m \in Q\). Then a sequence of prime ideals in \(A\) is pseudo-finite if and only if it is convergent in \((\Pi \cup \{A\}, \sigma)\), and so a family of prime ideals in \(A\) is [relatively] compact if and only if it is [relatively] sequentially compact in \((\Pi \cup \{A\}, \sigma)\). However, in general, \(\sigma\) is neither Hausdorff nor locally compact. In the case where \(A = \mathcal{C}_0(\Omega)\), then \((\Pi \cup \{A\}, \sigma)\) is Hausdorff (but not locally compact).

4. **Homomorphisms from general commutative Banach algebras**

Let \(\theta : A \to B\) be a homomorphism from a commutative Banach algebra \(A\) into a Banach algebra \(B\). Let \((a_n : n \in \mathbb{N})\) be a sequence in \(A\). Then

\[
\mathcal{I}(\theta) : a_1a_2 \cdots a_n \subseteq \mathcal{I}(\theta) : a_1a_2 \cdots a_{n+1} \quad (n \in \mathbb{N}).
\]

It follows easily from the stability lemma (see [3, 5.2.7] or [17, 1.6] for the statement and proof) that there exists \(n_0\) such that

\[
\mathcal{I}(\theta) : a_1a_2 \cdots a_n = \mathcal{I}(\theta) : a_1a_2 \cdots a_{n+1} \quad (n \geq n_0).
\]

Thus \(\mathcal{I}(\theta)\) is an abstract continuity ideal in the following sense.

**Definition 4.1.** Let \(A\) be a commutative algebra. An ideal \(I\) is an abstract continuity ideal if, for each sequence \((a_n)\) in \(A\), there exists \(n_0\) such that

\[
I:a_1a_2 \cdots a_n = I:a_1a_2 \cdots a_{n+1} \quad (n \geq n_0).
\]

**Proposition 4.2.** Let \(\mathfrak{P}\) be a relatively compact family of prime ideals in a commutative algebra \(A\). Then \(\bigcap \{P : P \in \mathfrak{P}\}\) is an abstract continuity ideal in \(A\).

**Proof.** Set \(I = \bigcap \{P : P \in \mathfrak{P}\}\). Assume toward a contradiction that \(I\) is not an abstract continuity ideal. Then there exists a sequence \((a_n)\) in \(A\) such that

\[
I:a_1a_2 \cdots a_n \not\subseteq I:a_1a_2 \cdots a_{n+1} \quad (n \in \mathbb{N}).
\]

For each \(n\), we see that

\[
I:a_1 \cdots a_n = \bigcap \{P \in \mathfrak{P} : a_1 \cdots a_n \notin P\}.
\]
Thus, it follows that, there exists $P_n \in \mathfrak{P}$ such that $a_1 \cdots a_n \notin P_n$ but $a_1 \cdots a_{n+1} \in P_n$. The relative compactness implies that there exists $n_1 < n_2 < \cdots$ such that $(P_{n_i})$ is pseudo-finite. However, we see that $a_1 a_2 \cdots a_{n_2} \in P_{n_1}$, but $a_1 a_2 \cdots a_{n_2} \notin P_{n_i}$ ($i \geq 2$); this contradicts the pseudo-finiteness. \hfill $\square$

The remaining of this section is devoted to a converse of the above proposition. Let $I$ be an abstract continuity ideal of a commutative algebra $A$. Denote by $\mathfrak{P}$ the set of prime ideals of the form $I: a$ for some $a \in A$. The following is a generalisation of [14, 4.3].

Lemma 4.3. For each cardinal $\kappa \leq |\mathfrak{P}|$, there exists a sub-family $\mathcal{G} \subset \mathfrak{P}$ with the properties that $|\mathcal{G}| \geq \kappa$ and that $|\{P \in \mathcal{G} : a \notin P\}| < \kappa$ for each $a \in \bigcup\{P : P \in \mathcal{G}\}$.

Proof. For each $a \in A \cup \{e_A\}$, let $\mathfrak{P}_a$ be the set of prime ideals of the form $I: ab$ for some $b \in A$. We claim that there exists $a_0 \in A \cup \{e_A\}$ such that $|\mathfrak{P}_{a_0}| \geq \kappa$ and such that, for each $a \in A$, either $|\mathfrak{P}_{a_0 a}| < \kappa$ or $I_{a_0 a} = I_{a_0}$. Indeed, assume the contrary. Then, since $|\mathfrak{P}_{e_A}| \geq \kappa$, by induction, there exists a sequence $(a_n) \subset A$ such that $|\mathfrak{P}_{a_0 \cdots a_n}| \geq \kappa$ and such that

$$I_{a_0} \cdots a_n \subset I_{a_0} a_{n+1} \quad (n \in \mathbb{N}).$$

This contradicts the definition of an abstract continuity ideal. Hence the claim holds.

Put $\mathcal{G} = \mathfrak{P}_{a_0}$; this obviously satisfies $|\mathcal{G}| \geq \kappa$. Suppose that $a \in A$ and that $\mathcal{G}' = \{P \in \mathcal{G} : a \notin P\}$ has cardinality at least $\kappa$. Then, for each $P \in \mathcal{G}'$, because $a \notin P$ we have $P: a = P$. Thus $\mathcal{G}' \subset \mathfrak{P}_{a_0 a}$, and hence $|\mathfrak{P}_{a_0 a}| \geq \kappa$. Therefore, by the claim, we must have $I_{a_0 a} = I_{a_0}$. We now show that $\mathcal{G}' = \mathcal{G}$. Assume towards a contradiction that $\mathcal{G}' \neq \mathcal{G}$, and let $P \in \mathcal{G} \setminus \mathcal{G}'$, say $P = I_{a_0 a_1}$ for some $a_1 \in A$. Then, since $a \in P$ we have $a_1 \in I_{a_0 a} = I_{a_0}$, so that $a_0 a_1 \in I$. This implies that $P = A$, a contradiction. This proves that $\mathcal{G}$ has the desired property. \hfill $\square$

Lemma 4.4. $\sqrt{I}$ is the intersection of the prime ideals in $\mathfrak{P}$.

Proof. This is based on the commutative prime kernel theorem due to Sinclair (see [3, Theorem 5.3.15] or [17, Theorem 11.4]; the proof in [14, Lemma 4.1] works almost verbatim). \hfill $\square$

Lemma 4.5. Every element in $\mathfrak{P}$ contains a minimal element.

Proof. Assume toward a contradiction that there exists $(P_n = I : f_n) \subset \mathfrak{P}$ such that

$$P_1 \supseteq P_2 \supseteq \cdots \supseteq P_n \supseteq \cdots.$$

For each $n$, choose $a_n \in A$ such that $a_n \in P_n \setminus P_{n+1}$. Then we see that $a_1 \cdots a_n f_n \in I$ but $a_1 \cdots a_n f_{n+1} \notin I$. Thus

$$I_{a_1 a_2 \cdots a_n} \subset I_{a_1 a_2 \cdots a_{n+1}} \quad (n \in \mathbb{N});$$

a contradiction to $I$ being an abstract continuity ideal. \hfill $\square$

Lemma 4.6. Let $P$ be in $\mathfrak{P}$. Then there exists $a \notin P$ but $a \in Q$ for all $Q \in \mathfrak{P}$ such that $Q \not\subseteq P$.

Proof. Assume the contrary. Pick $a_1 \notin P$. Suppose that we have already picked $a_1, \ldots, a_n \notin P$. By the assumption, we can find $Q_n \in \mathfrak{P}$ such that $a_1, \ldots, a_n \notin Q_n$ and $Q_n \not\subseteq P$. We can then choose $a_{n+1} \in Q_n \setminus P$. The induction can be continued. We see that $(a_n)$, $(Q_n)$ constructed satisfy $a_1 \cdots a_n \notin Q_n$ but $a_1 \cdots a_{n+1} \in Q_n$.
Let \( Q_n = I : f_n \). Then we see that \( f_n \in I : a_1 a_2 \cdots a_{n+1} \setminus I : a_1 a_2 \cdots a_n \); a contradiction to \( I \) being an abstract continuity ideal. \( \square \)

Lemma 4.7. Let \( \{ P_\alpha : \alpha \in S \} \) be a subfamily of \( \mathfrak{P} \). Then \( \bigcap_{\alpha \in S} P_\alpha \) is also an abstract continuity ideal.

Proof. Assume the contrary. Then there exists \( (a_n) \) such that
\[
(\bigcap_{\alpha \in S} P_\alpha) : a_1 a_2 \cdots a_n \subseteq (\bigcap_{\alpha \in S} P_\alpha) : a_1 a_2 \cdots a_{n+1} \quad (n \in \mathbb{N}).
\]
For each \( n \), choose \( b_n \in A \) such that \( a_1 \cdots a_n b_n \notin \bigcap_{\alpha \in S} P_\alpha \) but \( a_1 \cdots a_{n+1} b_n \in \bigcap_{\alpha \in S} P_\alpha \). Then choose \( \alpha_n \in S \) such that \( a_1 \cdots a_n b_n \notin P_{\alpha_n} \). We have \( P_{\alpha_n} = I : f_{\alpha_n} \) for some \( f_{\alpha_n} \in A \). We see that \( a_1 \cdots a_n b_n f_{\alpha_n} \notin I \) but \( a_1 \cdots a_{n+1} b_n f_{\alpha_n} \in I \). Thus
\[
I : a_1 a_2 \cdots a_n \subseteq I : a_1 a_2 \cdots a_{n+1} \quad (n \in \mathbb{N});
\]
a contradiction to \( I \) being an abstract continuity ideal. \( \square \)

Lemma 4.8. Let \( J \) be a semiprime ideal in \( A \). Let \( a, b \in A \) be such that \( J : a \) and \( J : b \) are prime ideals. Then the following are equivalent:
\[
\begin{align*}
(a) & \quad J : a \subseteq J : b; \\
(b) & \quad ab \notin J; \\
(c) & \quad J : a = J : b.
\end{align*}
\]
Proof. (a) \( \Rightarrow \) (b): Since \( J \) is semiprime and \( J : b \) is a proper ideal in \( A \), we see that \( b \notin J : b \). So \( b \notin J : a \), and therefore \( ab \notin J \).

(b) \( \Rightarrow \) (c): Condition (b) implies that \( a \notin J : b \). Let \( f \in J : a \). Then \( fa \in J \subseteq J : b \), and so, by the primeness of \( J : b \), \( f \in J : b \). Thus \( J : a \subseteq J : b \). Similarly, we have \( J : b \subseteq J : a \). \( \square \)

Remark. In the case where \( I \) is semiprime, the above lemma shows that, for each \( P = I : a \in \mathfrak{P} \), \( P \) is minimal in \( \mathfrak{P} \) and \( a \notin P \) but \( a \in Q \) whenever \( Q \in \mathfrak{P} \setminus \{ P \} \).

We can now state the main result of this section.

