# ON THE WITT GROUP OF THE PUNCTURED SPECTRUM OF A REGULAR SEMILOCAL RING

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Abstract. Let $R$ be a regular semilocal ring of dimension $4q+1\geq 5$, which contains $\frac{1}{2}$. Let $l\geq 1$ be the number of maximal ideals of $R$ and $U$ the punctured spectrum of $R$, *i.e.* $\operatorname{Spec} R$ without the maximal ideals. We show that the Witt group $\mathrm{W}(U)$ of $U$ has $l$ generators $\mathfrak{E}_1,\ldots,\mathfrak{E}_l$ as $\mathrm{W}(R)$-algebra, which satisfy $\mathfrak{E}_i\mathfrak{E}_j=0$ for all $1\leq i,j\leq l$. If $R$ is integral then these generators become trivial over the fraction field $K$ of $R$. In particular, the natural morphism $\mathrm{W}(U)\longrightarrow\mathrm{W}(K)$ is not injective.

## 1. Introdcution

Let $R$ be a regular semilocal ring of dimension $n$, which contains $\frac{1}{2}$. Denote by $\mathfrak{m}_1,\ldots,\mathfrak{m}_l$, $l\geq 1$, the maximal ideals of $R$ and let $\kappa: U\hookrightarrow\operatorname{Spec} R$ be the punctured spectrum of $R$, *i.e.* $\operatorname{Spec} R$ without the maximal ideals. We assume that all maximal ideals of $R$ have the same height $n=\dim R$. It has been shown by Balmer and Walter [7] if $R$ is local and by Balmer [4] in the semilocal case that if $n\not\equiv 1 \bmod 4$ then the pullback $\kappa^*:\mathrm{W}(R)\longrightarrow\mathrm{W}(U)$ is an isomorphism, and if $n\equiv 1 \bmod 4$ then there is an exact sequence

$$0\longrightarrow\mathrm{W}(R)\xrightarrow{\kappa^*}\mathrm{W}(U)\xrightarrow{\partial}\mathrm{W}^1_{\mathfrak{J}(R)}(R)\longrightarrow 0\,,$$

where $\mathrm{W}^1_{\mathfrak{J}(R)}(R)$ is Balmer's [3] first derived Witt group of $R$ with support in the Jacobson radical $\mathfrak{J}(R)=\bigcap\limits_{i=1}^{l}\mathfrak{m}_i$ of $R$.

The aim of this note is to show the following.

**Theorem A.** *Let $R$ be a regular semilocal ring of dimension $n=4q+1\geq 5$, which contains $\frac{1}{2}$, and whose maximal ideals all have the same height. Denote by $U$ the punctured spectrum of $R$ and let $l\geq 1$ be the number of maximal ideals of $R$. Then the Witt ring $\mathrm{W}(U)$ of $U$ has $l$ generators $\mathfrak{E}_1,\ldots,\mathfrak{E}_l$ as $\mathrm{W}(R)$-algebra, which are all locally trivial, i.e. their images in $\mathrm{W}(R_P)$ are zero for all $P\in U$, and which satisfy the identity*

$$\mathfrak{E}_i\mathfrak{E}_j\,=\,0$$

*for all $1\leq i,j\leq l$. In particular, if $R$ is an integral domain with fraction field $K$ then the natural morphism $\mathrm{W}(U)\longrightarrow\mathrm{W}(K)$ is not injective.*

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Note that an old conjecture, which is attributed to Knebusch, asserts that the homomorphism $\mathrm{W}(R) \longrightarrow \mathrm{W}(K)$ is injective if $R$ is a regular and local ring with fraction field $K$. This has been proven if $R$ contains a field by Balmer, Walter, and the authors [6], and recently by the authors [18] if $R$ is geometrically regular over a discrete valuation ring.

We consider also $\epsilon$-hermitian Witt groups, $\epsilon \in \{\pm 1\}$, where we prove:

**Theorem B.** *Let $R$ be a complete regular local ring of odd dimension $n \geq 1$, which contains $\frac{1}{2}$, and $(A, \tau)$ an $R$-Azumaya algebra with involution of the first- or second kind over $R$. Denote by $U$ the punctured spectrum of $R$. Then*

$$\mathrm{W}_\epsilon(A|_U, \tau|_U) \simeq \mathrm{W}_\epsilon(A_k, \tau_k) \oplus \mathrm{W}_{\epsilon\epsilon_0}(A_k, \tau_k)\,,$$

*where $k$ is the residue field of $R$ and $\epsilon_0 = (-1)^{\frac{n-1}{2}}$.*

There are two main ingredients in the proof of these results. First the computation of the Witt groups of symmetric forms over the punctured affine space due to Balmer and the first named author [5], and second a new factorization theorem for the transfer of coherent (hermitian) Witt groups along a regular immersion (of rank one). The latter result, proven in Section 5, can be seen as a generalization of the (so called) zero theorem for the transfer, a theorem which is crucial for our [18] proof of the Gersten conjecture for hermitian Witt groups of Azumaya algebras with involution over a local ring which is geometrically regular over a discrete valuation ring.

The content of the paper is as follows. The main results are proven in the last two sections after Section 5 on the factorization lemma. Sections 2 and 3 review definitions and results of triangular-, derived-, and coherent Witt theory in an effort to make this paper as self contained as feasible. However we refer for details and more information about these theories to the fundamental work of Balmer [2, 3] as well as to the articles [9, 12, 14, 18]. In Section 4 we recall results from the paper [5] on the Witt group of the punctured affine space over a ring.

*We assume throughout this work that* Hom*-sets of additive categories are uniquely* 2*-divisible, and so in particular that the global sections of schemes contain $\frac{1}{2}$.*

## 2. Preliminaries I: Triangular Witt theory

**2.1.** Let $(\mathcal{K}, \mathfrak{D}, \delta, \varpi)$, or shorter $(\mathcal{K}, \mathfrak{D}, \varpi)$, be a triangulated category with duality, *i.e.* $\mathfrak{D}$ is a $\delta$-exact contravariant functor, $\delta \in \{\pm 1\}$, and $\varpi : \mathrm{id}_{\mathcal{K}} \xrightarrow{\simeq} \mathfrak{D} \circ \mathfrak{D}$ is a natural isomorphism, such that $\varpi_{\mathfrak{D}M} = \mathfrak{D}(\varpi_M)^{-1}$ for all $M \in \mathcal{K}$.

A symmetric space in the triangulated category with duality $(\mathcal{K}, \mathfrak{D}, \varpi)$ is defined in the usual way as a pair $(M, \varphi)$, where $M \in \mathcal{K}$ and $\varphi : M \longrightarrow \mathfrak{D}M$ is a symmetric isomorphism, *i.e.* $\varphi$ is an isomorphism and we have $\varphi = \mathfrak{D}(\varphi) \circ \varpi_M$. The Witt group $\mathrm{W}(\mathcal{K}, \mathfrak{D}, \varpi) = W^0(\mathcal{K}, \mathfrak{D}, \varpi)$ is the Grothendieck group of the isomorphism classes of symmetric spaces in $(\mathcal{K}, \mathfrak{D}, \varpi)$ with the orthogonal sum as addition modulo the subgroup of neutral spaces. The 'higher' triangular Witt groups $\mathrm{W}^i(\mathcal{K}, \mathfrak{D}, \varpi)$, or shorter $\mathrm{W}^i(\mathcal{K})$ if the duality structure is clear from the context, are the Witt groups of the $i$th shifted triangulated category with duality

$$(\mathcal{K}, \mathfrak{D}, \delta, \varpi)^{(i)} := (\mathcal{K}, T^i\mathfrak{D}, (-1)^i\delta, (-1)^{\frac{i(i+1)}{2}}\varpi)\,,$$

$i \in \mathbb{Z}$, where $T$ denotes the translation functor of $\mathcal{K}$. The elements of the latter are represented by so called $i$-symmetric spaces. We denote the class of a space $(M,\varphi)$ in its Witt group by $[M,\varphi]$.

