Primitive spectrum and representations of plactic algebras

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(joint work with Lukasz Kubat)

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1. Motivation and background
   i) plactic monoid - definition, origin and applications
   ii) open problems on finitely presented algebras
   iii) algebras defined by homogeneous semigroup presentations; motivating results on Chinese algebras
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3. New results (focus on the case of algebras of rank 3)
   i) minimal spectrum
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For any integer $n \geq 1$, the finitely presented monoid $M_n = \langle a_1, \ldots, a_n \rangle$ defined by the relations

\[ a_ia_kaj = a_ka_iaj \quad \text{for} \quad i \leq j < k, \]
\[ ajaiak = ajakai \quad \text{for} \quad i < j \leq k \]

is called the plactic monoid of rank $n$.

For any fixed $i < j < k$ write $a = a_i$, $b = a_j$, $c = a_k$. Then the defining relations look like:

\[ (ac)b = c(ab), \quad b(ac) = b(ca) \]
\[ b(ba) = (ba)b, \quad a(ba) = (ba)a \]

If $K$ is a field then $K[M_n]$ denotes the corresponding semigroup algebra. In particular:

\[ K[M_1] = K[x], \quad K[M_2] = K\langle x, y \mid yx \text{ is central} \rangle. \]
The construction of the plactic monoid originated from the papers of Schensted and Knuth concerned with certain combinatorial problems and operations on Young tableaux.

Schensted: An algorithm for determining of the length of a longest nondecreasing (decreasing) subsequence of a sequence with elements in \( \{1, 2, \ldots, n\} \).

Knuth: a semigroup structure associate to this algorithm.

This construction was then systematically studied by Lascoux and Schützenberger (1978) and became a very important tool in the theory of symmetric functions, algebraic combinatorics, Young tableaux and various aspects of representation theory.

For example: the formula for decomposition of the tensor product of irreducible representations of \( GL_n(K) \) is based on an application of the plactic monoid (Littlewood - Richardson coefficients).
By a row in $M_n$ we mean an element which is a word $a_{i_1} \cdots a_{i_r}$ such that $r \geq 1$ and $i_1 \leq i_2 \leq \cdots \leq i_r$. A column is defined as a word with strictly decreasing indices.

We say that a row $u = a_{i_1} \cdots a_{i_r}$ dominates a row $v = a_{k_1} \cdots a_{k_s}$ if $r \leq s$ and $i_j > k_j$ for every $j = 1, \ldots, r$. We write $u \triangleright v$ in this case.

A tableau is a word $w = u_1 \cdots u_t$, where $u_i$ are rows such that $u_1 \triangleright u_2 \triangleright \cdots \triangleright u_t$. One of the main features of $M_n$ is that every element $w \in M_n, w \neq 1$, is equal to a unique tableau. For example

$$w = a_5 \ a_3 \ a_4 \ a_4 \ a_2 \ a_3 \ a_3 \ a_3 \ a_3 \ a_1 \ a_1 \ a_2 \ a_2 \ a_2 \ a_3$$

is a tableau with the subsequent rows

$$v_1 = a_5, \ v_2 = a_3 \ a_4 \ a_4, \ v_3 = a_2 \ a_3 \ a_3 \ a_3, \ v_4 = a_1 \ a_1 \ a_2 \ a_2 \ a_2 \ a_3.$$

Such tableaux can be interpreted as planar objects.

$$w = \begin{array}{c}
  a_5 \\
  a_3 \ a_4 \ a_4 \\
  a_2 \ a_3 \ a_3 \ a_3 \\
  a_1 \ a_1 \ a_2 \ a_2 \ a_2 \ a_3
\end{array}$$
and the subsequent columns of \( w \) are

\[
\begin{align*}
  w_1 &= a_5a_3a_2a_1, \\
  w_2 &= a_4a_3a_1, \\
  w_3 &= a_4a_3a_2, \\
  w_4 &= a_3a_2, \\
  w_5 &= a_2, \\
  w_6 &= a_3.
\end{align*}
\]

Here the element is read row by row from left to right.