Theorem 4.9. Let \( I \) be an abstract continuity ideal of a commutative algebra \( A \). Denote by \( \mathfrak{P}_0 \) the set of minimal ideals among the prime ideals of the form \( I : a \) for some \( a \in A \). Then:
\[
\begin{align*}
(i) & \quad \sqrt{I} = \bigcap \{ P : P \in \mathfrak{P}_0 \}; \\
(ii) & \quad \mathfrak{P}_0 \ is \ a \ relatively \ compact \ family \ of \ prime \ ideals.
\end{align*}
\]
Proof. The first assertion follows from Lemmas 4.4 and 4.5. For the second one, let \( (P_n) \subset \mathfrak{P}_0 \). We can assume that \( P_n \ (n \in \mathbb{N}) \) are distinct. Set \( J = \bigcap_{n=1}^\infty P_n \). By Lemma 4.6, there exists \( a_n \in \bigcap_{i \neq n} P_i \setminus P_n \), and so \( P_n = J : a_n \). Let \( a \in A \) be such that \( J : a \) is a prime ideal. We claim that \( J : a \in \{ P_n \} \). Indeed, we see that \( a \notin J \), and thus \( a \notin P_n \) for some \( n_0 \). So, \( aa_n \notin J \). By Lemma 4.8, we deduce that \( J : a = P_n \). It then follows from Lemmas 4.7 and 4.3 (applied to \( J \) and with \( \kappa \) being the first infinite cardinal \( \aleph_0 \)) that \( (P_n) \) must have a pseudo-finite subsequence. \( \square \)

Corollary 4.10. Let \( \theta : A \rightarrow B \) be a homomorphism from a Banach algebra \( A \) into a commutative Banach algebra \( B \). Then \( \sqrt{I(\theta)} \) is the intersection of a relatively compact family of prime ideals of the form \( I(\theta) : a \) for \( a \in A \).

Remark. Since \( \ker \theta \subset I(\theta) \) and \( A/\ker \theta \) is commutative, all the previous definitions and results still make sense in this case.
Proof. It can be seen that
\[(I(\theta)/\ker \theta):(a + \ker \theta) = (I(\theta):a)/\ker \theta \quad (a \in A).\]
So, by the stability lemma, we see that \(I(\theta)/\ker \theta\) is an abstract continuity ideal in \(A/\ker \theta\). The rest follows from the theorem. \(\square\)

Corollary 4.11. Let \(\theta : A \to B\) be an epimorphism from a Banach algebra \(A\) onto a commutative Banach algebra \(B\). Then \(\sqrt{I(\theta)}\) is the intersection of a finite number of prime ideals of the form \(I(\theta):a\) for \(a \in A\) and there exists \(k \in \mathbb{N}\) such that \(\sqrt{I(\theta)} = \{a \in A : a^k \in I(\theta)\}\).

Proof. This is proved in the same way as [14, Corollary 4.7]. \(\square\)

Lemma 4.12. Let \(I\) be an abstract continuity ideal of \(C_0(\Omega)\) for a locally compact space \(\Omega\). Then \(I\) is either a semiprime ideal or the whole of \(C_0(\Omega)\).

Proof. The proof is the same as the proof that the continuity ideal of a discontinuous homomorphism from \(C_0(\Omega)\) into a Banach algebra is semiprime ([6],[16], cf. [3, Theorem 5.4.31]). \(\square\)

Corollary 4.13. Let \(\Omega\) be a locally compact space.

(i) Let \(I\) be an abstract continuity ideal in \(C_0(\Omega)\). Denote by \(\mathfrak{P}\) the set of prime ideals of the form \(I:f\) for some \(f \in C_0(\Omega)\). Then:

(a) \(I = \bigcap\{P : P \in \mathfrak{P}\};\)
(b) \(\mathfrak{P}\) is a relatively compact family of prime ideals.

(ii) Conversely, let \(\mathfrak{P}\) be a relatively compact family of prime ideals in \(C_0(\Omega)\). Then \(\bigcap\{P : P \in \mathfrak{P}\}\) is an abstract continuity ideal in \(C_0(\Omega)\).

Corollary 4.14. Let \(\Omega\) be a locally compact space. Then each homomorphism from \(C_0(\Omega)\) into a Banach algebra is continuous on the intersection of a relatively compact family of prime ideals of the form \(I(\theta):f\) for \(f \in C_0(\Omega)\).

5. (Relatively) Compact families of prime ideals in \(C_0(\Omega)\)

In this section, let \(\Omega\) be a locally compact space, and let \(\mathfrak{P}\) be a non-empty relatively compact family of prime ideals in \(C_0(\Omega)\). Denote by \(\mathfrak{Q}\) the collection of all ideals that are unions of countably many ideals in \(\mathfrak{P}\). We call \(\mathfrak{Q}\) the closure of \(\mathfrak{P}\): we shall show that it is indeed the smallest compact family of prime ideals containing \(\mathfrak{P}\).

Note that an ideal in \(\mathfrak{Q}\) is automatically prime in \(C_0(\Omega)\), and that the union of each pseudo-finite sequence of prime ideals in \(C_0(\Omega)\) is again a prime ideal in \(C_0(\Omega)\); this follows from the next lemma.

Lemma 5.1. The union of finitely many prime ideals in \(C_0(\Omega)\) either is one of the given prime ideal or is not even a linear space. The union of countably many prime ideals in \(C_0(\Omega)\) is not equal \(C_0(\Omega)\).

Proof. We prove the second clause only; the proof of the first one is similar. Let \(P_n\) \((n \in \mathbb{N})\) be prime ideals in \(C_0(\Omega)\). Choose \(f_n \in C_0(\Omega) \setminus P_n\). We can assume that \(0 \leq f_n \leq 2^{-n}\). Set \(f = \sum_{n=1}^{\infty} f_n\). Then \(f \in C_0(\Omega)\) but \(f \notin P_n\) since \(f \geq f_n\) \((n \in \mathbb{N})\). \(\square\)

Lemma 5.2. Each chain in \(\mathfrak{Q}\) is well-ordered with respect to the inclusion; that is, each non-empty chain in \(\mathfrak{Q}\) has a smallest element.
Assume the contrary, then we can find an infinite chain $\cdots \subseteq Q_n \subseteq \cdots \subseteq Q_1$ in $\mathcal{Q}$. For each $n$, choose $P_n \in \mathcal{P}$ such that $P_n \subseteq Q_n$ but $P_n \not\subseteq Q_{n+1}$. By the relative compactness of $\mathcal{P}$ and without loss of generality, we can assume that $(P_n : n \in \mathbb{N})$ is a pseudo-finite sequence. Set $Q = \bigcup_{n=1}^\infty P_n$. Then $Q \in \mathcal{Q}$, and for each $n \in \mathbb{N}$, either $Q_n \subseteq Q$ or $Q \subseteq Q_n$ (since both contain $P_n$). Since $P_{n-1} \not\subseteq Q_n$, we must have $Q_n \subseteq Q$ ($n \geq 2$). Choose $a \in Q_2 \setminus Q_1$. Then $a \notin Q_n$, and so $a \notin P_n$ ($n \geq 3$). However, $a \in Q = \bigcup_{n=1}^\infty P_n$. This contradicts the pseudo-finiteness of $(P_n)$. □

**Lemma 5.3.** $\mathcal{Q}$ is compact.

Proof. Let $(Q_n)$ be a sequence in $\mathcal{Q}$. Let $P_n \in \mathcal{P}$ such that $P_n \subseteq Q_n$. Since $\mathcal{P}$ is relatively compact, without loss of generality, we can assume that $(P_n)$ is pseudo-finite; the union of which is denoted by $Q$. We see that either $Q_n \subseteq Q$ or $Q \subseteq Q_n$. If there are infinitely many $Q_n$ contained in $Q$, then those $Q_n$ form a pseudo-finite sequence, following from the pseudo-finiteness of $(P_n)$. On the other hand, if there are infinitely many $Q_n$ containing $Q$, then those $Q_n$ form a chain, and the previous lemma enable us to find a non-decreasing sequence of ideals. Thus $\mathcal{Q}$ is relatively compact. The result then follows from the definition of $\mathcal{Q}$. □

**Lemma 5.4.** $\mathcal{Q}$ is the set of unions of pseudo-finite sequences of ideals in $\mathcal{P}$.

Proof. We only need to prove that each ideal $Q \in \mathcal{Q}$ is the union of a pseudo-finite sequence in $\mathcal{P}$. For this purpose, without loss of generality, we can suppose that $\mathcal{P}$ is countable and that $Q$ is the union of $\mathcal{P}$. It is obvious that, in this case, any chain in $\mathcal{Q}$ is countable.

Case 1: $Q$ is the union of a chain of ideals in $\Omega \setminus \{Q\}$. By the countability and well-ordering of the chain, there exist $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q$ such that $Q = \bigcup_{n=1}^\infty Q_n$. For each $n$, choose $P_n \in \mathcal{P}$ such that $P_n \subseteq Q_{n+1}$ but $P_n \not\subseteq Q_n$. Since $\mathcal{P}$ is relatively compact, without loss of generality, we can assume that $(P_n)$ is pseudo-finite; its union is denoted by $Q'$. Then $Q' \subseteq Q$, and for each $n \geq 2$, either $Q_n \subseteq Q'$ or $Q' \subseteq Q_n$ (since both contain $P_{n-1}$). Since $P_n \not\subseteq Q_n$, we must have $Q_n \subseteq Q'$ ($n \geq 2$). So $Q = Q'$.

Case 2: $Q$ is not the union of any chain of ideals in $\Omega \setminus \{Q\}$. Then any $P \in \mathcal{Q} \setminus \{Q\}$ is contained in a maximal element of $\mathcal{Q} \setminus \{Q\}$. Since $Q$ cannot be the union of any finite number of prime ideals properly contained in $Q$, each $\Omega = \{Q\}$ which implies that $Q \in \mathcal{P}$ or there exists infinitely many maximal elements of $\Omega \setminus \{Q\}$. In the latter case, let $Q_n (n \in \mathbb{N})$ be distinct maximal elements of $\mathcal{Q} \setminus \{Q\}$. Choose $P_n \in \mathcal{P}$ such that $P_n \subseteq Q_n$. By the relative compactness of $\mathcal{P}$ and without loss of generality, we can assume that $(P_n : n \in \mathbb{N})$ is a pseudo-finite sequence. Set $Q' = \bigcup_{n=1}^\infty P_n$. Then $Q' \in \mathcal{Q}$, and for each $n \in \mathbb{N}$, either $Q_n \subseteq Q'$ or $Q' \subseteq Q_n$. Assume toward a contradiction that $Q' \neq Q$. The maximality of the ideal $Q_n$ in $\mathcal{Q} \setminus \{Q\}$ implies that $Q' \subseteq Q_n (n \in \mathbb{N})$. This implies that $\{Q_n : n \in \mathbb{N}\}$ form a chain, contradicting the maximality and distinctness of the ideals $Q_n$. Thus $Q = Q$.

In both cases, we see that $Q$ is the union of a pseudo-finite sequence in $\mathcal{P}$. □

In summary, we have the following.

**Proposition 5.5.** The closure $\mathcal{Q}$ of $\mathcal{P}$ satisfies the following:

(i) $\mathcal{Q}$ is the set of unions of pseudo-finite sequences of ideals in $\mathcal{P}$;
(ii) $\mathcal{Q}$ is compact;
(iii) every chain in $\mathcal{Q}$ is well-ordered with respect to the inclusion;
(iv) $\bigcap \mathfrak{P} = \bigcap \Omega$.