**2.2.** Let $(\mathcal{K}_1,\mathfrak{D}_1,\delta_1,\varpi_1)$ be another triangulated category with duality. A *duality preserving functor* $(\mathcal{K},\mathfrak{D},\delta,\varpi) \longrightarrow (\mathcal{K}_1,\mathfrak{D}_1,\delta_1,\varpi_1)$ is a pair $(F,\eta)$, where $F:\mathcal{K} \longrightarrow \mathcal{K}_1$ is an exact covariant functor and $\eta : F\mathfrak{D} \xrightarrow{\simeq} \mathfrak{D}_1 F$ a natural isomorphism, the *duality transformation*, such that (a) $T_1^{-1}\eta_X = (\delta\cdot\delta_1)\eta_{T_\mathcal{K} X}$ for all $X\in\mathcal{K}$, where $T_1$ denotes the translation functor in $\mathcal{K}_1$, and (b) $\mathfrak{D}_1(\eta)\circ\varpi_1 = \eta_\mathfrak{D}\circ F(\varpi)$. Such a functor induces a homomorphism $\mathrm{W}^i(\mathcal{K}) \longrightarrow \mathrm{W}^i(\mathcal{K}_1)$ for all $i\in\mathbb{Z}$, mapping the class of the $i$-symmetric space $(M,\varphi)$ onto the class of the space

$$(F,\eta)_*(M,\varphi) \;:=\; (F(M),(\delta_1\delta)^i\cdot T_1^i(\eta_M)\circ F(\varphi))\,.$$

If $(G,\theta)$ is a duality preserving functor with the same target and domain as $(F,\eta)$ we say following [10, Def. 1.2] that these are *isometric* if there exists an isomorphism of functors $s:F\xrightarrow{\simeq} G$, which commutes with the respective translation functors and satisfies

$$\mathfrak{D}_1(s)\circ\theta\circ s_\mathfrak{D} \;=\; \eta\,.$$

We then have an isometry $s_M : (F,\eta)_*(M,\varphi)\xrightarrow{\simeq}(G,\theta)_*(M,\varphi)$ for all $i$-symmetric spaces $(M,\varphi)$ in $(\mathcal{K},\mathfrak{D},\delta,\varpi)$.

If $(F_1,\eta_1) : (\mathcal{K}_1,\mathfrak{D}_1,\delta_1,\varpi_1) \longrightarrow (\mathcal{K}_2,\mathfrak{D}_2,\delta_2,\varpi_2)$ is another duality preserving functor then the composition with $(F,\eta)$ is defined by

$$(F_1,\eta_1)\circ(F,\eta) \;:=\; (F_1\circ F\,,\; \eta_{1\,F}\circ F_1(\eta))\,.$$

**Example.** An important example of a duality preserving functor is the following. Let as above $(\mathcal{K},\mathfrak{D},\delta,\varpi)$ be a triangulated category with duality. Then the second power of the translation functor is duality preserving: $T^2\mathfrak{D} = \mathfrak{D}T^2$. We get a duality preserving functor $(T^2,\mathrm{id}) : (\mathcal{K},\mathfrak{D},\delta,\varpi) \longrightarrow (\mathcal{K},\mathfrak{D},\delta,\varpi)^{(4)}$, and so an isomorphism of triangular Witt groups $\rho : \mathrm{W}^i(\mathcal{K}) \xrightarrow{\simeq} \mathrm{W}^{i+4}(\mathcal{K})$ for all $i\in\mathbb{Z}$, the (here so called) 4-*periodicity isomorphism*.

**2.3. Notations and (sign-)conventions.** If $\mathcal{E}$ is an exact category we denote by $\mathrm{D}^b(\mathcal{E})$ its bounded derived category. As usual in derived Witt theory we use homological complexes.

Let $R$ be a commutative ring. We use the following sign conventions for the (total) Hom- and $\otimes$-complex of two bounded complexes of $R$-modules $M_\bullet$ and $N_\bullet$.

In degree $l$ we have

$$\mathrm{Hom}_R(M_\bullet,N_\bullet)_l \;=\; \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(M_{-l-i},N_{-i})\,,$$

and the differential maps $g\in\mathrm{Hom}_R(M_{-l-i},N_{-i})$ onto $g\circ d^M_{-l-i+1} + (-1)^{l+1} d^N_{-i}\circ g$.

The tensor product of $M_\bullet$ and $N_\bullet$ is given in degree $l$ by

$$(M_\bullet\otimes_R N_\bullet)_l \;=\; \bigoplus_{i+j=l} M_i\otimes_R N_j\,,$$

and the differential maps $m\otimes n\in M_i\otimes_R N_j$ onto $d^M_i(m)\otimes n + (-1)^i m\otimes d^N_j(n)$.

There is a morphism of complexes $M_\bullet \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M_\bullet, N_\bullet), N_\bullet)$, which we denote $\varpi^N$. It is in degree $l$ the homomorphism

$$M_l \longrightarrow \bigoplus_{i,j} \mathrm{Hom}_R(\mathrm{Hom}_R(M_{l+j-i}, N_{-i}), N_{-j})$$

with $ij$-component 0 if $i \neq j$ and $(-1)^{\frac{i(i+1)}{2}}$-times the evaluation map otherwise.

A bounded complex $I_\bullet$ of injective $R$-modules is called a *dualizing complex* if $\varpi^I$ is an isomorphism of functors in the bounded derived category of complexes of $R$-modules with finitely generated homology modules.

Following the conventions in [18] we understand by an $R$-Azumaya algebra with involution $(A, \tau)$ an $R$-algebra with $R$-linear involution, which is an Azumaya algebra (in the usual sense) over its centre, which is assumed to be either equal $R$, or a quadratic étale extension of $R$. The involution is called of the *first kind* if the centre is $R$, and of the *second kind* otherwise. An algebra with involution over a scheme is called Azumaya algebra with involution if it is locally so.

## 3. Preliminaries II: Derived-, and coherent Witt theory

**3.1.** Let $(\mathcal{A}, \tau)$ be an Azumaya algebra with involution over a scheme $X$. By an $\mathcal{A}$-module we understand a left $\mathcal{A}$-module. Given a right $\mathcal{A}$-module $\mathcal{F}$ we use the involution $\tau$ to equip $\mathcal{F}$ with a left $\mathcal{A}$-module structure. This left $\mathcal{A}$-module is denoted $\overline{\mathcal{F}}^{\tau}$, or $\overline{\mathcal{F}}$ only if $\tau$ is clear from the context.

Let $\mathcal{P}(\mathcal{A})$ be the category of coherent $\mathcal{A}$-modules, which are locally free as $\mathcal{O}_X$-modules. There are two dualities on $\mathcal{P}(\mathcal{A})$:

$$\mathfrak{D}^{\mathcal{A},\tau} := \overline{\mathcal{H}om_{\mathcal{A}}(\,-\,,\mathcal{A})} \qquad \text{and} \qquad \mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{O}_X} := \overline{\mathcal{H}om_{\mathcal{O}_X}(\,-\,,\mathcal{O}_X)}\,.$$

We denote the associated *hermitian Witt groups* by $\mathrm{W}_\epsilon(\mathcal{A},\tau)$ and $\mathrm{W}_\epsilon(\mathcal{A},\tau,\mathcal{O}_X)$, respectively, for $\epsilon \in \{\pm 1\}$. The derived functors of these dualities make $\mathrm{D}^b(\mathcal{P}(\mathcal{A}))$ a triangulated category with duality, whose associated Witt groups are denoted $\mathrm{W}^i(\mathcal{A},\tau)$ and $\mathrm{W}^i(\mathcal{A},\tau,\mathcal{O}_X)$, respectively, and called *derived hermitian Witt groups.* The reduced trace induces an isomorphism of functors $\mathfrak{D}^{\mathcal{A},\tau} \xrightarrow{\simeq} \mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{O}_X}$, and so we have natural isomorphisms of usual- and derived hermitian Witt groups

$$\mathrm{W}_\epsilon(\mathcal{A},\tau) \xrightarrow{\simeq} \mathrm{W}_\epsilon(\mathcal{A},\tau,\mathcal{O}_X) \quad \text{and} \quad \mathrm{W}^i(\mathcal{A},\tau) \xrightarrow{\simeq} \mathrm{W}^i(\mathcal{A},\tau,\mathcal{O}_X)$$

for all $\epsilon \in \{\pm 1\}$ and $i \in \mathbb{Z}$, see [13, App.]. If $(\mathcal{A},\tau) = (\mathcal{O}_X, \mathrm{id}_{\mathcal{O}_X})$ we have $\mathfrak{D}^{\mathcal{O}_X,\mathrm{id}_{\mathcal{O}_X}} = \mathfrak{D}^{(\mathcal{O}_X,\mathrm{id}_{\mathcal{O}_X})}_{\mathcal{O}_X}$, and we write $\mathrm{W}(X)$ instead of $\mathrm{W}_1(X,\mathrm{id}_{\mathcal{O}_X})$, $\mathrm{W}_-(X)$ instead of $\mathrm{W}_{-1}(\mathcal{O}_X,\mathrm{id}_{\mathcal{O}_X})$, $\mathrm{W}^i(X)$ instead of $\mathrm{W}^i(X,\mathrm{id}_{\mathcal{O}_X})$, and so on.