A surprising fact: if \( w_1, \ldots, w_k \) denote the columns of a tableau, read from the top to the bottom, then the element \( w_1 \cdots w_k \) called the ‘column reading’ of \( w \) satisfies \( w = w_1 \cdots w_k \) in \( M_n \).

For example

\[
w = a_5a_3a_2a_1 \ a_4a_3a_1 \ a_4a_3a_2 \ a_3a_2 \ a_2 \ a_3
\]

for the tableau \( w \) displayed above.

Given a sequence \( k_i, \ i = 1, 2, \ldots, m \), with \( k_i \in \{1, 2, \ldots, n\} \), let \( u \in M_n \) be the tableau form of \( a_{k_1} \cdots a_{k_m} \in M_n \). Then:

- the length of the first column of \( u \) is the maximal length of a decreasing subsequence of the sequence \( k_i \) and the length of the last row of \( u \) is the maximal length of a nondecreasing subsequence of \( k_i \).
The elements of $M_n$ are in a one-to-one correspondence with Young tableaux of the above type. Because of its strong relations to Young tableaux, the plactic monoid has already become a classical tool in several areas of representation theory and algebraic combinatorics (cf. W.Fulton, *Young Tableaux*, Cambridge University Press, 1997).

The combinatorics of $M_n$ has been extensively studied but there are only a few preliminary results on the algebraic structure of the monoid algebra $K[M_n]$ of $M_n$ over a field $K$ ([Cedó,JO], 2004). In particular, if $n < 3$ then $K[M_n]$ the structure of $K[M_n]$ is pretty well understood.

Our aim is to present recent results on the structure and representations of the algebra $K[M_n]$. We focus on the case $n = 3$. 
Some general open problems on finitely presented algebras

\( \mathcal{K} \) will denote a field

\( X \) - a free finitely generated monoid

\( \mathcal{K}\langle X\rangle \) - the corresponding free (associative) algebra

a \( \mathcal{K} \)-algebra \( A \) is finitely presented if it is of the form \( A = \mathcal{K}\langle X\rangle/J \) for a finitely generated ideal \( J \) of \( \mathcal{K}\langle X\rangle \).

A special class - algebras defined by homogeneous semigroup relations:

\( J \) is an ideal of \( \mathcal{K}\langle X\rangle \) generated by a set of the form

\[ \{ w - u \mid (w, u) \in R \} \]

for a subset \( R \subseteq X \times X \) such that \( |w| = |u| \) for all \( (w, u) \in R \).

So, \( A = \mathcal{K}[M] \), where \( M \) is the monoid defined by \( M = X/\rho \), where \( \rho \) is the congruence on \( X \) generated by the set \( R \).
Assume \( A \) is a finitely presented \( K \)-algebra.

1. Is the Jacobson radical \( J(A) \) a nil (a locally nilpotent) ideal? (Amitsur)

2. Let \( A \) be a nil algebra. Is \( A \) nilpotent (hence, finite dimensional)? (Latyshev, Zelmanov)

Positive answers are known only for some very special classes, in particular if \( A \) is a PI-algebra.

2. What about algebras defined by homogeneous semigroup presentations?

These problems seem very difficult in full generality. So, any partial results (for special classes of algebras) would be of interest.

3. Special nature of minimal prime ideals of finitely presented algebras defined by homogeneous semigroup presentations?

(some evidence: Chinese algebras, algebras yielding set theoretical solutions of the Yang-Baxter equation, ... )

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Plactic algebras
A related algebra [Jaszuńska, JO], [Cedó, JO]

Let $R_n = K[C_n]$ be the Chinese algebra of rank $n$. (This is an algebra defined by a homogeneous finite presentation similar to that for $K[M_n]$ [Cassaigne at al.]. If $n = 1, 2$ then $R_n = K[M_n]$.)

1. Every minimal prime ideal of $R_n$ is generated by a finite set of homogeneous elements of degree 2 or 3 that are of the form $s - t$ for some $s, t \in C_n$.