From (i), we see that $\Omega$ is the smallest compact family of prime ideals containing $\mathfrak{P}$. Property (iii) also shows that the intersection of $\mathfrak{P}$ is equal the intersection of its minimal elements.

In the remaining of the section, we consider $\Omega$ to be any compact family of prime ideals in $\mathcal{C}_0(\Omega)$.

**Lemma 5.6.** Let $P$ be in $\Omega$. Then there exists $a \not\in P$ but $a \in Q$ for all $Q \in \Omega$ such that $Q \not\subset P$.

**Proof.** Assume the contrary. As in Lemma 4.6, we can construct $(a_n) \subset A$, $(Q_n) \subset \Omega$ satisfying $a_1 \ldots a_n \not\in Q_n$ but $a_1 \ldots a_{n+1} \in Q_n$ ($n \in \mathbb{N}$). By the compactness, $(Q_n)$ has a pseudo-finite subsequence $(Q_{n_i})$. However, $a_1 \ldots a_{n_2} \in Q_{n_1}$, but $a_1 \ldots a_{n_2} \not\in Q_{n_i}$ ($i \geq 2$); a contradiction to the pseudo-finiteness. $\square$

We say that an ideal $Q$ is a roof of $\Omega$ if it is the union of the ideals in a maximal chain in $\Omega$. A roof must be either a prime ideal in $\mathcal{C}_0(\Omega)$ or $\mathcal{C}_0(\Omega)$ itself.

**Lemma 5.7.** $\Omega$ has only finitely many roofs. Also, there are only finitely many maximal modular ideals in $\mathcal{C}_0(\Omega)$ such that each of them contains an ideal in $\Omega$.

**Proof.** We shall prove that there are only finitely many disjoint maximal chains in $\Omega \setminus \{\mathcal{C}_0(\Omega)\}$; the lemma then follows. Assume the contrary that $\mathfrak{E}_n$ $(n \in \mathbb{N})$ are disjoint maximal chains in $\Omega \setminus \{\mathcal{C}_0(\Omega)\}$. Pick $Q_n \in \mathfrak{E}_n$. Without loss of generality, we can suppose that $(Q_n)$ is pseudo-finite; the union is denoted by $Q$. We see that $Q \in \Omega \setminus \{\mathcal{C}_0(\Omega)\}$, and since $Q \supset Q_n$ and $\mathfrak{E}_n$ is maximal, we have $Q \in \mathfrak{E}_n$ $(n \in \mathbb{N})$, contradicting the disjointness of the chains $\mathfrak{E}_n$. $\square$

For a function $f \in \mathcal{C}_0(\Omega)$, we can define a new function $\frac{f}{\sqrt{|f|}}$, also in $\mathcal{C}_0(\Omega)$, as follows:

$$\left(\frac{f}{\sqrt{|f|}}\right)(x) = \begin{cases} \frac{f(x)}{\sqrt{|f(x)|}} & \text{for } x \in \Omega \text{ with } f(x) \neq 0, \\ 0 & \text{for } x \in \Omega \text{ with } f(x) = 0 \end{cases}$$

The following lemma and proposition are inspired by a suggestion of an anonymous referee of an initial version of [14].

**Lemma 5.8.** Suppose that $\Omega$ is a compact family of prime ideals in $\mathcal{C}_0(\Omega)$ with a maximum element $Q$. Set $I = \bigcap \Omega$. Let $a \in \mathcal{C}_0(\Omega) \setminus Q$ and let $b \in Q$. Then there exists $s \in Q$ such that $as - b \in I$.

**Proof.** It is standard that for each prime ideal $P \subset Q$ there exists $s \in Q$ such that $as - b \in P$.

Assume toward a contradiction that for all $s \in Q$ we have $as - b \not\in I$. Set $s_1 = 0$, $b_1 = b - as_1 = b$, and $\Omega_1 = \{P \in \Omega: b_1 \not\in P\}$. Suppose that we have already constructed $s_n \in Q$, $b_n = b - as_n$, and $\Omega_n = \{P \in \Omega: b_n \not\in P\}$ such that $\Omega_n \subseteq \cdots \subseteq \Omega_1$.

Since $b_n \not\in I$ by the assumption, we have $\Omega_n \neq \emptyset$. Choose $P_n \in \Omega_n$. We see that $b_n \in Q$, and so $\frac{b_n}{\sqrt{|b_n|}} \in Q$. Thus, there exists $s' \in Q$ such that $as' - \frac{b_n}{\sqrt{|b_n|}} \in P_n$. 


Set $s_{n+1} = s_n + s'\sqrt{b_n}$, 

$$b_{n+1} = b - as_{n+1} = \left( \frac{b_n}{\sqrt{|b_n|}} - as' \right) \sqrt{|b_n|},$$

and $\Omega_{n+1} = \{ P \in \Omega : b_{n+1} \notin P \}$. We see that $\Omega_{n+1} \subseteq \Omega_n$; $P_n \in \Omega_n \setminus \Omega_{n+1}$. Thus the construction can be continued inductively.

In particular, we have $b_m \in P_n (m > n \in \mathbb{N})$ but $b_m \notin P_n (m \leq n \in \mathbb{N})$. The compactness implies that there exists a pseudo-finite subsequence $(P_{n_i})$. However, this contradict the fact that $b_{n_i} \notin P_{n_{i-1}}$, but, for all $i \geq 2$, $b_{n_i} \notin P_{n_i}$.

\[ \square \]

**Proposition 5.9.** Suppose that $\Omega$ is non-compact and $\Omega$ is a compact family of non-modular prime ideals in $\mathcal{C}_0(\Omega)$ with a maximum element $Q$. Set $I = \bigcap \Omega$. Let $P \in \Omega$, and let $A$ be a subalgebra of $\mathcal{C}_0(\Omega)$. Suppose that $\mathcal{C}_0(\Omega) = A + P$ and $A \cap P$ is the intersection of a sub-family of $\Omega$. Let $B$ be a subalgebra of $A$ such that $B$ is maximal with respect to the property that $B \cap Q \subset I$. Then $\mathcal{C}_0(\Omega) = B + Q$.

**Proof.** When $a - b \in I$ we also write $a = b \pmod{I}$.

Note that $I \subset A \cap P$. By maximality of $B$, we see that $I \subset B$, so indeed $B \cap Q = I$. We can also check, using the maximality of $B$ again, that $B \neq I$.

Moreover, $I$ is non-modular in $B$; for otherwise, $Q$ is modular in $B + Q$ (because $B/I \cong (B + Q)/Q$), and from the primeness of $Q$ we can deduce that $Q$ is modular in $\mathcal{C}_0(\Omega)$ — a contradiction. Thus $I$ is indeed a non-modular prime ideal in $B$, and hence, a prime ideal in $B^\#$.

**Claim:** for each $a \in A \setminus Q$ and each $b \in A \cap Q$, there exists $s \in A \cap Q$ such that $as - b \in I$. Indeed, by the previous lemma, there exists $s \in Q$ such that $as - b \in I$.

Write $s = c + p$ where $c \in A$ and $p \in P$. Then $ap + (ac - b) \in I \subset A$ implies that $ap \in A \cap P$. Since $a \notin Q$ and $A \cap P$ is the intersection of a family of prime ideals contained in $Q$, we must have $p \in A \cap P$. Thus $s = c + p \in A \cap Q$.

We shall prove that $A = B + (A \cap Q)$; the proposition then follows.

Assume toward a contradiction. Let $a \in A$ but $a \notin B + Q$. By the maximality of $B$, there exists a polynomial $q(X)$ in $B^\#(X) \setminus I[X]$ such that $q(a) \in Q$. Let $q(X)$ be such a polynomial with smallest degree. By multiply with some element in $B \setminus I$, we can further suppose that the coefficients of $q(X)$ are in $B$. Then, we see that $q'(a) \in A \setminus Q$, where $q'(X)$ is the formal derivative. Let $s \in \mathcal{C}_0(\Omega)$. Then

$$q(a + q'(a)s) = q(a) + q'(a)^2s + \ldots + q'(a)^nq^{(n)}(a)\frac{s^n}{n!};$$

where $q^{(k)}(X)$ is the formal $k^{th}$ derivative of $q(X)$. By the claim, there exists

$$d \in A \cap Q \text{ such that } q(a) = q'(a)^2d \pmod{I}.\tag{1}$$

So, for all $s \in \mathcal{C}_0(\Omega)$,

$$q(a + q'(a)s) = q'(a)^2\left( d + s + \ldots + q'(a)^{n-2}q^{(n)}(a)\frac{s^n}{n!} \right) \pmod{I}. $$

Since $(A/(A \cap P)) \cong \mathcal{C}_0(\Omega)/P$ is radical and $(\mathcal{C}_0(\Omega)/P)^\#$ is Henselian, there exists $s \in A$ such that

$$d + s + \ldots + q'(a)^{n-2}q^{(n)}(a)\frac{s^n}{n!} \in A \cap P \subset Q;$$
For clarity, we shall use $U \subseteq \pi(C)$.

Let $Q \subseteq C$ be prime in $p$ is radical and deduce that $\mathbb{Q} \in B$ such that polynomial with the smallest degree among all polynomials in $\mathbb{Q}$ such that $q(b) \in Q$.

Hence, without loss of generality, we can assume from the start that $q(a) \in P$.

The case where $q(a) \in P$: Then $d \in P$ by (1), so $d \in A \cap P$. Again, since $\mathcal{C}_0(\Omega)/I$ is radical and $\mathcal{C}_0(\Omega)/I$ is Henselian, there exists $t \in \mathcal{C}_0(\Omega)$ such that $d + t + \ldots + q'(a)^{n-2}q^{(n)}(a)\frac{t^n}{n!} \in I \subset A \cap P$.

Since $A \cap P$ is an ideal in $\mathcal{C}_0(\Omega)$, it follows that $t \in A \cap P$. Set $c = a + q'(a)t$.

Then $c \in A$ but $c \notin B + Q$, $q(c) \in I$, and $q(X)$ is a polynomial with the smallest degree among all polynomials in $B^a[X] \setminus I[X]$ such that $q(c) \in Q$. The maximality of $B$ implies that there exists a polynomial $p(X)$ with coefficients in $B^a$ such that $p(c) \in Q \setminus I$. We see that there exist an element $u \in B \setminus I$ and a polynomial $h(X)$ with coefficient in $B^a$ such that $up(X) = q(X)h(X)$ (mod $I$); this is possible since $I$ is prime in $B^a$. Then $up(c) = q(c)h(c) = 0$ (mod $I$). Since $u \notin Q$ (otherwise, $u \notin B \cap Q = I$) and $I$ is the intersection of some prime ideals contained in $Q$, we deduce that $p(c) \in I$; a contradiction. 

\[\Box\]

A special case of the previous proposition is when $A = \mathcal{C}_0(\Omega)$.

6. HOMOMORPHISMS FROM $\mathcal{C}_0(\Omega)$

In this section, we shall show the connection between continuity ideals as well as the kernels of homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras and intersections of (relatively) compact families of prime ideals. One direction is an immediate consequence of the results in §4, so most of this section concerns the converse.