Let $Z \subseteq X$ be a closed subset and $\mathrm{D}^b_Z(\mathcal{P}(\mathcal{A}))$ be the full triangulated subcategory of $\mathrm{D}^b(\mathcal{P}(\mathcal{A}))$ consisting of complexes whose homology modules have support in $Z$. The functors $\mathfrak{D}^{\mathcal{A},\tau}$ and $\mathfrak{D}^{(\mathcal{A},\tau)}_{\mathcal{O}_X}$, respectively, are also dualities on these subcategories. The associated derived Witt groups are called *Witt groups with support* in $Z$, and denoted $\mathrm{W}^i_Z(\mathcal{A},\tau)$ and $\mathrm{W}^i_Z(\mathcal{A},\tau,\mathcal{O}_X)$, respectively.

If $X = \operatorname{Spec} R$ is an affine scheme and $Z$ is defined by the ideal $\mathfrak{a} \subseteq R$ we use affine notations. Then $A = \Gamma(X,\mathcal{A})$ is an $R$-Azumaya algebra with involution $\tau$ of the first or second kind, and we denote the respective Witt groups by $\mathrm{W}_\epsilon(A,\tau)$, $\mathrm{W}_\epsilon(A,\tau,R)$, $\mathrm{W}^i(A,\tau)$, $\mathrm{W}^i(A,\tau,R)$, $\mathrm{W}^i_{\mathfrak{a}}(A,\tau)$, and $\mathrm{W}^i_{\mathfrak{a}}(A,\tau,R)$.

Finally we recall that by the main result of Balmer [3] there are natural isomorphisms

$$\mathrm{W}_\epsilon(\mathcal{A},\tau) \xrightarrow{\simeq} \mathrm{W}^{1-\epsilon}(\mathcal{A},\tau) \quad \text{and} \quad \mathrm{W}_\epsilon(\mathcal{A},\tau,\mathcal{O}_X) \xrightarrow{\simeq} \mathrm{W}^{1-\epsilon}(\mathcal{A},\tau,\mathcal{O}_X)$$

for all $\epsilon \in \{\pm 1\}$.

**3.2.** Let now $R$ be a commutative noetherian ring with a dualizing complex $I_\bullet$, and $(A,\tau)$ and $R$-Azumaya algebra with involution. Let $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ be the full subcategory of the bounded derived category of all $A$-modules consisting of complexes with finitely generated homology modules.

On $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ the 1-exact contravariant functor

$$\mathfrak{D}_I^{(A,\tau)} \;:=\; \overline{\mathrm{Hom}_R(\,-\,,I_\bullet)}^{\,\tau}$$

is a duality, such that $(\mathrm{D}^b_c(\mathcal{M}_{qc}(A)),\mathfrak{D}_I^{(A,\tau)},1,\varpi^I)$ is a triangulated category with duality. The associated triangular Witt groups are called the *coherent Witt groups* of $(A,\tau)$ with respect to the duality $\mathfrak{D}_I^{(A,\tau)}$, and are denote by $\tilde{\mathrm{W}}^i(A,\tau,I_\bullet)$, $i\in\mathbb{Z}$.

Replacing $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ by $\mathrm{D}^b_{c,\,\mathfrak{a}}(\mathcal{M}_{qc}(A))$, the full subcategory of complexes whose homology modules are annihilated by a power of the ideal $\mathfrak{a}\subset R$, we get the coherent Witt groups with support $\tilde{W}^i_{\mathfrak{a}}(A,\tau,I_\bullet)$, $i\in\mathbb{Z}$.

**3.3.** Let $\pi : R \longrightarrow S$ be a finite morphism of rings. Then $S$ has a dualizing complex as well, which is given by $\pi^\natural(I_\bullet) := \mathrm{Hom}_R(S,I_\bullet)$. Set $(B,\nu) := S\otimes_R (A,\tau)$.

The homomorphism of complexes $\pi^\natural(I_\bullet)\longrightarrow I_\bullet$, which is given in degree $l$ by

$$\mathrm{Hom}_R(S,I_l) \;\longrightarrow\; I_l\,,\; g \;\longmapsto\; (-1)^{\frac{(l+1)(l+2)}{2}} g(1)\,,$$

induces an isomorphism of functors $\pi_*\mathfrak{D}^{(B,\nu)}_{\pi^\natural(I)} \xrightarrow{\simeq} \mathfrak{D}_I^{(A,\tau)}\pi_*$, which is a duality transformation for the push-forward $\pi_*$. Hence we have a duality preserving functor

$$\mathrm{Tr}_\pi \;:\; (\mathrm{D}^b_c(\mathcal{M}_{qc}(B)),\mathfrak{D}^{(B,\nu)}_{\pi^\natural(I)},1,\varpi^{\pi^\natural(I)}) \;\longrightarrow\; (\mathrm{D}^b_c(\mathcal{M}_{qc}(A)),\mathfrak{D}_I^{(A,\tau)},1,\varpi^I)\,.$$

**3.4. Dévissage Theorem.** *Let $\mathfrak{a}\subseteq\mathfrak{b}\subset R$ be ideals, $\pi : R\longrightarrow R/\mathfrak{a}$ the quotient morphism, and $I_\bullet$ a dualizing complex of $R$. Then*

$$\mathrm{Tr}_\pi \;:\; \tilde{W}^i_{\mathfrak{b}/\mathfrak{a}}(R/\mathfrak{a}\otimes(A,\tau),\pi^\natural(I_\bullet)) \;\longrightarrow\; \tilde{W}^i_{\mathfrak{b}}(A,\tau,I_\bullet)$$

*is an isomorphism.*

*Proof.* In case $\mathfrak{b}=\mathfrak{a}$ this is [12, Thm. 5.2]. The same proof with some obvious modifications works in the more general case. □

**3.5.** For the rest of this section we assume that $R$ is a Gorenstein ring of finite Krull dimension. Let $0\longrightarrow R\xrightarrow{\iota} I_0 \xrightarrow{d_0^I} I_{-1}\longrightarrow \ldots \longrightarrow I_{-\dim R}\longrightarrow 0$ be an injective resolution of the $R$-module $R$. Then $I_\bullet$ considered as a complex concentrated in degrees $0,-1,\ldots,-\dim R$ is a dualizing complex of $R$, and $\iota : R\longrightarrow I_\bullet$, where we consider $R$ as a complex concentrated in degree 0, is a quasi-isomorphism.

We denote by $\mathcal{E}(A)$ the category of finitely generated $A$-modules $M$, which are reflexive as $R$-modules and satisfy

$$\mathrm{Ext}^i_R(M,R) \;=\; \mathrm{Ext}^i_R(\mathrm{Hom}_R(M,R),R) \;=\; 0$$

for all $i\geq 1$. The bounded derived category $\mathrm{D}^b(\mathcal{E}(A))$ of $\mathcal{E}(A)$ is a triangulated category with 1-exact duality $\mathfrak{D}_R^{(A,\tau)} = \overline{\mathrm{Hom}_R(\,-\,,R)}^{\,\tau}$ and bidual isomorphism $\varpi^R$.