2. The minimal spectrum $\mathcal{P}$ of $R_n$ is finite and for $P \in \mathcal{P}$ we have $R_n/P = K[C_n/\rho_P]$ for a congruence $\rho_P$ on $C_n$ and hence $R_n/P$ is again defined by a homogeneous semigroup presentation.

3. The Jacobson radical of $R_n$ is nilpotent. It is nonzero if $n \geq 3$.

An application - a new representation for $C_n$: 

$$C_n \subseteq \prod_{P \in \mathcal{P}} C_n/\rho_P \subseteq \mathbb{Z}^i \times B^j$$

for some $i, j \geq 1$ where $B$ is the bicyclic monoid. Consequently, $C_n$ satisfies an (explicitly given) semigroup identity.
Lemma

Let $z = a_n \cdots a_1 \in M_n$. Then the center of $K[M_n]$ is

$$Z(K[M_n]) = K[z]$$

and the nonzero elements of $K[z]$ are regular elements of the algebra $K[M_n]$.

Moreover, if $w \in M_n$ then $w \in zM_n$ if and only if $w = w_n a_n \cdots w_1 a_1 w_0$ for some $w_0, \ldots, w_n \in M_n$.

The bicyclic monoid $B = \langle p, q \mid qp = 1 \rangle$ plays an important role in semigroup theory and in ring theory.

It is easy to see that $M_2 \langle a_2 a_1 \rangle^{-1} \cong B \times \mathbb{Z}$, via the map

$$(p, 1) \mapsto a_1 (a_2 a_1)^{-1}, (q, 1) \mapsto a_2, (1, g) \mapsto a_2 a_1$$

where $g$ is a generator of $\mathbb{Z}$. 
$K[B]$ is left (and right) primitive. Indeed: if $V$ is a $K$-space with basis $e_1, e_2, \ldots$ then

$$p(e_i) = e_{i+1}, \quad q(e_i) = \begin{cases} e_{i-1} & \text{for } i > 0, \\ 0 & \text{for } i > 0, \end{cases}$$

defines a structure of a left $K[B]$-module, which is simple and faithful.

The algebra $K[B]$ contains an ideal $I$ isomorphic to the algebra $\mathcal{M}_\infty(K)$ of $\mathbb{IN} \times \mathbb{IN}$ matrices with finitely many nonzero entries. Moreover, $K[B]/I \simeq K[x, x^{-1}]$.

The algebra $K[M_n]$ is not right (left) noetherian and does not satisfy any polynomial identity.

The GK-dimension of $K[M_n]$ is $n(n+1)/2$. 
Theorem

\( K[M_2] \) is a prime \( K \)-algebra.

**Proof:** Since \( M'_2 = M_2/(a_2a_1 = 1) \) is the bicyclic monoid, \( L[M'_2] \) is right primitive for every field \( L \). One shows that

\[
K[M_n](K[z] \setminus \{0\})^{-1} \cong K(x)[M'_n].
\]

Hence \( K[M_2] \) is prime. \( \square \).

Theorem

\( K[M_2] \) is a semiprimitive \( K \)-algebra.

**Proof:** Is based on the observation that \( K[B]/I \cong K[x, x^{-1}] \) for \( I = M_{\infty}(K) \) and \( M_2\langle a_2a_1 \rangle^{-1} \cong B \times \mathbb{Z} \). It also uses the \( \mathbb{Z} \)-gradations on \( K[B \times \mathbb{Z}] \). \( \square \).
Theorem

Let $K[M_n]$ be the plactic algebra of rank $n$. Then

1. if $n \geq 3$ then $K[M_n]$ is not prime,
2. if $n \geq 4$ then $K[M_n]$ is not semiprime.

Consider the following nonzero elements of $K[M_n]$

\[ \alpha = (a_n a_{n-1} \cdots a_2 a_1)(a_{n-1} \cdots a_2) - (a_{n-1} \cdots a_2 a_1)(a_n a_{n-1} \cdots a_2), \]
\[ \beta = (a_n a_1) - (a_1)(a_n), \quad \gamma = (a_n a_2) - (a_2)(a_n). \]

Using the normal form of elements of $M_n$ it can be verified that $\alpha M_n \beta = 0$. Therefore $K[M_n]$ is not prime.