We shall need some basic complex algebraic-geometry results. Our references for algebraic geometry will be [13]. For a set $S \subseteq \mathbb{C}[Z_1, Z_2, \ldots, Z_n]$, denote by $\mathcal{V}(S)$ the variety (i.e., common zero set) of $S$ in $\mathbb{C}^n$. For each prime ideal $Q$ in $\mathbb{C}[Z_1, \ldots, Z_n]$, the variety $\mathcal{V}(Q)$ is irreducible. The topology considered on complex spaces will be the Euclidean topology. We shall need the fact that, for each irreducible variety $V$ and each variety $W$ not containing $V$, $V \setminus W$ is dense and (relatively) open in $V$ [13, Chapter IV, Theorem 2.11].

Notation. For clarity, we shall use $X_i, Y_j$ for variables, $x_i, y_j$ for complex numbers, and $a_i, b_j$ for elements of an algebra. When there is no ambiguity, we shall use boldface characters to denote tuples of elements of the same type; for example, we set

$X = (X_1, X_2, \ldots, X_m)$ or $y = (y_1, \ldots, y_n)$.

In the case where $X = (X_1, \ldots, X_m)$, we also denote by $\mathcal{C}_X$ the corresponding space $C^m$.

Lemma 6.1. Let $m, n \in \mathbb{N}$, and let $Q$ be a prime ideal in $\mathbb{C}[X, Y]$, where $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$. Consider $Q_X = Q \cap \mathbb{C}[X]$ as a prime ideal in $\mathbb{C}[X]$. Let $V$ be the variety of $Q$, and let $V_X$ be the variety of $Q_X$. Let $\pi$ be the natural projection $\mathcal{C}_{X, Y} \twoheadrightarrow \mathcal{C}_X$. Then $\pi : V \rightarrow V_X$ and there exists a dense open subset $U$ of $V$ such that $\pi : U \rightarrow V_X$ is an open map.
Proof. Obviously, \( \pi : V \to V_X \). Without loss of generality, let \( (X_1, \ldots, X_k) \) be a transcendental basis for \( \mathbb{C}[X] \) modulo \( Q_X \). We consider \( \mathbb{C}^k = \mathbb{C}_{X_1, \ldots, X_k} \). Denote by \( \pi_1 \) the natural projection \( \mathbb{C}_X \to \mathbb{C}^k \).

By [14, Lemma 6.3], there exist dense open subsets \( U \) and \( V_X \) respectively, such that \( \pi_1 : U \to \mathbb{C}^k \) and \( \pi_2 : U_X \to \mathbb{C}^k \) are open maps. Inspecting the proof of [14, Lemma 6.3], we see that \( U_X \) can be chosen as \( V_X \setminus V_0 \), where \( V_0 \) is a proper subvariety of \( V_X \), and that \( \pi_2 \) is even a local homeomorphism from \( U_X \) onto an open subset of \( \mathbb{C}^k \). Since \( V_0 \) has dimension at most \( k - 1 \) [13, Chapter IV], we can further require (we may need to shrink \( U_X \)) that

\[
\pi_2(U_X) \cap \pi_2(U_X \setminus U_X) = \emptyset.
\]

Let \( W = \pi_1(U) \cap \pi_2(U_X) \). Then \( W \) is an open set in \( \mathbb{C}^k \). It can be seen that \( W \) is dense in \( \pi_1(U) \). Set

\[
U' = U \cap \pi_1^{-1}(W).
\]

Then \( U' \) is a dense open subset of \( U \), by the openness of \( \pi_1 : U \to \mathbb{C}^k \), and hence a dense open subset of \( V \).

We claim that \( \pi : U' \to V_X \) is an open map. Indeed, let \( (x, y) \in U' \) be arbitrary. Then \( (x_1, \ldots, x_k) \in W \), and so \( x \in U_X \) by (2). Choose \( \Delta_X \) be a neighbourhood of \( x \) in \( U_X \), such that \( \pi_2 \) is a homeomorphism from \( \Delta_X \) onto an open subset of \( \mathbb{C}^k \). Let \( \Delta \) be any neighbourhood of \( (x, y) \) in \( U' \) contained in \( \pi^{-1}(\Delta_X) \cap U'. \) Then, we see that \( \pi(\Delta) \subset \Delta_X \) and \( \pi_2(\pi(\Delta)) = \pi_1(\Delta) \), which is open in \( \mathbb{C}^k \) by the openness of \( \pi_1 \) on \( U. \) It follows that \( \pi(\Delta) \) is open in \( \Delta_X \), and thus, in \( V_X \).

\[ \square \]

**Proposition 6.2.** Let \( A = C_0(\Omega) \) for a locally compact space \( \Omega \), and let \( Q \) be a non-empty compact family of non-modular prime ideals in \( A \). Suppose that every chain in \( Q \) is countable. Then there exist a cardinal \( \kappa \), a free ultrafilter \( \mathcal{U} \) on \( \kappa \), and, for each \( P \in \mathcal{U} \), a homomorphism \( \theta_P : A \to (C^\kappa/\mathcal{U})^\kappa \) such that:

(a) \( \ker \theta_P = P \) (\( P \in \mathcal{U} \)), and

(b) the set \{\( \theta_P(a) : P \in \mathcal{U} \)\} is finite for each \( a \in A \).

Note that \( \Omega \) must be non-compact and \( A \) must be non-unital. Since each \( P \in \mathcal{U} \) is a non-modular prime ideal in \( A \), it is a prime ideal in \( A^\# \). For each \( Q \in \mathcal{U} \), set

\[ \Omega_Q = \{P \in \mathcal{U} : P \subset Q\}, \]

and set \( I_Q = \bigcap \Omega_Q \). We begin the proof of Proposition 6.2 with some lemmas:

**Lemma 6.3.** Let \( Q_* \subset Q^* \subset \mathcal{U} \). Let \( A_* \supset A^* \) be subalgebras of \( A \) such that \( A^* \cap Q* \subset I_Q^* \), and \( A_* \cap Q_* = I_Q^* \). Suppose further that \( A_* + Q_* \subset A \). Let \( \mathcal{F} \) be a chain in \( \Omega_Q \), where each ideal in \( \mathcal{F} \) contains \( Q_* \). Then, we can find subalgebras \( A_Q \subset A \) (\( Q \in \mathcal{F} \)) satisfying the following conditions:

(i) \( A_Q \cap Q = I_Q \) and \( A = A_Q + Q \) (\( Q \in \mathcal{F} \));

(ii) for each \( Q_1 \subset Q_2 \in \mathcal{F} \), we have \( A_* \supset A_{Q_1} \supset A_{Q_2} \supset A^* \).

\[ \text{Proof.} \] Assume toward a contradiction that the lemma fails for a particular instance of \( Q_* \), \( Q^* \), \( A_* \), \( A^* \), and \( \mathcal{F} \); we can further require that \( \kappa(\mathcal{F}) \) is smallest among such instances (note that \( \mathcal{F} \) must be well-ordered by Proposition 5.5(iii)).

Note that, for each \( Q \in \mathcal{F} \), \( A_* \cap Q_* = \bigcap \Omega_Q \) is the intersection of a sub-family of \( \Omega_Q \), and

\[ A^* \cap Q \subset A^* \cap Q_* \subset I_Q^* \subset I_Q. \]
Thus, we are in situation where Proposition 5.9 can be applied.

Proposition 5.9 implies that \( C \) must be infinite; otherwise, we can construct \( A_Q \) \((Q \in C)\) by finite induction. If \( C \) is order isomorphic to \( \omega \), the first infinite ordinal, say

\[
C = \{Q_1 \subset Q_2 \subset \cdots\},
\]

then Proposition 5.9 also enables us to construct \((A_{Q_n})\) inductively.

In general, there exists a sequence \((Q_n)\) in \( C \) converging in the order topology to \( Q_\infty = \bigcup C \). (This is where our proof needs the countability condition in Proposition 6.2.) If \( Q_\infty \in C \), we can, by Proposition 5.9 construct \( A_{Q_\infty} \), first. As in the previous paragraph, we can (then) find \( A_{Q_n} \) \((n \in \mathbb{N})\) satisfying both the conditions (i) and (ii). The ideals \( Q_n \) divide \( C \setminus \{Q_\infty\} \) into chains which are order isomorphic to ordinals strictly smaller than \( o(C) \). The minimality of \( o(C) \) (and the assumption) then imply that we can extend the present collection \( \{A_Q\} \) to the whole \( Q \in C \) that satisfies the conditions (i) and (ii).

Thus, in any case, we have a contradiction. \( \square \)

**Lemma 6.4.** We can find subalgebras \( A_Q \subset A \) \((Q \in \Omega)\) satisfying the following conditions:

(i) \( A_Q \cap Q = I_Q \) and \( A = A_Q + Q \) \((Q \in \Omega)\);

(ii) for each \( Q_1 \subset Q_2 \in \Omega \), we have \( A_{Q_1} \supset A_{Q_2} \).

**Proof.** This follows from the previous lemma, Zorn’s lemma, and the fact that all prime ideals containing a given prime ideal in \( A = C_0(\Omega) \) form a chain. \( \square \)

Let \( \kappa \) be the set of all tuples of the form \((\delta; \mathcal{G}; a_1, \ldots, a_m)\), where \( \delta > 0 \), \( \mathcal{G} \) is a non-empty finite subsets of \( \Omega \), and \((a_1, \ldots, a_m)\) is a non-empty finite sequence of distinct elements in \( A \). Define a partial order \(<\) on \( \kappa \) by setting

\[
(\delta; \mathcal{G}; a_1, a_2, \ldots, a_m) < (\delta'; \mathcal{G}'; a'_1, a'_2, \ldots, a'_m)
\]

if \( \delta > \delta', \mathcal{G} \subset \mathcal{G}', \{a_1, \ldots, a_m\} \) is a subset of \( \{a'_1, a'_2, \ldots, a'_m\} \). Then \((\kappa, <)\) is a net. Fix an ultrafilter \( \mathcal{U} \) on \( \kappa \) majorizing this net.

**Lemma 6.5.** Let \( A_Q \) \((Q \in \Omega)\) be as in the previous lemma. Let \( w = (\delta; \mathcal{G}; a_1, \ldots, a_m) \in \kappa \). Then, for each \( P \in \mathcal{G} \), we can find a tuple

\[
\tau_P(w) = x^{(P)_m} = (x^{(P)}_1, \ldots, x^{(P)}_m)
\]

in \( C^m \) satisfying all the following conditions:

(i) \( p(x^{(P)}_1, \ldots, x^{(P)}_m) = 0 \) for each \( p \in \mathbb{C}[X_1, \ldots, X_m] \) with \( p(a_1, \ldots, a_m) \in P \);

(ii) for each \( 1 \leq k \leq m \) with \( a_k \notin P \), we have \( x^{(P)}_k \neq 0 \);

(iii) \( |x^{(P)}_k| \leq \delta \) \((1 \leq k \leq m)\).

(iv) for each \( 1 \leq k \leq m \) and each \( P \subset Q \in \mathcal{G} \) such that \( a_k \in A_Q \), we have \( x^{(P)}_k = x^{(Q)}_k \).