By a result of Bass [8, Thm. 8.2], *cf.* [9, pp 113-114] and [12, Sect. 2.11], the natural functor $\mathrm{D}^b(\mathcal{E}(A)) \longrightarrow \mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ is an equivalence, which becomes duality preserving by the isomorphism of functors $\mathfrak{D}_R^{(A,\tau)} \xrightarrow{\simeq} \mathfrak{D}_I^{(A,\tau)}$ induced by $\iota$. We denote the resulting duality preserving functor $F_\iota$. It induces an isomorphism

$$\mathrm{W}^i(\mathrm{D}^b(\mathcal{E}(A)), \mathfrak{D}_R^{(A,\tau)}, \varpi^R) \xrightarrow{\simeq} \tilde{\mathrm{W}}^i(A, \tau, I_\bullet)$$

for all $i \in \mathbb{Z}$, which depends on $\iota$.

If $R$ is regular we can replace here $\mathcal{E}(A)$ by the 'smaller' category $\mathcal{P}(A)$.

**3.6.** Let $f : R \longrightarrow S$ be a flat morphism of rings with $S$ a Gorenstein ring of finite Krull dimension. Set $(B,\nu) := S \otimes_R (A,\tau)$. The (derived) pull-back $f^* := S \otimes_R -$ maps $\mathrm{D}^b(\mathcal{E}(A))$ into $\mathrm{D}^b(\mathcal{E}(B))$ and is duality preserving via the natural isomorphism of functors $f^*\mathfrak{D}_R^{(A,\tau)} \xrightarrow{\simeq} \mathfrak{D}_S^{(B,\nu)} f^*$. As the duality transformation is canonical we denote the associated duality preserving functor by $f^*$ only.

**3.7.** Finally we recall the left pairing between (derived) symmetric spaces over $R$ and (coherent) hermitian spaces over $(A,\tau)$, referring to [17] for details.

Let for this $(P_\bullet, \varphi)$ be an $i$-symmetric space in $(\mathrm{D}^b(\mathcal{P}(R)), \mathfrak{D}^R, 1, \varpi^R)$. Then the functor $\mathrm{D}^b_c(\mathcal{M}_{qc}(A)) \longrightarrow \mathrm{D}^b_c(\mathcal{M}_{qc}(A))$, $M_\bullet \mapsto P_\bullet \otimes_R M_\bullet$ becomes duality preserving using the $i$-symmetric form $\varphi$ on $P_\bullet$ as follows: The natural isomorphism

$$P_\bullet \otimes_R \mathfrak{D}_I^{(A,\tau)}(\,-\,) \xrightarrow{\varphi\otimes\mathrm{id}} T^i\mathfrak{D}^R P_\bullet \otimes_R \mathfrak{D}_I^{(A,\tau)}(\,-\,) \xrightarrow{\simeq} T^i\big(\mathfrak{D}_I^{(A,\tau)}(P_\bullet \otimes_R -)\big)\,,$$

where the isomorphism on the right hand side is the natural one (no signs involved), is a duality transformation. We get a duality preserving functor $(P_\bullet, \varphi) \star - :$

$$(\mathrm{D}^b_c(\mathcal{M}_{qc}(A)), \mathfrak{D}_I^{(A,\tau)}, 1, \varpi^I) \longrightarrow (\mathrm{D}^b_c(\mathcal{M}_{qc}(A)), T^i\mathfrak{D}_I^{(A,\tau)}, (-1)^i, (-1)^{\frac{i(i+1)}{2}}\varpi^I)\,.$$

This is the definition of the left pairing. It maps a $l$-symmetric space $(M_\bullet, \psi)$ in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(A)), \mathfrak{D}_I^{(A,\tau)}, 1, \varpi^I)$ onto the left product $(P_\bullet, \varphi) \star (M_\bullet, \psi)$, which is a $(i+l)$-symmetric space.

Analogously, we have the left product $(P_\bullet, \varphi) \star -\ :$

$$(\mathrm{D}^b(\mathcal{E}(A)), \mathfrak{D}_R^{(A,\tau)}, 1, \varpi^R) \longrightarrow (\mathrm{D}^b(\mathcal{E}(A)), T^i\mathfrak{D}_I^{(A,\tau)}, (-1)^i, (-1)^{\frac{i(i+1)}{2}}\varpi^I)\,,$$

and it is straightforward to check that there is an isometry

$$F_\iota((P_\bullet, \varphi) \star (M_\bullet, \psi)) \simeq (P_\bullet, \varphi) \star F_\iota(M_\bullet, \psi) \tag{1}$$

for all $l$-symmetric spaces $(M_\bullet, \varphi)$ in $(\mathrm{D}^b(\mathcal{E}(A)), \mathfrak{D}_R^{(A,\tau)}, 1, \varpi^R)$.

## 4. The standard form on the Koszul complex and punctured spectra

**4.1.** Let $t$ be an element of the ring $R$, and $\mathrm{K}_\bullet(t)$ the associated Koszul complex $R \xrightarrow{\cdot t} R$, which we consider as an element of $\mathrm{D}^b(\mathcal{P}(R))$ living in degrees 0 and 1. On this complex we have the (here so called) 1-*symmetric standard form*:

$$\begin{array}{ccccl} R & \xrightarrow{\cdot(-1)} & R & & = \mathrm{Hom}_R(R,R) \\ \big\downarrow{\scriptstyle \cdot t} & & \big\downarrow{\scriptstyle \cdot(-t)} & & \\ R & \xrightarrow{\cdot 1} & R & & = \mathrm{Hom}_R(R,R)\ . \end{array}$$

We denote this 1-symmetric space by $\mathfrak{Kos}(R,t)$, or $\mathfrak{Kos}(t)$ only if the ring $R$ is clear from the context. It is the negative of the cone of the symmetric morphism $R \xrightarrow{\cdot t} R = \mathrm{Hom}_R(R,R)$ and so neutral in in the triangulated category with duality $(\mathrm{D}^b_c(\mathcal{M}_{qc}(R)), \mathfrak{D}^R, \varpi^R)$, but not necessarily in the triangulated category with duality $(\mathrm{D}^b_{c,\,Rt}(\mathcal{M}_{qc}(R)), \mathfrak{D}^R, \varpi^R)$.

More general, if $\underline{t} = (t_1, \ldots, t_n)$ is a sequence of elements of $R$ then we set

$$\mathfrak{Kos}(\underline{t}) \;=\; \mathfrak{Kos}(R,\underline{t}) \;:=\; \mathfrak{Kos}(R,t_1) \star \mathfrak{Kos}(R,t_2) \star \ldots \star \mathfrak{Kos}(R,t_n)\,.$$

This is a $n$-symmetric space in $(\mathrm{D}^b(\mathcal{P}(R)), \mathfrak{D}^R, \varpi^R)$. The order of the product matters here as $\mathfrak{Kos}(x) \star \mathfrak{Kos}(y) \simeq -\mathfrak{Kos}(y) \star \mathfrak{Kos}(x)$ by [17, Sect. 3.1].

**4.2.** Let

$$S_n \;:=\; \mathbb{Z}[\frac{1}{2}][T_1, \ldots, T_n]$$

be the polynomial ring in $n \geq 1$ variables over $\mathbb{Z}[\frac{1}{2}]$. Following [5] we write $n = 4q + r - 1$ with $q \geq 0$ an integer and $r \in \{-1, 0, 1, 2\}$.

Let further

$$\mathbb{U}^n_{\mathbb{Z}[\frac{1}{2}]} \;:=\; \bigcup_{i=1}^{n} \mathrm{Spec}\, S_n[T_i^{-1}] \;\subset\; \mathbb{A}^n_{\mathbb{Z}[\frac{1}{2}]}$$

be the punctured affine $n$-space over $\mathbb{Z}[\frac{1}{2}]$. We denote for $1 \leq i \leq n$ by $\vartheta_i$ the open embedding $\mathrm{Spec}\, S_n[T_i^{-1}] \hookrightarrow \mathbb{U}^n_{\mathbb{Z}[\frac{1}{2}]}$ and let $\partial^n : \mathrm{W}^r(\mathbb{U}^n_{\mathbb{Z}[\frac{1}{2}]}) \longrightarrow \mathrm{W}^{r+1}_{\sum_{i=1}^{n} S_n T_i}(S_n)$ be the connecting homomorphism in Balmer's [2] localization sequence for the open embedding $\mathbb{U}^n_{\mathbb{Z}[\frac{1}{2}]} \hookrightarrow \mathbb{A}^n_{\mathbb{Z}[\frac{1}{2}]}$.