Moreover, if $n \geq 4$ then it is easy to see that $\alpha \gamma K[M_n] \alpha \gamma = 0$ and $\alpha \gamma \neq 0$. Hence $K[M_n]$ is not semiprime.

Theorem

$K[M_3]$ is a semiprimitive $K$-algebra.

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New results: case $n = 3$

Let $M = M_3$ and $a = a_1$, $b = a_2$, $c = a_3$. So $M = \langle a, b, c \rangle$ with the convention that $a < b < c$ (when applying the defining relations of $M$).

The canonical form of an element $w \in M$ looks as follows:

$$w = (cba)^{k_1}(ba)^{k_2}(ca)^{k_3}(cb)^{k_4}a^{k_5}b^{k_6}c^{k_7},$$

where $k_i \geq 0$ such that either $k_4 = 0$ or $k_5 = 0$. Let

$$N_1 = M/(ac = ca), \quad N_2 = M/(bacb = cbab),$$

We use the same notation for the elements of $N_1$ and $N_2$ as for $M$, if unambiguous.

**Lemma**

*Every element* $u \in N_1$ *can be uniquely written in the form*

$$u = (cba)^{k_1}(ba)^{k_2}a^{k_3}(cb)^{k_4}b^{k_5}c^{k_6},$$

*where* $k_i \geq 0$. *Also,* $cba$ *is a central and regular element of* $K[N_1]$. 
Consider the monoid

\[ M'_n = M_n/(z = 1) \]

where \( z = a_n \cdots a_1 \). So \( K[M'_n] \cong K[M_n]/(z - 1) \).

Certain natural antiendomorphisms of \( K[M] \) are very useful.

**Lemma**

*There exists an involution \( g : K[M] \to K[M] \) such that*

\[ g(a) = c, \quad g(b) = b, \quad g(c) = a. \]

*Moreover, for \( 0 \neq \lambda \in K \), there exists an antimonomorphism \( f_\lambda : K[M] \to K[M] \) satisfying*

\[ f_\lambda(a) = \lambda^{-1}cb, \quad f_\lambda(b) = ca, \quad f_\lambda(c) = ba, \]

*which induces an involution of \( K[M]/(cba - \lambda) \).*
**Theorem**

Let $K[M]$ be the plactic algebra of rank 3 over a field $K$. Then the principal ideals $P_1 = (ac - ca)$ and $P_2 = (bacb - cbab)$ are the only minimal prime ideals of $K[M]$.

**Sketch:**

As noticed above, we have $\alpha K[M] \beta = 0$ for $\alpha = bacb - cbab$ and $\beta = ac - ca$. Thus every prime ideal of $K[M]$ contains $P_1$ or $P_2$.

Using the canonical form of elements of $K[M]/P_1$ one can show that $P_1$ is a prime ideal of $K[M]$.

Let $z = cba$. We know that $K[M]\langle z \rangle^{-1} \cong K[M'][x, x^{-1}]$. 
Because $z$ is a nonzero central element in $K[M]/P_1 \cong K[N_1]$, we get a commuting diagram

$$
\begin{array}{ccc}
K[M]\langle z \rangle^{-1} & \cong & K[M'][x, x^{-1}] \\
\downarrow & & \downarrow \\
K[N_1]\langle z \rangle^{-1} & \cong & K[M'/ac = ca][x, x^{-1}]
\end{array}
$$

The central localization

$$
K[N_1]\langle z \rangle^{-1} \cong K[M'/(ac = ca)][x, x^{-1}]
$$

is prime, so the algebra $K[M'/(ac = ca)]$ is also prime.

Applying the antiautomorphism $f = f_1$ constructed above we get

$$
f(\beta') = f(ac - ca) = bacb - b = (bacb - cbab)' = \alpha'.
$$

Hence the algebra $K[M'/(bacb = cbab)]$ is prime, so we conclude that

$$
K[N_2]\langle z \rangle^{-1} \cong K[M'/(bacb = cbab)][x, x^{-1}]
$$

is also prime. Since this is a central localization of the algebra $K[N_2] \cong K[M]/P_2$, the latter must be prime. Therefore $P_2$ is a prime ideal of $K[M]$. 
We will assume that $K$ is nondenumerable and algebraically closed. The following can be proved by applying the density theorem.