**Proof.** Set \( X = (X_1, \ldots, X_m) \). For each \( P \in \mathcal{G} \), set \( a^P = (a_1, \ldots, a_m) \cap A_P \) and set \( X^P = (X_i : a_i \in A_P) \). Without loss of generality, we can assume that none of these is empty. Note that, when \( P \subset Q \in \mathcal{G} \) then \( a^Q \subset a^P \) and \( X^Q \subset X^P \).

Denote by \( \pi_P \) the projection from \( C_X \) onto \( C_{X^P} \), and \( \pi_{PQ} \) the projection from \( C_{X^P} \) onto \( C_{X^Q} \) \((P \subset Q \in \mathcal{G})\). We also conveniently consider \( C_{X^P} \) as a subspace of \( C_X \).
For each \( Q \in \mathfrak{S} \), define \( \tilde{Q} = \{ p \in \mathbb{C}[X] : p(a) \in Q \} \), and
\[
\tilde{Q} = \{ p \in \mathbb{C}[X^Q] : p(a^Q) \in Q \}.
\]
We see that for \( P \subset Q \in \mathfrak{S} \), \( \tilde{P} \subset \tilde{Q} \) are prime ideals in \( \mathbb{C}[X] \), and \( \tilde{Q} = \tilde{Q} \cap \mathbb{C}[X^Q] \) is a prime ideal in \( \mathbb{C}[X^Q] \). Also, since \( A_Q \cap Q = A_Q \cap P \) for each \( P \subset Q \in \mathfrak{S} \), we see that
\[
\tilde{Q} = \{ p \in \mathbb{C}[X^Q] : p(a^Q) \in P \} = \tilde{P} \cap \mathbb{C}[X^Q];
\]
and thus \( \tilde{Q} = \mathbb{C}[X^Q] \cap \tilde{P} \).

It follows from Lemma 6.1 that there exist, for each \( Q \in \mathfrak{S} \), dense open subsets \( U_Q \) of \( \mathcal{V}(\tilde{Q}) \) and \( W_Q \) of \( \mathcal{V}(\tilde{Q}) \), respectively, such that:
1. \( \pi_Q : U_Q \to W_Q \) is an open map;
2. \( \pi_{PQ} : W_P \to W_Q \) is an open map \( (P \subset Q \in \mathfrak{S}) \).
(We have only finitely many varieties here.)

For each \( Q \in \mathfrak{S} \), set
\[
V_Q = \bigcup_{1 \leq r \leq m, a_r \notin Q} \{ x = (x_1, \ldots, x_m) : x_r = 0 \}.
\]
Then \( V_Q \) is a variety which does not contain \( \mathcal{V}(\tilde{Q}) \), and so \( \mathcal{V}(\tilde{Q}) \setminus V_Q \) is also a dense open subset of \( \mathcal{V}(\tilde{Q}) \) because \( \mathcal{V}(\tilde{Q}) \) is irreducible. Therefore, \( U_Q \setminus V_Q \) is again a dense open subset of \( \mathcal{V}(\tilde{Q}) \).

Set
\[
\Delta = \{ x = (x_1, \ldots, x_m) : |x_r| < \delta (1 \leq r \leq m) \}.
\]
Finally, set \( U'_Q = (U_Q \setminus V_Q) \cap \Delta \) and \( W'_Q = W_Q \cap \Delta \). Note that the origin \( 0 \) is in \( \mathcal{V}(\tilde{Q}) \) and \( \mathcal{V}(\tilde{Q}) (Q \in \mathfrak{S}) \). So \( U'_Q \) (respectively, \( W'_Q \)) is a non-empty open subset of \( \mathcal{V}(\tilde{Q}) \) (respectively, \( \mathcal{V}(\tilde{Q}) \)).

The conditions (i), (ii), and (iii) will be satisfied as long as we choose \( x^{(Q)} \in U'_Q \) \((Q \in \mathfrak{S}) \). We are now concerning with condition (iv).

Fix \( P \subset Q \in \mathfrak{S} \). Since \( U_P \setminus V_P \) is dense in \( \mathcal{V}(\tilde{P}) \) and \( U'_Q \subset \mathcal{V}(\tilde{Q}) \subset \mathcal{V}(\tilde{P}) \), we have \( \pi_Q(U'_P \cap \pi_Q(U'_Q)) \) is dense (and open) in \( \pi_Q(U'_Q) \) (which is open in \( W_Q \)). Note that \( \pi_Q(U'_P) = \pi_{PQ}([\pi_P(U'_P)]) \). In fact, we see that for every dense open subset \( D \) of \( \pi_P(U'_P) \), \( \pi_{PQ}(D) \cap \pi_Q(U'_Q) \) is dense and open in \( \pi_Q(U'_Q) \).

Thus, we can define a dense open subset \( D_Q \) of \( \pi_Q(U'_Q) \) such as \( x^{(Q)} \in D_Q \) \((Q \in \mathfrak{S}) \) as follows: For \( P \) minimal in \( \mathfrak{S} \), set \( D_P = \pi_P(U'_P) \). Then, define inductively,
\[
D_Q = \bigcap_{P \in \mathfrak{S}, P \subset Q} \pi_{PQ}(D_P) \cap \pi_Q(U'_Q).
\]
The non-emptiness of the sets \( D_P \) then allows us to choose tuples \( \alpha^Q \in D_Q \) \((Q \in \mathfrak{S}) \) such that \( \alpha^Q = \pi_{PQ}(\alpha^P) \) whenever \( P \subset Q \in \mathfrak{S} \). (This can be carried out inductively, starting from the maximal elements in \( \mathfrak{S} \); note that \( \mathfrak{S} \) is finite and that the prime ideals containing a given prime ideal form a chain.) Finally choose \( x^{(Q)} \in U'_Q \) \((Q \in \mathfrak{S}) \) such that \( \pi_Q(x^{(Q)}) = \alpha^Q \). It can be checked that these are the desired tuples.

\[ \square \]

Proof of Proposition 6.2. We keep the same notations as in the previous lemmas.

Define \( \xi_P : A \to \mathbb{C}^n (P \in \mathfrak{Q}) \) as follows: For each \( a \in A \) and each \( w = (\delta; \mathfrak{S}; a_1, \ldots, a_m) \in \kappa \), if \( P \in \mathfrak{S} \) and if \( a \) is in \( (a_1, \ldots, a_m) \), say \( a = a_k \) (there
The idea of mapping into an ultrapower in the above proposition generalise (CH) since Let Let Let (i) Since Let I continuity ideal in a compact family of prime ideals with the same intersection. Then 5.5 then allows us to pass from a relatively compact family of prime ideals to see that every chain in elements in I for each I is eventually constant.

Proof. (i) Since \( \theta \) maps into a radical algebra, we see that ker \( \theta \) is non-modular for each \( f \in \mathcal{C}_0(\Omega) \). Theorem 1.1 shows that ker \( \theta = \mathcal{I}(\theta) \), and so, it is an abstract continuity ideal in \( \mathcal{C}_0(\Omega) \). The result then follows from Corollary 4.13(i). Proposition 5.5 then allows us to pass from a relatively compact family of prime ideals to a compact family of prime ideals with the same intersection.

(ii) Let \( \mathfrak{P} \) be a relatively compact family of non-modular prime ideals in \( \mathcal{C}_0(\Omega) \) such that \( I = \bigcap \mathfrak{P} \). We can assume that \( \mathfrak{P} \neq \emptyset \). By keeping only the minimal elements in \( \mathfrak{P} \), we can further suppose that \( P \not\subset Q \) (\( P \neq Q \in \mathfrak{P} \)) (cf. Proposition 5.5(iii)). Since \( I \) is the intersection of a countable family of prime ideals, we see that \( \mathfrak{P} \) is countable (using Lemma 5.6). Let \( \Omega \) be the closure of \( \mathfrak{P} \). It is easy to see that every chain in \( \Omega \) is countable.

Let \( \theta_P : \mathcal{C}_0(\Omega) \to (\mathcal{C}^*/\mathcal{U})^\circ \) (\( P \in \Omega \)) be homomorphisms as in Proposition 6.2. Let \( B \) be the subalgebra of \( (\mathcal{C}^*/\mathcal{U})^\circ \) generated by all the images of \( \theta_P \) (\( P \in \Omega \)). Then \( B \) is a non-unital integral domain. We also have

\[
|B| = \left| \bigcup_{P \in \Omega} \theta_P(\mathcal{C}_0(\Omega)) \right| = \left| \bigcup_{a \in \mathcal{C}_0(\Omega)} \{ \theta_P(a) : P \in \Omega \} \right| = \varepsilon;
\]


We are now ready to prove our main results.

**Theorem 6.6.** Let \( \Omega \) be a locally compact space.

(i) Let \( \theta \) be a homomorphism from \( \mathcal{C}_0(\Omega) \) into a radical Banach algebra \( \mathcal{R} \). Then ker \( \theta \) is the intersection of a (relatively) compact family of non-modular prime ideals in \( \mathcal{C}_0(\Omega) \).

(ii) (CH) Let \( I \) be the intersection of a relatively compact family of non-modular prime ideals in \( \mathcal{C}_0(\Omega) \) such that \( I \) is also the intersection of a countable family of prime ideals. Suppose that

\[
|\mathcal{C}_0(\Omega)/I| = \varepsilon.
\]

Then there exists a homomorphism \( \theta \) from \( \mathcal{C}_0(\Omega) \) into a radical Banach algebra such that ker \( \theta = I \).

Proof. (i) Since \( \theta \) maps into a radical algebra, we see that ker \( \theta \) is non-modular for each \( f \in \mathcal{C}_0(\Omega) \). Theorem 1.1 shows that ker \( \theta = \mathcal{I}(\theta) \), and so, it is an abstract continuity ideal in \( \mathcal{C}_0(\Omega) \). The result then follows from Corollary 4.13(i). Proposition 5.5 then allows us to pass from a relatively compact family of prime ideals to a compact family of prime ideals with the same intersection.

(ii) Let \( \mathfrak{P} \) be a relatively compact family of non-modular prime ideals in \( \mathcal{C}_0(\Omega) \) such that \( I = \bigcap \mathfrak{P} \). We can assume that \( \mathfrak{P} \neq \emptyset \). By keeping only the minimal elements in \( \mathfrak{P} \), we can further suppose that \( P \not\subset Q \) (\( P \neq Q \in \mathfrak{P} \)) (cf. Proposition 5.5(iii)). Since \( I \) is the intersection of a countable family of prime ideals, we see that \( \mathfrak{P} \) is countable (using Lemma 5.6). Let \( \Omega \) be the closure of \( \mathfrak{P} \). It is easy to see that every chain in \( \Omega \) is countable.
since, for each \( b \in a + I \), we have

\[
\{ \theta_P(b) : P \in \Omega \} = \{ \theta_P(a) : P \in \Omega \},
\]

which is finite. Thus [9] there exists an embedding \( \psi : B \to R_0 \) where \( R_0 \) is any universal radical Banach algebra. Then, the following map

\[
\theta : C_0(\Omega) \to \prod_{P \in \Omega} R_0, \quad a \mapsto ((\psi \circ \theta_P)(a) : P \in \Omega)
\]

is a homomorphism with kernel \( \bigcap \Omega = I \). It can then be deduced from Proposition 6.2 that the image of \( \theta \) is in fact contained in the radical of \( \ell^\infty(\Omega, R_0) \).