In [5] it is shown that there exists a $r$-symmetric space $\mathfrak{E}_n$ over $\mathbb{U}^n_{\mathbb{Z}[\frac{1}{2}]}$ satisfying

$$\mathfrak{Kos}(S_n, \underline{T}) \;=\; \rho^q\big(\partial^n(\mathfrak{E}_n)\big)\,,$$

where $\rho$ is the 4-periodicity isomorphism recalled in the example in 2.2. It is further shown in [5, Sect. 9] that if $n \geq 2$ then the class of the space $\mathfrak{E}_n$ in $\mathrm{W}^r(\mathbb{U}^n_{\mathbb{Z}[\frac{1}{2}]})$ satisfies:

(a) $\mathfrak{E}_n \star \mathfrak{E}_n = 0$, and
(b) $\vartheta_i^*(\mathfrak{E}_n) = 0$ for all $1 \leq i \leq n$.

**4.3. Lemma.** *Let $R$ be a regular ring, $\underline{t} = (t_1, \ldots, t_n)$ a sequence of elements, which do not generate $R$, and $\mathfrak{a}$ the ideal generated by $\underline{t}$. Denote by $U \subset \mathrm{Spec}\, R$ the open subscheme of $P \in \mathrm{Spec}\, R$ with $P \not\supseteq \underline{t}$, and write $n = 4q + r + 1$ with $q \in \mathbb{Z}$ and $r \in \{-1, 0, 1, 2\}$.*

*Then there exists $\mathfrak{E}_{\underline{t}} \in \mathrm{W}^r(U)$, such that $[\mathfrak{Kos}(R,\underline{t})] = \rho^q\big(\partial(\mathfrak{E}_{\underline{t}})\big)$, where $\partial : \mathrm{W}^r(U) \longrightarrow \mathrm{W}^{r+1}_{\mathfrak{a}}(R)$ is the connecting homomorphism in Balmer's [2] localization sequence and $\rho$ is the 4-periodicity isomorphism, see the example in 2.2.*

*If $n \geq 2$ we have $\mathfrak{e}_{\underline{t}} \star \mathfrak{e}_{\underline{t}} = 0$ and $\beta_i^*(\mathfrak{e}_{\underline{t}}) = 0$, where $\beta_i$ is the open immersion $\mathrm{Spec}\, R[t_i^{-1}] \hookrightarrow U$ for all $1 \leq i \leq n$.*

*Proof.* As we assume that $\frac{1}{2} \in R$ we have the homomorphism of $\mathbb{Z}[\frac{1}{2}]$-algebras

$$S_n \;:=\; \mathbb{Z}[\frac{1}{2}][T_1, \ldots, T_n] \longrightarrow R\,,\; T_i \longmapsto t_i\,,$$

which induces a morphism of schemes $\gamma : \operatorname{Spec} R \longrightarrow \mathbb{A}^n_{\mathbb{Z}[\frac{1}{2}]}$. We denote the restriction of $\gamma$ to $U$ by $\gamma$ as well. This is a morphism $U \longrightarrow \mathbb{U}^n_{\mathbb{Z}[\frac{1}{2}]}$.

We now set $\mathfrak{E}_{\underline{t}} := \gamma^*(\mathfrak{E}_n)$. The first part of the lemma follows from the commutative diagram

$$\begin{tikzcd} \mathrm{W}^r(\mathbb{U}^n_{\mathbb{Z}[\frac{1}{2}]}) \arrow[r, "\partial^n"] \arrow[d, "\gamma^*"'] & \mathrm{W}^{r+1}_{\mathfrak{m}_n}(S_n) \arrow[r, "\rho^q", "\simeq"'] \arrow[d, "\gamma^*"'] & \mathrm{W}^n_{\mathfrak{m}_n}(S_n) \arrow[d, "\gamma^*"] \\ \mathrm{W}^r(U) \arrow[r, "\partial"] & \mathrm{W}^{r+1}_{\mathfrak{a}}(R) \arrow[r, "\rho^q", "\simeq"'] & \mathrm{W}^n_{\mathfrak{a}}(R) \end{tikzcd} ,$$

since $\gamma^*(\mathfrak{Kos}(S_n, T_1, \ldots, T_n)) = \mathfrak{Kos}(R, \underline{t})$.

The last part follows form the analogous results for $\mathfrak{E}_n$, see 4.2, the fact that the pull-back $\gamma^*$ commutes with the left product by [17, Thm. 3.2], and since the diagram

$$\begin{tikzcd} \mathrm{W}(\mathbb{U}^n_{\mathbb{Z}[\frac{1}{2}]}) \arrow[r, "\vartheta_i^*"] \arrow[d, "\gamma^*"'] & \mathrm{W}(S_n[T_i^{-1}]) \arrow[d, "(\gamma|_{\operatorname{Spec} S_n[T_i^{-1}]})^*"] \\ \mathrm{W}(U) \arrow[r, "\beta_i^*"'] & \mathrm{W}(R[t_i^{-1}]) \end{tikzcd}$$

commutes for all $1 \leq i \leq n$. $\square$

## 5. The factorization lemma

**5.1.** We consider in this section the following commutative diagram of morphisms of Gorenstein rings of finite Krull dimension:

$$\begin{tikzcd} & \tilde{R} \\ S \arrow[ur, "\pi"] & R \arrow[l, "u"] \arrow[u, "p"'] \end{tikzcd} , \qquad (2)$$

where $u$ and $p$ are assumed to be flat if $R$ is not regular, and $\pi$ is generated by a non zero divisor $t \in S$, *i.e.* $\tilde{R} = S/St$.

Let $J_\bullet$ be a minimal injective resolution of the $S$-module $S$, and $j : S \longrightarrow J_\bullet$ a quasi-isomorphism. Since $t \in S$ is not a zero divisor the complex

$$\tilde{I}_\bullet \;=\; \pi^\natural(J_\bullet) \;:=\; \operatorname{Hom}_S(\tilde{R}, J_\bullet)$$

is a minimal injective resolution of $\tilde{R}$ (starting in degree $-1$). We identify here the $\tilde{R} = S/St$-module $\tilde{I}_m$ with $\Big\{ x \in J_m \,\Big|\, t \cdot x = 0 \Big\}$. We have $\tilde{I}_0 = \{0\}$ as $J_0 \xrightarrow{\cdot t} J_0$ is an isomorphism, whose inverse we denote by $\cdot t^{-1}$ (by some abuse of notation).

The composition $S \xrightarrow{j} J_0 \xrightarrow{\cdot t^{-1}} J_0 \xrightarrow{d_0^J} J_{-1}$ induces an embedding $\tilde{\imath} : S/St = \tilde{R} \hookrightarrow \tilde{I}_{-1}$, such that

$$0 \longrightarrow \tilde{R} \xrightarrow{\tilde{\imath}} \tilde{I}_{-1} \xrightarrow{d^{\tilde{I}}_{-1}} \tilde{I}_{-2} \longrightarrow \ldots$$

is a minimal injective resolution of the $\tilde{R}$-module $\tilde{R}$, see [16, Lem. 2.4].

**5.2.** In above situation let $(A,\tau)$ be an $R$-Azumaya algebra with involution of the first- or second kind, and set $(\tilde{A},\tilde{\tau}) := \tilde{R}\otimes_R (A,\sigma)$ and $(B,\nu) := S\otimes_R (A,\tau) = S\otimes_{\tilde{R}} (\tilde{A},\tilde{\tau})$.