**Proposition**

Let $L$ be an algebraically closed field. Let $A$ be an $L$-algebra such that $\dim_L A < |L|$ and $Z(A) \neq L$. Then $A$ is not left primitive.

Hence, if $R = K[M]/Q$ is a primitive image of $K[M]$ then $z - \lambda \in Q$ for some $\lambda \in K$. 
Every finite-dimensional irreducible representation $\varrho$ of $K[M]$ is 1-dimensional.

If $\varrho(cba) \neq 0$ then, since $cba$ is central in $K[M]$, we get $\varrho(cba) = \lambda$ for some $0 \neq \lambda \in K$. Therefore $\varrho(a)$, $\varrho(b)$, $\varrho(c)$ are invertible. Now, the relation $aba = baa$ in $M$ yields $\varrho(aba) = \varrho(baa)$. So, canceling $\varrho(a)$, we get $\varrho(ab) = \varrho(ba)$. Similarly $\varrho(ac) = \varrho(ca)$ and $\varrho(bc) = \varrho(cb)$. Hence $\varrho(M)$ is commutative and $n = 1$.

If $\varrho(cba) = 0$ then using the canonical form of elements of $M$ one can show that $cbMa = cbaM \subseteq \ker(\varrho)$.

If $\varrho(cb) = 0$ then similarly one shows that $cMb \subseteq \ker(\varrho)$ and hence $\varrho(b) = 0$ or $\varrho(c) = 0$. Consequently, $\varrho(a) = 0$ or $\varrho(b) = 0$ or $\varrho(c) = 0$.

This gives a reduction to a representation of $K[M_2]$, and the assertion easily follows.
The involution \( g \) gives a one-to-one correspondence between left and right primitive ideals of \( K[M] \). Similarly, for \( 0 \neq \lambda \in K \), the involution induced by \( f_\lambda \) gives a one-to-one correspondence between left and right primitive ideals of \( K[M] \) containing \( cba - \lambda \).

We first describe all primitive ideals \( P \) of \( K[M] \) such that \( cba \in P \).

**Theorem**

Let \( P \) be a left or right primitive ideal of the plactic algebra \( K[M] \) of rank 3 over a field \( K \). If \( cba \in P \) then \( P \) is one of the following ideals:

1. \((a - \alpha, b - \beta, c - \gamma)\) for \( \alpha, \beta, \gamma \in K \) with \( \alpha \beta \gamma = 0 \),
2. \((a, cb - \delta)\) or \((b, ca - \delta)\) or \((c, ba - \delta)\) for \( 0 \neq \delta \in K \).

Conversely, each of these ideals is a left and right primitive ideal of \( K[M] \).
Let $P$ be a left primitive ideal of $K[M]$ such that $cba \notin P$. Then $cba - \lambda \in P$ for some $0 \neq \lambda \in K$. Moreover, $P_1 \subseteq P$ or $P_2 \subseteq P$. We first construct some examples of simple left $K[M]$-modules with annihilators of this type.

**Proposition**

Let $V$ be a vector space over $K$ with basis $\{e_{ij} : i, j \geq 0\}$. Let the action of $a, b, c \in M$ on $V$ be given by

$$ae_{ij} = e_{i,j+1}, \quad be_{ij} = \begin{cases} \beta e_{ij} & \text{for } j = 0, \\ e_{i+1,j-1} & \text{for } j > 0, \end{cases} \quad ce_{ij} = \begin{cases} 0 & \text{for } i = 0, \\ \gamma e_{i-1,j} & \text{for } i > 0, \end{cases}$$

where $\beta \in K$ and $0 \neq \gamma \in K$. Then this action makes $V$ a simple left $K[M]$-module. Moreover, if $P$ denotes the annihilator of $V$ then

$$P = (ac - ca, (b - \beta)(acb - \gamma) + \beta(abc - bac), cba - \gamma).$$
Next, we construct another class of primitive ideals of $K[M]$ not containing the element $cba$.