**Theorem 6.7.** Let \( \Omega \) be a locally compact space.

(i) Let \( \theta \) be a homomorphism from \( C_0(\Omega) \) into a Banach algebra \( B \). Then \( \mathcal{I}(\theta) \) is the intersection of a (relatively) compact family of prime ideals in \( C_0(\Omega) \).

(ii) (CH) Let \( I \) be the intersection of a relatively compact family of prime ideals in \( C_0(\Omega) \) such that \( I \) is also the intersection of a countable family of prime ideals. Suppose that

\[
|C_0(\Omega)/I| = \epsilon.
\]

Then there exists a homomorphism \( \theta \) from \( C_0(\Omega) \) into a Banach algebra such that \( \mathcal{I}(\theta) = I \).

**Proof.** (i) The continuity ideal \( \mathcal{I}(\theta) \) is an abstract continuity ideal in \( C_0(\Omega) \). The result then follows from Corollary 4.13(i) and Proposition 5.5.

(ii) Let \( \mathfrak{P} \) be a relatively compact family of prime ideals in \( C_0(\Omega) \) such that \( I = \bigcap \mathfrak{P} \). We can assume that \( \mathfrak{P} \neq \emptyset \). As in the previous proof, we can assume that \( \mathfrak{P} \) is countable.

Denote by \( \mathfrak{P}_0 \) the set of non-modular ideals in \( \mathfrak{P} \). Let \( \Omega' \) be the closure of \( \mathfrak{P} \setminus \mathfrak{P}_0 \). By Lemma 5.7, \( \Omega' \) has only finitely many roofs. Denote by \( Q_1, \ldots, Q_n \) the roofs of \( \Omega' \), and set \( \mathfrak{P}_k = \{ P \in \mathfrak{P} \setminus \mathfrak{P}_0 : P \subseteq Q_k \} \).

Let \( 1 \leq k \leq n \). First, it is easy to see that \( Q_k \) is a modular prime ideal. Pick a modular identity \( u \) for \( Q_k \), and pick \( a \notin Q_k \). Then \( a - au \in Q_k \), and so, by Lemma 5.8, there exists \( v \in Q_k \) such that \( a - au - av \in \bigcap \mathfrak{P}_k \). It follows easily that \( u + v \) is a modular identity for \( \bigcap \mathfrak{P}_k \); denote it by \( u_k \).

Theorem 6.6 shows that there exists a homomorphism \( \theta_0 \) from \( C_0(\Omega) \) into a radical Banach algebra \( R_0 \) such that \( \ker \theta_0 = \bigcap \mathfrak{P}_0 \). Similarly, for each \( 1 \leq k \leq n \), there exists a homomorphism \( \theta_k \) from \( M_k \) into \( R_k \) such that \( \ker \theta_k = \bigcap \mathfrak{P}_k \); where \( M_k \) is the maximal modular ideal containing \( Q_k \). We extend \( \theta_k \) to a homomorphism from \( C_0(\Omega) \) into \( R^\# \) by setting \( \theta_k(u_k) = e_{R_k} \). It still remains true that \( \ker \theta_k = \bigcap \mathfrak{P}_k \).

It follows from the result of Bade and Curtis that \( \mathcal{I}(\theta_k) = \ker \theta_k (0 \leq k \leq n) \). Thus the homomorphism \( \theta : C_0(\Omega) \to \bigcap_{k=0}^n R_k^\# \) defined by \( \theta(a) = (\theta_0(a), \ldots, \theta_n(a)) \) satisfies \( \mathcal{I}(\theta) = \mathcal{I}(\theta_k) = \bigcap \mathfrak{P} = I \).

**Remark.** In Parts (ii) of Theorems 6.6 and 6.7, we only need that \( I \) is the intersection of a relatively compact family \( \mathfrak{P} \) of prime ideals where every chain in the closure of \( \mathfrak{P} \) is countable (see Proposition 6.2).
7. Examples on metrisable locally compact spaces

For examples of pseudo-finite sequences of prime ideals in $C_0(\Omega)$, see [14]. In this section, we shall construct relatively compact families of prime ideals that are not unions of finitely many pseudo-finite families.

For the entire section, fix a well-ordered set $\kappa$. Set $\kappa^{(0)} = \kappa$. For each $n \in \mathbb{N}$, define inductively $\kappa^{(n)}$ as the set of limiting elements in $\kappa^{(n-1)}$. We shall only consider those $\kappa$ for which $\kappa^{(n)} = \emptyset$ for some $n \in \mathbb{N}$. This condition forces $\kappa$ to be countable. Let $d$ be the largest integer for which $\kappa^{(d)} \neq \emptyset$; we call it the depth of $\kappa$. For simplicity, we also suppose that $\kappa$ has the largest element, $\max \kappa$, and that $\kappa^{(d)} = \{\max \kappa\}$. Otherwise, we can always replace $\kappa$ by a bigger well-ordered set.

We also set

$$\kappa_0 = \{\beta \in \kappa : l(\beta) = 0\}.$$

For each $\alpha \in \kappa$ define $l(\alpha)$ to be the largest integer $l$ for which $\alpha \in \kappa^{(l)}$. We define a relation $\prec$ on $\kappa$ as follows: For each $\alpha, \beta \in \kappa$, we write $\alpha \prec \beta$ if $\beta$ is the smallest element in $\kappa$ with the properties that $\beta \geq \alpha$ and that $l(\beta) = l(\alpha) + 1$. We then define a partial order $\ll$ on $\kappa$ as follows: For each $\alpha, \beta \in \kappa$, we write $\alpha \ll \beta$ if there exists a finite sequence $\alpha = \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_n = \beta$. Note that, for each $\alpha \in \kappa$, there is exactly one chain $\alpha = \gamma_1 \prec \gamma_2 \prec \cdots$, which must be finite and the ultimate end point of this chain is $\max \kappa$.

The next lemma lists some basic properties of the relation $\ll$ on $\kappa$.

**Lemma 7.1.**

(i) If $\beta \ll \alpha$ and $\alpha \in \kappa_0$ then $\beta = \alpha$.

(ii) If $\gamma \ll \alpha$ and $\gamma \ll \beta$ then either $\alpha \ll \beta$ or $\beta \ll \alpha$.

(iii) If $\beta \ll \alpha$ and $\beta \ll \gamma \ll \alpha$ then $\gamma \ll \alpha$.

(iv) If $\beta \ll \alpha$ and $\beta \neq \alpha$ then there exists $\gamma \in \kappa_0$ such that $\beta < \gamma < \alpha$ (and so $\gamma \ll \alpha$).

**Proof.** The proof is routine. \qed

We start our construction with the following general lemma.

**Lemma 7.2.** Let $A$ be a commutative algebra and $Q$ be an ideal which either is prime in $A$ or is $A$ itself. Suppose that we have $(f_\alpha : \alpha \in \kappa) \subseteq Q$ and a semiprime ideal $I \subseteq Q$ such that

(i) $f_\alpha \notin I$ and $I : f_\alpha \subseteq Q$ ($\alpha \in \kappa$);

(ii) $f_\alpha f_\beta \in I$ whenever both $\alpha \ll \beta$ and $\beta \ll \alpha$ in $\kappa$;

(iii) if $g f_\alpha \in I$ for some $\alpha \in \kappa$ then $g f_\beta \in I$ for all $\beta \ll \alpha$;

(iv) if $g f_\beta \in I$ for some $\beta$ and $\beta \ll \alpha$, then, for each $\alpha \gg \beta$, there exists $\beta_1 \in \kappa_0$ with $\beta_1 \ll \alpha$ such that $g f_\beta \in I$ for all $\beta \in \kappa_0$ with $\beta_1 \ll \beta \ll \alpha$.

Then there exist prime ideals $(P_\alpha : \alpha \in \kappa)$ satisfying that:

(a) $f_\alpha \notin P_\alpha$ and $I : f_\alpha \subseteq P_\alpha \subseteq Q$ ($\alpha \in \kappa$);

(b) $P_\alpha = \bigcup_{\beta \ll \alpha, \beta \neq \alpha} P_\beta$ ($\alpha \in \kappa$);

(c) if $g \in P_\alpha$ for some $\alpha \in \kappa$ then there exists $\beta_1 \in \kappa_0$ with $\beta_1 \ll \alpha$ such that $g \in P_\beta$ for all $\beta \in \kappa_0$ with $\beta_1 \ll \beta \ll \alpha$.

**Proof.** We prove by induction on the depth $d$ of $\kappa$.

When $d = 0$, $\kappa = \{0\}$. The conditions (i)-(iv) reduce to $I$ being semiprime, $f_0 \notin I$ and $I : f_0 \subseteq Q$. It follows that $I \cap S = \emptyset$, where $S = \{f_0^k, f_0^k f : k \geq 1, f \in A \setminus Q\}$ is closed under multiplication. Therefore, we can find a prime ideal $P_0$ such that
Put $I_0 \subset P_0$ and that $P_0 \cap S = \emptyset$, and so $P_0 \subset Q$ and $f_0 \notin P_0$. We see that $P_0$ is the required prime ideal.

Now, suppose that the result holds whenever the depth is less than $d$. By Zorn’s lemma, we can find a semiprime ideal $J$ containing $I$ such that $J$ is maximal with respect to conditions (i)-(iv).

Claim 1: If $f \notin Q$ then $J : f = J$. Indeed, it is easy to see that $J : f$ is semiprime and satisfies conditions (i)-(iv). So the maximality of $J$ implies $J : f = J$.

Claim 2: If $f \notin J$ then $J : f \subset Q$. For otherwise, there would exist $g \in J : f \setminus Q$, and so $f \in J : g = J$, by Claim 1; a contradiction.

Set $P = \bigcup_{\alpha \in \kappa} J : f_\alpha$. Then $P \subset Q$. Condition (iii) implies that

$$P = \bigcup_{\alpha \in \kappa_0} J : f_\alpha$$

and condition (iv) (by choosing $\alpha = \max \kappa$) implies that $P$ is an ideal.

Claim 3: If $f \notin P$ then $J : f = J$. Indeed, it is easy to see that $J : f$ is semiprime and satisfies conditions (i)-(iv) (the less obvious one is (i)), however, since $f \notin P$, $ff_\alpha \notin J$, and so $f_\alpha \notin J : f$ and $(J : f) : f_\alpha = J : f_\alpha \subset Q$ by Claim 2. So the maximality of $J$ implies $J : f = J$.

Claim 4: $P$ is either prime in $A$ or $A$ itself. Indeed, if $f, g \notin P$, then, by Claim 3,

$$g \notin \bigcup_{\alpha \in \kappa} J : f_\alpha = \bigcup_{\alpha \in \kappa} (J : f) : f_\alpha = P : f.$$ 

Thus $fg \notin P$.

Let $\alpha_1 < \alpha_2 < \ldots$ be the non-limiting elements in $\kappa^{(d-1)}$; their limit is $\max \kappa$. Set $\kappa_1 = \{ \alpha \in \kappa : \alpha < \alpha_1 \}$, and, for each $n \geq 2$, set $\kappa_n = \{ \alpha \in \kappa : \alpha_{n-1} < \alpha \leq \alpha_n \}$. Each $\kappa_n$ has depth $d - 1$, and $\kappa = \bigcup_{n=1}^{\infty} \kappa_n \cup \{ \max \kappa \}$.