Associated with these data we have two duality preserving functors

$$(F,\eta)\,,\ (G,\theta)\ :\ (\mathrm{D}^b(\mathcal{E}(A)),\mathfrak{D}_R^{(A,\tau)},1,\varpi^R)\ \longrightarrow\ (\mathrm{D}^b_c(\mathcal{M}_{qc}(B)),T\mathfrak{D}_J^{(B,\nu)},-1,-\varpi^J)\,,$$

which are defined as follows:

(a) $(F,\eta) := F_j\circ\big[\,\mathfrak{K}os(t)\star u^*(\,-\,)\,\big]$; and

(b) $(G,\theta)$ is defined by the commutative diagram of duality preserving functors

$$\begin{tikzcd}
(\mathrm{D}^b(\mathcal{E}(A)),\mathfrak{D}_R^{(A,\tau)},1,\varpi^R) \arrow[r, "p^*"] \arrow[dd, "{(G,\theta)}"'] & (\mathrm{D}^b(\mathcal{E}(\tilde{A})),\mathfrak{D}_{\tilde{R}}^{(\tilde{A},\tilde{\tau})},1,\varpi^{\tilde{R}}) \arrow[d, "F_{\tilde{\iota}}"] \\
 & (\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A})),\mathfrak{D}_{T\tilde{I}}^{(\tilde{A},\tilde{\tau})},1,\varpi^{T\tilde{I}}) \arrow[d, "\mathfrak{sh}"] \\
(\mathrm{D}^b_c(\mathcal{M}_{qc}(B)),T\mathfrak{D}_J^{(B,\nu)},-1,-\varpi^J) & (\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A})),T\mathfrak{D}_{\tilde{I}}^{(\tilde{A},\tilde{\tau})},-1,-\varpi^{\tilde{I}})\,. \arrow[l, "\mathrm{Tr}_\pi"]
\end{tikzcd}\,,$$

*i.e.* $(G,\theta) := \mathrm{Tr}_\pi\circ\mathfrak{sh}\circ F_{\tilde{\iota}}\circ p^*$.

Recall here the definition of the duality preserving functor $\mathfrak{sh}$: The underlying functor is the identity functor, and the duality transformation $\mathfrak{D}_{T\tilde{I}}^{(\tilde{A},\tilde{\tau})}\xrightarrow{\simeq} T\mathfrak{D}_{\tilde{I}}^{(\tilde{A},\tilde{\tau})}$ is $(-1)^l$-times the identity in degree $l$, see [14, Sect. 1.2].

The quasi-isomorphism of complexes

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "\cdot t"'] & 0 \arrow[d] \\
S \arrow[r, "\pi"] & \tilde{R}
\end{tikzcd}$$

induces a quasi-isomorphism

$$s_M\ :\ F(M_\bullet) = \mathrm{K}_\bullet(t)\otimes_R M_\bullet \longrightarrow G(M_\bullet) = \tilde{R}\otimes_R M_\bullet\,,$$

which is natural in $M_\bullet\in\mathrm{D}^b(\mathcal{E}(A))$.

In other words, we have an isomorphism of functors $s: F\xrightarrow{\simeq} G$.

**5.3. Theorem.** *The isomorphism of functors $s: F\xrightarrow{\simeq} G$ is an isometry between the two duality preserving functors $(F,\eta)$ and $(G,\theta)$.*

*Proof.* We compute first

$$\ell_M\ :=\ T\mathfrak{D}_J^{(B,\nu)}(s_M)\circ\theta_M\circ s_{\mathfrak{D}_R^{(A,\tau)}M}\ -\ \eta_M$$

for $M_\bullet\in\mathrm{D}^b(\mathcal{E}(A))$. For ease of notation we set

$$D(N)\ :=\ \overline{\mathrm{Hom}_R(N,R)}^{\,\tau} = \mathfrak{D}_R^{(A,\tau)}(N)$$

for an $A$-module $N$, and

$$D_i(N)\ :=\ \overline{\mathrm{Hom}_S(N,J_{-i})}^{\,\nu}$$

for a $B$-module $N$, $0 \le i \le \dim S = d$. We also set $M_S := S \otimes_R M$ for an $R$-module $M$. With these notations $\ell_M$ is in degree $l \in \mathbb{Z}$ the homomorphism

$$\begin{aligned}(\ell_M)_l \,:\, & D(M_{-l})_S \oplus D(M_{-(l-1)})_S \\ & \longrightarrow \bigoplus_{i=0}^{d} \Big[ D_i\big((M_{-(l-1)-i})_S\big) \oplus D_i\big((M_{-l-i})_S\big) \Big] \\ & \qquad = \Big[ D_0\big((M_{-(l-1)-i})_S\big) \oplus D_0\big((M_{-l-i})_S\big) \Big] \oplus \ldots ,\end{aligned}$$

which maps $(s_1 \otimes f_1, s_2 \otimes f_2) \in D(M_{-l})_S \oplus D(M_{-(l-1)})_S$ onto

$$\Big( j[s_2 \otimes (u \circ f_2)] \,,\, -j[s_1 \otimes (u \circ f_1)] \,,\, (-1)^l \tilde{\imath}[\pi(s_1) \otimes (q \circ f_1)] \,,\, 0, \ldots , 0) \,.$$

We now define a chain homotopy between $\ell_M$ and the zero map as follows. Let for $l \in \mathbb{Z}$

$$(h_M)_l \,:\, \big[\, \mathrm{K}_\bullet(t) \otimes_R \mathfrak{D}_R^{(A,\tau)}(M_\bullet) \big]_l \longrightarrow \big[\, T\mathfrak{D}_J^{(B,\nu)}(S \otimes_R M_\bullet) \big]_{l+1} \,,$$

be the homomorphism sending $(s_1 \otimes f_1, s_2 \otimes f_2) \in D(M_{-l})_S \oplus D(M_{-(l-1)})_S$ to

$$(t^{-1} \cdot j[s_1 \otimes (u \circ f_1)] \,, 0, \ldots , 0) \;\in\; \bigoplus_{i=0}^{d} \Big[ D_i\big((M_{-l-i})_S\big) \oplus D_i\big((M_{-(l+1)-i})_S\big) \Big] \,.$$

To verify that this is a chain homotopy between $\ell_M$ and the zero map we have to show that

$$(\ell_M)_l \;=\; h_{l-1} \circ d_l^{K_\bullet(t)\otimes_R \mathfrak{D}_R^{(A,\tau)}(M_\bullet)} \;+\; d_{l+1}^{T\mathfrak{D}_J^{(B,\nu)}(S\otimes_R M_\bullet)} \circ h_l \tag{3}$$

for all $l \in \mathbb{Z}$ and $M_\bullet \in \mathrm{D}^b(\mathcal{E}(A))$.

The verification of (3) is a straightforward computation using that

$$\begin{aligned} & d_l^{K_\bullet(t)\otimes_R \mathfrak{D}_R^{(A,\tau)}(M_\bullet)}(s_1 \otimes f_1, s_2 \otimes f_2) \\ & \qquad = \Big( s_1 \otimes (f_1 \circ d^M_{-(l-1)}) \,+\, ts_2 \otimes f_2, -s_2 \otimes (f_2 \circ d^M_{-(l-2)}) \Big)\end{aligned}$$

and

$$\begin{aligned} & d_{l+1}^{T\mathfrak{D}_J^{(B,\nu)}(S\otimes_R M_\bullet)}(g, 0, \ldots , 0) \\ & \quad = \Big( -g \circ (\mathrm{id}_S \otimes d^M_{-(l-1)}) \,,\, -g \circ ([\cdot t] \otimes \mathrm{id}_{M_{-l}}) \,,\, (-1)^l d_0^J \circ g \,,\, 0, \ldots , 0 \Big) \,,\end{aligned}$$

as well as that by definition of $\tilde{\imath}$ we have

$$\begin{aligned} \tilde{\imath}\big[\pi(s_1) \otimes (q \circ f_1)\big] &= \tilde{\imath}\big[\pi(s_1) \otimes (\pi \circ u \circ f_1)\big] \\ &= \tilde{\imath}\Big(\pi\big[\, s_1 \otimes (u \circ f_1)\big]\Big) \;=\; d_0^J \Big(\, t^{-1} \cdot j\big(s_1 \otimes (u \circ f_1)\big) \Big) \,.\end{aligned}$$

We are done. $\square$

**5.4.** Our first corollary of Theorem 5.3 is the following generalization of the zero theorem [14, Thm. 6.3]. We keep above notation and denote for an integer $h \ge 0$ by $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))^{(h)}$ and $\mathrm{D}^b_c(\mathcal{M}_{qc}(B))^{(h)}$ the subcategories of $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ and $\mathrm{D}^b_c(\mathcal{M}_{qc}(B))$, respectively, consisting of complexes whose homology modules have support in codimension $\ge h$. These are triangulated categories with (the restriction of the) duality $\mathfrak{D}_I^{(A,\tau)}$ and $\mathfrak{D}_J^{(B,\nu)}$, respectively. Let further $\iota : R \longrightarrow I_\bullet$ be a minimal injective resolution.