**Proposition**

Let $V$ be a vector space over $K$ with basis $\{e_i : i \geq 0\}$. Let the action of $a, b, c \in M$ on $V$ be given by

$$ae_i = \alpha e_i, \quad be_i = \beta e_{i+1}, \quad ce_i = \begin{cases} 0 & \text{for } i = 0, \\ \gamma e_{i-1} & \text{for } i > 0, \end{cases}$$

where $0 \neq \alpha, \beta, \gamma \in K$. Then this action makes $V$ a simple left $K[M]$-module. Moreover, if $P$ denotes the annihilator of $V$ then $P = (a - \alpha, cb - \beta \gamma)$. 
As a consequence of the results obtained above, we get the following characterization of all left and right primitive ideals $P$ of $K[M]$ such that $cba \notin P$.

**Theorem**

Let $P$ be a left or right primitive ideal of the plactic algebra $K[M]$ of rank 3 over a field $K$. If $cba \notin P$ then $P$ is one of the following ideals:

1. $(a - \alpha, b - \beta, c - \gamma)$ for $0 \neq \alpha, \beta, \gamma \in K$,
2. $(a - \alpha, cb - \delta)$ or $(c - \gamma, ba - \delta)$ for $0 \neq \alpha, \gamma, \delta \in K$,
3. $(ac - ca, bacb - \lambda b, cba - \lambda)$ for $0 \neq \lambda \in K$,
4. $(ac - ca, (b - \beta)(acb - \lambda) + \beta(abc - bac), cba - \lambda)$ for $0 \neq \beta, \lambda \in K$,
5. $(bacb - \lambda b, (acb - \lambda)(ca - \beta) + \beta(\lambda^{-1}bacacb - bac), cba - \lambda)$ for $0 \neq \beta, \lambda \in K$.

Conversely, each of these ideals is a left and right primitive ideal of $K[M]$. 
A representation $K[M] \to \text{End}_K(V)$ is said to be monomial, if there exists a basis $E$ of $V$ over $K$ such that for every $w \in M$ and every $e \in E$ there exist $\lambda \in K$ and $e' \in E$ satisfying $we = \lambda e'$.

The proof of the main theorem shows that every simple $K[M]$-module is isomorphic to one of the modules constructed in the foregoing propositions. We also know that $K[M]$ is semiprimitive. Hence, we get the following consequence.

**Corollary**

*Irreducible representations of $K[M]$ are monomial. Hence $M$ can be presented as a subdirect product of its monomial images.*
Using families (4) and (5) of primitive ideals of $K[M]$ from the above theorem, it is possible to give a new proof of semiprimitivity of $K[M]$. It is a consequence of the following two results.

**Proposition**

Let $P_1$ and $P_2$ be the minimal prime ideals of the plactic algebra $K[M]$ of rank 3 over any field $K$. Then $P_1 \cap P_2 = 0$. Hence $K[M]$ is semiprime.

**Proposition**

Let $P_1$ and $P_2$ be the minimal prime ideals of the plactic algebra $K[M]$ of rank 3 over an infinite field $K$. Then

\[ P_1 = \bigcap_{\beta, \lambda \in K \setminus \{0\}} (ac - ca, (b - \beta)(acb - \lambda) + \beta(abc - bac), cba - \lambda) \]

\[ P_2 = \bigcap_{\beta, \lambda \in K \setminus \{0\}} (bacb - \lambda b, (acb - \lambda)(ca - \beta) + \beta(\lambda^{-1} bacacb - bac), cba - \lambda). \]
Plactic algebra $K[M_n]$ of arbitrary rank $n$:
- minimal spectrum
- irreducible representations and primitive spectrum
- nilpotency of the Jacobson radical
- applications: a new representation of $M_n$, ...

based on the fact that

$$M_n \hookrightarrow K[M_n]/J(K[M_n]) \subseteq \prod_{Q} K[M_n]/Q,$$

where $Q$ runs through the set of primitive ideals of $K[M_n]$, even though $K[M_n]$ is not semiprimitive.