For each $n \in \mathbb{N}$, we see that $(f_\alpha : \alpha \in \kappa_n), J$, and $P$ satisfy $(f_\alpha : \alpha \in \kappa_n) \subset P$, $J \subset P$, and conditions (i)-(iv) (with $\kappa_n$ replacing $\kappa$, $J$ replacing $I$, and $P$ replacing $Q$). So, by induction, there exist prime ideals $P_\alpha (\alpha \in \kappa_n)$ satisfying the conditions (a)-(c) (with obvious modification). Set $P_{\max \kappa} = P$.

Note that if $\beta \ll \alpha < \max \kappa$ then both $\alpha$ and $\beta$ belong to the same $\kappa_n$ for some $n \in \mathbb{N}$. We see that the combined sequence $(P_\alpha : \alpha \in \kappa)$ obviously satisfies the conditions (a)-(c) (with $J$ replacing $I$); the only one need to really check is condition (c) when $\alpha = \max \kappa$, however, this case follows from the facts that $J : \beta \subset P_\beta \subset P_{\max \kappa}$, that

$$P_{\max \kappa} = \bigcup_{\beta \in \kappa_0} J : f_\beta,$$

and that $J$ satisfies condition (iv).

Now, let $\Omega$ be a metrisable locally compact space. We define a non-increasing sequence $(\partial^{(n)} \Omega^p : n \in \mathbb{Z}^+)$ of compact subsets of $\Omega^p$ as follows:

(i) put $\partial^{(0)} \Omega^p = \Omega^p$;

(ii) for each $n \in \mathbb{Z}^+$, define $\partial^{(n+1)} \Omega^p$ to be the set of all limit points of $\partial^{(n)} \Omega^p$.

Define $\partial^{(\infty)} \Omega^p = \bigcap \{ \partial^{(n)} \Omega^p : n \in \mathbb{Z}^+ \}$. By the compactness, either $\partial^{(\infty)} \Omega^p$ is non-empty or $\partial^{(l)} \Omega^p$ is empty for some $l \in \mathbb{Z}^+$.

To construct non-trivial pseudo-finite sequences of prime ideals in $C_0(\Omega)$ it is necessary that there exists $p \in \partial^{(\infty)} \Omega^p$; this follows from [14, Proposition 8.7].
Remark that all uncountable Polish spaces possess such point \( p \), and there are even countable compact subspaces of \( \mathbb{R} \) satisfying this condition.

We need some further preparations: For each \( \beta \in \kappa \setminus \{ \max \kappa \} \), there exists a unique \( \alpha \in \kappa \) such that \( \beta < \alpha \). We can define \( t(\beta) \) to be the number of \( \gamma \in \kappa \) such that \( \gamma < \alpha \) and \( \gamma \leq \beta \); there are only finitely many such \( \gamma \). For each \( \beta \in \kappa_0 \), there exists a unique \( (\alpha_1, \ldots, \alpha_{d-1}) \in \kappa \) such that

\[
\beta < \alpha_1 < \cdots < \alpha_{d-1} < \max \kappa;
\]

set \( w(\beta) = \max \{t(\beta), t(\alpha_1), \ldots, t(\alpha_{d-1})\} \). Note that, for each \( k \in \mathbb{N} \),

\[
\{\alpha \in \kappa_0 : w(\alpha) \leq k\} = k^d.
\]

Adjoin \( \infty \) to \( \mathbb{N} \) to obtain its one-point compactification \( \mathbb{N}^\infty \); the convention is that \( \infty > n \) (\( n \in \mathbb{N} \)) and \( 2^{-\infty} = 0 \). Define \( \Xi \) to be the subset of the product space \( (\mathbb{N}^\infty)^\kappa \) consisting of all elements \( (n_\alpha : \alpha \in \kappa_0) \) with the properties that there exists a finite set \( F \subset \kappa_0 \) such that \( n_\alpha = \infty \) (\( \alpha \in \kappa_0 \setminus F \)) and such that

\[
\max \{w(\beta) : \beta \in F\} (\alpha \in F).
\]

It is easy to see that \( \Xi \) is a closed subset of \( (\mathbb{N}^\infty)^\kappa \).

Lemma 7.3. Let \( \Omega \) be a metrisable locally compact space, and let \( p \in \partial^{(\infty)} \Omega^p \). Then, the space \( \Xi \) can be continuously embedded into \( \Omega^p \) such that the point \( \infty = (\infty, \infty, \ldots) \) is mapped into \( p \).

Proof. As proved in [14, Lemmas 9.1 and 9.2], the following compact subset of \( \mathbb{R} \)

\[
\Delta := \{0\} \cup \left\{ \sum_{i=1}^k 2^{-n_i} : k, n_1, n_2, \ldots, n_k \in \mathbb{N} \text{ and } k \leq n_1 < \cdots < n_k \right\}
\]

is continuously embedded in \( \Omega^p \) such that 0 is mapped into \( p \).

Choose an injective map \( r : \kappa_0 \times \mathbb{N} \to \mathbb{N} \) such that \( r(\alpha, j) \geq j^d \) (\( \alpha \in \kappa_0, j \in \mathbb{N} \)); this is possible since \( \kappa_0 \) is countable. For convenience, for each \( \alpha \in \kappa_0 \), we also set \( r(\alpha, \infty) = \infty \). Define a map \( \tau \) from \( \Xi \) into \([0, 1]\) as follows: for each \( (n_\alpha)_{\alpha \in \kappa_0} \in \Xi \), set

\[
\tau : (n_\alpha)_{\alpha \in \kappa_0} \mapsto \sum_{\alpha \in \kappa_0} 2^{-r(\alpha, n_\alpha)}.
\]

Then, we see that \( \tau \) is well-defined, injective and continuous.

Let \( (n_\alpha)_{\alpha \in \kappa_0} \in \Xi \) be arbitrary. Set \( F = \{\alpha \in \kappa_0 : n_\alpha < \infty\} \). Then \( F \) is a finite subset of \( \kappa_0 \), and \( n_\alpha \geq k (\alpha \in F) \) where \( k = \max \{w(\beta) : \beta \in F\} \). So, we see that, for \( \alpha \in F \),

\[
r(\alpha, n_\alpha) \geq n_\alpha^d \geq k^d = |\{\beta \in \kappa_0 : w(\beta) \leq k\}| \geq |F|.
\]

From this, we can deduce that \( \tau \) actually maps \( \Xi \) into \( \Delta \). The lemma then follows. \( \square \)

Lemma 7.4. Let \( \Omega \) be a metrisable locally compact space, and let \( p \in \partial^{(\infty)} \Omega^p \). Then, there exist a family of prime ideals \( (P_\alpha : \alpha \in \kappa) \) in \( \mathcal{C}_0(\Omega) \), where each ideal is supported at \( p \), and a family of functions \( (f_\alpha : \alpha \in \kappa) \) in \( \mathcal{C}_0(\Omega) \) satisfying the following:

(a) \( f_\alpha \notin P_\beta \) and \( f_\beta \in P_\alpha \) whenever both \( \beta \not< \alpha \) and \( \alpha \not< \beta \);
(b) \( P_\alpha = \bigcup_{\beta \in \alpha, \beta \neq \alpha} P_\beta \);
(c) if \( g \in P_\alpha \) then there exists \( \beta_1 \in \kappa_0 \) with \( \beta_1 \not< \alpha \) such that \( g \in P_\beta \) for all \( \beta \in \kappa_0 \) with \( \beta_1 \leq \beta \leq \alpha \).
It follows from the previous lemma that we only need to consider the case where \( \Omega = \Xi \) and \( p = \infty \). Thus, suppose that \( \Omega = \Xi \) and \( p = \infty \).

For each \( \alpha \in \kappa_0 \), define
\[
Z_\alpha = \{(n_\beta): n_\beta \in \Xi : \ n_\alpha = \infty \},
\]
and for \( \alpha \in \kappa \setminus \kappa_0 \), define
\[
Z_\alpha = \bigcap_{\beta \in \kappa_0, \ \beta \ll \alpha} Z_\beta.
\]
Then choose \( f_\alpha \in C(\Xi) \) such that \( Z_\alpha = \mathbb{Z}(f_\alpha) \). Let \( \mathcal{F} \) be the \( \mathbb{Z} \)-filter generated by all \( Z_\alpha \cup Z_\beta \) \((\alpha, \beta \in \kappa, \ \alpha \ll \beta \) and \( \beta \ll \alpha) \). Then define \( I = \mathbb{Z}^{-1}[\mathcal{F}] \). Obviously \( I \) is a semiprime ideal, \( I \subset M_\infty \), and \( (f_\alpha : \ \alpha \in \kappa) \subset M_\infty \).

It is sufficient to prove that \( (f_\alpha), I, \) and \( M_\infty \) satisfy conditions (i)-(iv) of Lemma 7.2.

First, for each \( \gamma \in \kappa \), \( f \in I: f \gamma \) if and only if
\[
\mathbb{Z}(f) \cup Z_\gamma \supset \bigcap_{k=1}^n \left(Z_{\alpha_k} \cup Z_{\beta_k}\right),
\]
where, for each \( k \), \( \alpha_k \ll \beta_k \) and \( \beta_k \ll \alpha_k \). We see from Lemma 7.1 that, for each \( k \), one of the following three cases must happen: (1) \( \alpha_k \ll \gamma \) and \( \gamma \ll \alpha_k \), (2) \( \gamma \ll \beta_k \) and \( \beta_k \ll \gamma \), (3) \( \alpha_k \ll \gamma \), \( \beta_k \ll \gamma \), \( \alpha_k \not\ll \beta_k \) and \( \beta_k \not\ll \alpha_k \).

Thus, we see that \( f \in I: f \gamma \) implies
\[
\mathbb{Z}(f) \cup Z_\gamma \supset \bigcap_{i=1}^r \left(Z_{\varphi_i} \cap \bigcap_{j=1}^s \left(Z_{\sigma_j} \cup Z_{\zeta_j}\right)\right),
\]
where, for each \( i \), \( \varphi_i \ll \gamma \) and \( \gamma \ll \varphi_i \), and for each \( j \), \( \sigma_j \ll \gamma \), \( \zeta_j \ll \gamma \), \( \sigma_j \not\ll \zeta_j \), and \( \zeta_j \not\ll \sigma_j \). In particular, we see that \( \sigma_j < \gamma \) \((1 \leq j \leq s) \), and so, by Lemma 7.1(iv), there exists \( \beta \in \kappa_0 \) such that
\[
\beta > \max \{\sigma_j : 1 \leq j \leq s\} \quad \text{and} \quad \beta \ll \gamma.
\]
These also imply that \( \beta \ll \sigma_j \) \((1 \leq j \leq s) \) and \( \beta \ll \varphi_i \) \((1 \leq i \leq r) \). Thus, if, for each \( k \geq w(\beta) \), set \( n^{(k)}_\beta = k \) and \( n^{(k)}_\alpha = \infty \) \((\alpha \neq \beta) \), then
\[
(n^{(k)}_\alpha : \ \alpha \in \kappa_0) \in \left(\bigcap_{i=1}^r \left(Z_{\varphi_i} \cap \bigcap_{j=1}^s \left(Z_{\sigma_j} \cup Z_{\zeta_j}\right)\right)\right) \backslash Z_\gamma \subset \mathbb{Z}(f).
\]
On the other hand, we see that \( \lim_{k \to \infty} (n^{(k)}_\alpha : \ \alpha \in \kappa_0) = \infty \) in \( \Xi \). It follows that \( \infty \in \mathbb{Z}(f) \). Hence, \( I: f \gamma \subset M_\infty \) and \( f \not\in I \), so condition (i) of Lemma 7.2 holds.