**5.5. Generalized Zero Theorem.** *Let $(M_\bullet, \varphi)$ be a $i$-symmetric space in the triangulated category with duality $(\mathrm{D}^b_c(\mathcal{M}_{qc}(A))^{(h)}, \mathfrak{D}_I^{(A,\tau)}, \varpi^I)$. Then the $(i+1)$-symmetric space*

$$\mathrm{Tr}_\pi\Big(\mathfrak{sh}\big(p^*(M_\bullet, \varphi)\big)\Big)$$

*is neutral in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(B))^{(h)}, \mathfrak{D}_J^{(B,\nu)}, 1, \varpi^J)^{(i+1)}$.*

*Proof.* We can assume that $(M_\bullet, \varphi) \simeq F_\iota(N_\bullet, \psi)$ for some $i$-symmetric space $(N_\bullet, \psi)$ in $(\mathrm{D}^b(\mathcal{E}(A))^{(h)}, \mathfrak{D}_R^{(A,\tau)}, 1, \varpi^R)$.

Since

$$\begin{tikzcd} & S \arrow[d, "\cdot t"] \\ S \arrow[r, "\mathrm{id}"] & S \end{tikzcd}$$

is a Lagrangian of $\mathfrak{Kos}(t) = \mathfrak{Kos}(S, t)$ the canonical morphism of complexes

$$S \otimes_R M_\bullet = u^*(N_\bullet) \longrightarrow \mathrm{K}_\bullet(t) \otimes u^* N_\bullet$$

is a Lagrangian for $\mathfrak{Kos}(t) \star u^*(N_\bullet, \psi)$. As $u$ is flat we have $u^*(M_\bullet) \in \mathrm{D}^b(\mathcal{E}(B))^{(h)}$, and so $\mathfrak{Kos}(t) \star u^*(N_\bullet, \psi)$ is neutral in $(\mathrm{D}^b(\mathcal{E}(B))^{(h)}, \mathfrak{D}_S^{(B,\nu)}, 1, \varpi^S)^{(i+1)}$. It follows that the $(i+1)$-symmetric space $F_j(\mathfrak{Kos}(t) \star u^*(N_\bullet, \psi)$ is neutral in the triangulated category with duality $(\mathrm{D}^b_c(\mathcal{M}_{qc}(B))^{(h)}, \mathfrak{D}_J^{(B,\nu)}, 1, \varpi^J)^{(i+1)}$.

By Theorem 5.3 this space is isometric in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(B))^{(h)}, \mathfrak{D}_J^{(B,\nu)}, 1, \varpi^J)^{(i+1)}$ to $\mathrm{Tr}_\pi\Big(\mathfrak{sh}\big(p^*(M_\bullet, \varphi)\big)\Big) = \mathrm{Tr}_\pi\Big(\mathfrak{sh}\big(F_{\tilde{\iota}}(p^*(N_\bullet, \psi))\big)\Big)$. $\square$

**5.6.** In the rest of this section we assume that $R$ is regular. Let $\underline{t} = (t_1, \ldots, t_n)$ be a regular sequence. We set $\mathfrak{a}_0 := \{0\}$ and $\mathfrak{a}_r := \sum\limits_{l=n-r+1}^{n} Rt_l$, and denote by $\pi_r : R/\mathfrak{a}_{r-1} \longrightarrow R/\mathfrak{a}_r$ the quotient morphism for $1 \le r \le n$. We assume that the quotient rings $R_r := R/\mathfrak{a}_r$ are regular for all $0 \le r \le n$.

Let $0 \longrightarrow R \xrightarrow{\iota} I_0 \xrightarrow{d_0^I} I_{-1} \longrightarrow \ \ldots \ \longrightarrow I_{-\dim R} \longrightarrow 0$ be a minimal injective resolution of $R$, and define inductively $I_\bullet^0 := I_\bullet$, and $I_\bullet^r := \pi_r^\sharp(I_\bullet^{r-1})$ for $1 \le r \le n$. We claim that there exists quasi-isomorphisms $\iota_n : R/\mathfrak{a}_n \longrightarrow I_\bullet^n$, such that the following diagram commutes:

$$\begin{tikzcd} & & \mathrm{W}^i(R/\mathfrak{a}_n \otimes_R (A,\tau), R/\mathfrak{a}_n) \arrow[dll, "\big((\mathrm{Tr}_{\pi_1} \circ \mathfrak{sh}) \circ \ldots \circ (\mathrm{Tr}_{\pi_n} \circ \mathfrak{sh})\big) \circ F_{\iota_n}"', "\simeq"] \\ \tilde{W}^{i+n}_{\mathfrak{a}_n}(A, \tau, I_\bullet) & & \mathrm{W}^i(A, \tau, R) \arrow[ll, "F_\iota \circ \big(\mathfrak{Kos}(R, \underline{t}) \star (-)\big)"] \arrow[u, "\pi_n^* \circ \pi_{n-1}^* \ldots \circ \pi_1^*"'] \end{tikzcd} \quad . \tag{4}$$

We prove this by induction on $n \ge 1$. The induction beginning $n = 1$ is a special case of Theorem 5.3 above. For the induction step we set $\underline{t}^1 := (t_2 + \mathfrak{a}_1, \ldots, t_n + \mathfrak{a}_1)$, which is a regular sequence in $R_1 = R/\mathfrak{a}_1$. We consider the following diagram:

$$\begin{tikzcd}
 & & \mathrm{W}^i(R_n \otimes (A,\tau), R_n) \arrow[dl, "\big((\mathrm{Tr}_{\pi_2}\circ\mathfrak{sh})\circ\ldots\circ(\mathrm{Tr}_{\pi_n}\circ\mathfrak{sh})\big)\circ F_{\iota_n}"', "\simeq"] \\
 & \tilde{W}^{i+n-1}_{\mathfrak{a}_n/\mathfrak{a}_1}(R_1 \otimes_R (A,\tau), I^1_\bullet) \arrow[dl, "\mathrm{Tr}_{\pi_1}\circ\mathfrak{sh}"', "\simeq"] & \mathrm{W}^i(R_1 \otimes_R (A,\tau), R_1) \arrow[l, "F_{\iota_1}\circ\big(\mathfrak{Kos}(R_1,\underline{t}^1)\star(-)\big)"] \arrow[u, "\pi_n^*\circ\ldots\circ\pi_2^*"'] \\
 & \mathrm{W}^{i+n-1}_{\mathfrak{a}_{n-1}}(A,\tau,R) \arrow[u, "F_{\iota_1}\circ\pi_1^*"] \arrow[dl, "F_\iota\circ\big(\mathfrak{Kos}(R,t_1)\star(-)\big)"] & \mathrm{W}^i(A,\tau,R) \arrow[l, "\mathfrak{Kos}(R,t_2,\ldots,t_n)\star(-)"] \arrow[u, "\pi_1^*"'] \\
\tilde{W}^{i+n}_{\mathfrak{a}_n}(A,\tau,I_\bullet) \ , & &
\end{tikzcd} \tag{5}$$

where the quasi-isomorphisms $\iota_1 : R_1 = R/\mathfrak{a}_1 \longrightarrow I^1_\bullet$ and $\iota_n : R_n = R/\mathfrak{a}_n \longrightarrow I^n_\bullet$ are to be determined, so that the diagram commutes.

Observe first that the lower right hand side square commutes since pull-backs are homomorphisms of the total derived Witt ring by [17, Thm. 3.4]. Now by Theorem 5.3 there exists a finite injective resolutions $\iota_1 : R_1 \longrightarrow I^1_\bullet$, such that the lower left hand side triangle commutes, and then by induction there exists such a resolution $\iota_n : R_n \longrightarrow I^n_\bullet$, such that the upper right hand side triangle commutes.

We arrive at the following result which generalizes [11, Thm. 9.3].