It is obvious that conditions (ii) and (iii) of Lemma 7.2 are satisfied by the definitions of \( I \) and the sets \( Z_\alpha \).

Now, let \( \beta_0, \alpha \in \kappa \) and let \( g \in C(\Xi) \) be such that \( l(\beta_0) = 0 \), \( \beta_0 \ll \alpha \), \( \beta_0 \neq \alpha \) (these imply that \( \alpha \notin \kappa_0 \)) and \( g \beta_0 \in I \). Then, from the previous discussion, noting that \( l(\beta_0) = 0 \), we have
\[
\mathbb{Z}(g) \cup Z_{\beta_0} \supset \bigcap_{i=1}^r Z_{\varphi_i} \quad \text{or} \quad \mathbb{Z}(g) \supset \bigcap_{i=1}^r Z_{\varphi_i} \cup Z_{\beta_0};
\]
where, for each $i$, $\beta_0 \nless g_i$, which implies that $\alpha \nless g_i$. We claim that

$$\mathbf{Z}(g) \supset \bigcap_{i=1}^{r} Z_{g_i} \cap \bigcap_{\gamma \in \kappa_0, w(\gamma) \leq w(\beta_0)} Z_{\gamma}.$$ 

Indeed, let $(n_\gamma : \gamma \in \kappa_0) \neq \infty$ be in the right-hand side set. Then, for each $\gamma \in \kappa_0$, $n_\gamma = \infty$ whenever $\gamma \nless g_i$ for some $i$ or $w(\gamma) \leq w(\beta_0)$. Set $F = \{ \gamma \in \kappa_0 : n_\gamma < \infty \}$. Thus, by setting $n_\gamma = \max \{ w(\beta) : \beta \in F \} > w(\beta_0)$ (\gamma \in F \}).

Thus, by setting $n_\gamma(k) = n_\gamma(\gamma \neq \beta_0) = k$ for each $k \geq \max \{ w(\beta) : \beta \in F \}$, we obtain

$$(n_\gamma(k) : \gamma \in \kappa_0) \in \bigcap_{i=1}^{r} Z_{g_i} \setminus Z_{\beta_0} \subset \mathbf{Z}(g).$$

On the other hand, we see that $\lim_{k \to \infty} (n_\gamma(k) : \gamma \in \kappa_0) = (n_\gamma : \gamma \in \kappa_0)$ in $\Xi$. It follows that $(n_\gamma : \gamma \in \kappa_0) \in \mathbf{Z}(g)$. The claim follows.

By Lemma 7.1(iv), we can choose $\beta_1 \in \kappa_0$ such that $\beta_1 \nless \alpha$, and that

$$\beta_1 > \max \left( \{ g_i : 1 \leq i \leq r, g_i \nless \alpha \} \cup \{ \gamma \in \kappa_0 : \gamma \nless \alpha \text{ and } w(\gamma) \leq w(\beta_0) \} \right);$$

the element on the right-hand side is $\nless \alpha$ and $\neq \alpha$. Let $\beta \in \kappa_0$ be such that $\beta_1 \leq \beta \leq \alpha$. Then $\beta \nless \alpha$. We see that $g_i \nless \beta$ ($1 \leq i \leq r$); otherwise, we see that $g_i = \beta \nless \alpha$ but then $\beta_1 \geq g_i$ by (3), contradicting $g_i = \beta$. We also see that $\beta \nless g_i$ ($1 \leq i \leq r$); otherwise, either $g_i \nless \alpha$ or $\alpha \nless g_i$, we have already ruled out $\alpha \nless g_i$ from the beginning of the previous paragraph, so $g_i \nless \alpha$, and again, this and (3) imply that $\beta \geq \beta_1 > g_i$, contradicting $\beta \nless g_i$. We also see that $\beta \nless \gamma$ and $\gamma \nless \beta$ for each $\gamma \in \kappa_0$ with $w(\gamma) \leq w(\beta_0)$; otherwise $\gamma = \beta \nless \alpha$, however, this and (3) imply that $\beta \geq \beta_1 > \gamma$. Thus

$$\mathbf{Z}(g) \cup Z_\beta \supset \bigcap_{i=1}^{r} (Z_{g_i} \cup Z_\beta) \cap \bigcap_{\gamma \in \kappa_0, w(\gamma) \leq w(\beta_0)} (Z_\gamma \cup Z_\beta) \in \mathcal{F};$$

and so $g f_\beta \in I$. Hence, condition (iv) of Lemma 7.2 is also satisfied.

We now state the main theorem of this section.

**Theorem 7.5.** Let $\Omega$ be a metrisable locally compact space, and let $p \in \partial^{(\infty)} \Omega^p$. Let $\kappa$ be a well-ordered set as above. Then, there exists a compact family of prime ideals $(P_\alpha : \alpha \in \kappa)$ in $\mathcal{C}_0(\Omega)$, where each ideal is supported at $p$, satisfying the following:

(a) $(P_{\alpha_n} : n \geq n_0)$ is a pseudo-finite sequence with union $P_\alpha$ for some $n_0 \in \mathbb{N}$ whenever $(\alpha_n)$ converges to $\alpha$ in the order topology of $\kappa$;

(b) $P_\alpha \subset P_\beta$ if and only if $\alpha \nless \beta$.

Let $\Psi$ be any relatively compact family of prime ideals with the same intersection as $\bigcap_{\alpha \in \kappa} P_\alpha$. Then the closure of $\Psi$ contains a chain of length $d + 1$. In particular, $\Psi$ is not the union of any finitely many pseudo-finite subfamilies of prime ideals when $d > 1$. 

\hfill $\Box$
Proof. Let \((P_\alpha : \alpha \in \kappa)\) be the family constructed in Lemma 7.4.

(a) Without loss of generality, we assume that \(\alpha_n \neq \alpha\) \((n \in \mathbb{N})\). There exists \(n_0\) such that \(\alpha_n \ll \alpha\) \((n \geq n_0)\). We see that \(P_{\alpha_n} \subset P_\alpha\) \((n \geq n_0)\) and

\[ P_\alpha = \bigcup_{\beta \in \kappa_0, \beta \ll \alpha} P_\beta. \]

So, for each \(g \in P_\alpha\), by Theorem 7.5(c), there exists \(\beta_1 \in \kappa_0\) with \(\beta_1 \ll \alpha\) such that \(g \in P_\beta\) for all \(\beta \in \kappa_0\) with \(\beta_1 \leq \beta \leq \alpha\). Choose \(n_1 \geq n_0\) such that \(\alpha_n \geq \beta_1\) \((n \geq n_1)\). Let \(n \geq n_1\) be arbitrary. Pick \(\beta' \in \kappa_0\) such that \(\beta' \ll \alpha_n\). If \(\beta' \leq \beta_1\), then \(\beta_1 \ll \alpha_n\) by Lemma 7.1(iii), and so \(g \in P_{\beta_1} \subset P_{\alpha_n}\). Otherwise, \(\beta_1 < \beta'\), then \(g \in P_{\beta'} \subset P_{\alpha_n}\). Thus, \((P_{\alpha_n} : n \geq n_0)\) is a pseudo-finite sequence whose union is \(P_\alpha\).

(b) This is a consequence of Lemma 7.4 (a) and (b).

In the order topology, \(\kappa\) is a compact metrisable space; each sequence in \(\kappa\) has a convergence subsequence. The compactness of \((P_\alpha : \alpha \in \kappa)\) thus follows from (a).

Let \(\mathfrak{P}\) be a relatively compact family of prime ideals such that \(\bigcap \mathfrak{P} = I = \bigcap_{\alpha \in \kappa} P_\alpha\). Denote by \(\mathfrak{P}_0\) the subfamily consisting of minimal elements of \(\mathfrak{P}\). By Proposition 5.5(iii), we see that \(\bigcap \mathfrak{P}_0 = I\). Lemma 5.6 shows that, for each \(P \in \mathfrak{P}_0\) there exists \(f_P \notin P\) but \(f_P \in Q\) for all \(Q \in \mathfrak{P}_0 \setminus \{P\}\); in particular, \(P = I : f_P\). This and Lemma 4.8 then imply that \(\mathfrak{P}_0\) is exactly the set of prime ideals of the form \(I : f\) for some \(f \in C_0(\Omega)\). Similarly, \(\{P_\alpha : \alpha \in \kappa_0\}\) is also exactly the set of prime ideals of the form \(I : f\) for some \(f \in C_0(\Omega)\). Thus

\[ \mathfrak{P}_0 = \{P_\alpha : \alpha \in \kappa_0\} \text{ and its closure is } \{P_\alpha : \alpha \in \kappa\}. \]

Therefore, for any \(\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_d = \max \kappa\), we have \(P_{\alpha_0} \subset \cdots \subset P_{\alpha_d}\) is a chain of length \(d + 1\) in the closure of \(\mathfrak{P}_0\). Obviously, this shows that \(\mathfrak{P}\) cannot be the union of any finite number of pseudo-finite families when the depth of \(\kappa\) is bigger than 1.

\[ \square \]

Remark. Since the cardinality of \(C(\Xi)\) is \(c\), we see that, for \((P_\alpha : \alpha \in \kappa)\) as in the above theorem,

\[ \left| C_0(\Omega) \div \bigcap_{\alpha \in \kappa} P_\alpha \right| = c. \]

Let \(\Omega\) be any metrisable locally compact space with \(\check{\sigma}(\infty)\Omega \neq \emptyset\). Let \(\mathfrak{P}\) be a relatively compact family of prime ideals in \(C_0(\Omega)\). Since we are mostly interested in \(\bigcap \mathfrak{P}\), by passing to the minimal elements of \(\mathfrak{P}\), we can suppose that \(P \not\subset Q\) \((P, Q \in \mathfrak{P})\). Let\(\mathfrak{P}\) call a family satisfying this condition reduced.

Question. Is there a reduced relatively compact family of prime ideals \(\mathfrak{P}\) in \(C_0(\Omega)\) whose closure contains an uncountable chain or merely an infinite chain? In the case that there exists such a reduced family \(\mathfrak{P}\) whose closure contains an uncountable chain, is it still possible to remove the countability condition from Proposition 6.2?

We know that every chain in the closure of a relatively compact family of prime ideals must be well-ordered (Proposition 5.5(iii)). However, we show in [15] the existence of uncountable well-ordered chains of prime ideals in \(C_0(\Omega)\). We also show in [15] that there is an uncountable non-redundant pseudo-finite family of prime ideals in \(C_0(\Omega)\), thus, in particular, showing that there exists an uncountable reduced relatively compact family of prime ideals (however, every chain in the closure of a pseudo-finite family of prime ideals has length at most 2).
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