**5.7. Theorem.** *For all $i \in \mathbb{Z}$ the homomorphism*

$$\mathrm{W}^i(A,\tau,R) \longrightarrow \mathrm{W}^{i+n}_{\mathfrak{a}_n}(A,\tau,R)\,,\ x \longmapsto \mathfrak{Kos}\big(R,(t_1,\ldots,t_n)\big)\star x$$

*is surjective, respectively an isomorphism, if and only if*

$$\pi^* \ :\ \mathrm{W}^i(A,\tau,R) \longrightarrow \mathrm{W}^i(R/\mathfrak{a}_n \otimes_R (A,\tau), R/\mathfrak{a}_n)$$

*is surjective, respectively an isomorphism, where $\pi : R \longrightarrow R/\mathfrak{a}_n$ is the quotient morphism, i.e. $\pi = \pi_1 \circ \pi_2 \circ \ldots \circ \pi_n$.*

*In particular, if $R$ is a complete regular local ring with maximal ideal $\mathfrak{m}$, and $t_1,\ldots,t_n$ is a regular system of parameters then we have*

(i) $\mathrm{W}^i_{\mathfrak{m}}(A,\tau,R) = 0$ *if $i-n$ is odd; and*

(ii) *the homomorphism*

$$\mathrm{W}^{1-\epsilon}(A,\tau,R) \xrightarrow{\mathfrak{Kos}(R,\underline{t})\star -} \mathrm{W}^{1-\epsilon+n}_{\mathfrak{m}}(A,\tau,R) \xrightarrow{\rho^l} \mathrm{W}^i_{\mathfrak{m}}(A,\tau,R)$$

*is an isomorphism if $i = n+1-\epsilon+4l$ for some $l \in \mathbb{Z}$ for all $\epsilon \in \{\pm 1\}$.*

*Hence in this case the homomorphism $\mathrm{W}^i_{\mathfrak{m}}(A,\tau,R) \longrightarrow \mathrm{W}^i(A,\tau,R)$ is the zero map for all $i \in \mathbb{Z}$.*

*Proof.* The first part is a consequence of Diagram (5) above and the fact that by the dévissage Theorem 3.4 the morphisms

$$\mathrm{Tr}_{\pi_r}\circ\mathfrak{sh} \ :\ \tilde{W}^j_{\mathfrak{a}_n/\mathfrak{a}_r}(R/\mathfrak{a}_r \otimes_R (A,\tau), I^r_\bullet) \longrightarrow \tilde{W}^{j+1}_{\mathfrak{a}_n/\mathfrak{a}_{r-1}}(R/\mathfrak{a}_{r-1} \otimes_R (A,\tau), I^{r-1}_\bullet)$$

are isomorphisms for all $j \in \mathbb{Z}$ and $1 \leq r \leq n$.

For the second part, assertion (i) is [12, Lem. 4.8], and (ii) follows since as $R$ is complete the natural homomophism

$$\mathrm{W}_\epsilon(A,\tau) \longrightarrow \mathrm{W}_\epsilon(R/\mathfrak{m} \otimes_R (A,\tau))$$

is an isomorphism by [19, Chap. II (4.6.1)] for all $\epsilon \in \{\pm 1\}$ (this uses also our assumption that 2 is invertible in $R$ and so also in $A$).

The last assertion follows from the fact that the class of $\mathfrak{Kos}(\underline{t}) = \mathfrak{Kos}(R,\underline{t})$ is trivial in $\mathrm{W}^n(R)$, and since the 4-periodicity map $\rho$ commutes with the product by [5, App. B], *i.e.* we have $\rho([\mathfrak{Kos}(\underline{t})] \star x) = [\mathfrak{Kos}(\underline{t})] \star \rho(x)$. □

## 6. Hermitian Witt groups of the punctured spectrum

**6.1.** Throughout this section $R$ denotes a regular local ring with maximal ideal $\mathfrak{m}$ and residue field $k$, and $(A,\tau)$ an $R$-Azumaya algebra with involution (recall our convention at the end of 2.3). Let $\kappa : U := \operatorname{Spec} R \setminus \{\mathfrak{m}\} \hookrightarrow \operatorname{Spec} R$ be the punctured spectrum. By Balmer's [2] localization sequence we have for $\epsilon \in \{\pm 1\}$ an exact sequence

$$\mathrm{W}^{1-\epsilon}_{\mathfrak{m}}(A,\tau,R) \longrightarrow \mathrm{W}_\epsilon(A,\tau) \xrightarrow{\kappa^*} \mathrm{W}_\epsilon(A|_U,\tau|_U) \xrightarrow{\partial} \mathrm{W}^{2-\epsilon}_{\mathfrak{m}}(A,\tau,R) \longrightarrow \mathrm{W}^{2-\epsilon}(A,\tau,R)\,.$$

If $R$ is moreover complete we know by Theorem 5.7 above that $\kappa^*$ is injective and that the connecting morphism $\partial$ is surjective. Hence we have a short exact sequence

$$0 \longrightarrow \mathrm{W}_\epsilon(A,\tau) \xrightarrow{\kappa^*} \mathrm{W}_\epsilon(A|_U,\tau|_U) \xrightarrow{\partial} \mathrm{W}^{2-\epsilon}_{\mathfrak{m}}(A,\tau,R) \longrightarrow 0 \tag{6}$$

for all $\epsilon \in \{\pm 1\}$. (Using the in [18] proven Gersten conjecture for hermitian Witt groups it would be enough to assume that $R$ is geometrically regular over a discrete valuation ring.)

Set now

$$n := \dim R = 4q + r + 1$$

with $q \in \mathbb{Z}$ and $r \in \{-1,0,1,2\}$, and let $\underline{t} = (t_1,\ldots,t_n) \subset R$ be a regular system of parameters for $R$.

If $n$ is even then $2-\epsilon-n$ is odd and so by Theorem 5.7 we have if $R$ is complete

$$\mathrm{W}^{2-\epsilon}_{\mathfrak{m}}(A,\tau,R) = 0\,.$$

Hence:

**Theorem.** *If* $\dim R$ *is even and* $R$ *complete then the pull-back along* $\kappa : U \hookrightarrow \operatorname{Spec} R$ *is an isomorphism*

$$\kappa^* : \mathrm{W}_\epsilon(A,\tau) \xrightarrow{\simeq} \mathrm{W}_\epsilon(A|_U,\tau|_U)\,.$$

*for all* $\epsilon \in \{\pm 1\}$.

For the rest of this section we assume that $n$ is odd, *i.e.* $r = 0$ or $r = 2$. We let $\mathfrak{E}_{\underline{t}} \in \mathrm{W}^r(U)$ be the $r$-symmetric space satisfying $\rho^q(\partial(\mathfrak{E}_{\underline{t}})) = \mathfrak{Kos}(R,\underline{t})$ defined in Lemma 4.3.

**6.2. Case $r=0$.** We consider the following diagram:

$$\begin{tikzcd}
\mathrm{W}_\epsilon(A|_U,\tau|_U) \arrow[r, "\simeq"] & \mathrm{W}^{1-\epsilon}(A|_U,\tau|_U,\mathcal{O}_U) \arrow[r, "\partial"] & \mathrm{W}^{2-\epsilon}_{\mathfrak{m}}(A,\tau,R) \\
& \mathrm{W}^{1-\epsilon}(A,\tau,R) \arrow[u, "x\mapsto \mathfrak{E}_{\underline{t}}\star(x|_U)"] \arrow[r, "\rho^{-q}", "\simeq"'] & \mathrm{W}^{2-\epsilon-(4q+1)}(A,\tau,R) \arrow[u, "y\mapsto \mathfrak{K}os(\underline{t})\star y"'] \ .
\end{tikzcd} \tag{7}$$

Note that if $R$ is complete the column arrow on the right hand side is an isomorphism by Theorem 5.7.

We claim that this diagram commutes. In fact, by [17, Thm. 2.9] we have $\mathfrak{E}_{\underline{t}}\star(x|_U)=\mathfrak{E}_{\underline{t}}\star_r(x|_U)$, where $\star_r$ denotes the right product, since $1-\epsilon$ is even. By the right product analog of [17, Thm. 2.11] we have a commutative diagram

$$\begin{tikzcd}
\mathrm{W}(U)\times \mathrm{W}^{1-\epsilon}(A,\tau,R) \arrow[r, "\star_r"] \arrow[d, "\partial\times\mathrm{id}"'] & \mathrm{W}^{1-\epsilon}(A,\tau,R) \arrow[d, "\partial"] \\
\mathrm{W}^1_{\mathfrak{m}}(R)\times \mathrm{W}^{1-\epsilon}(A,\tau,R) \arrow[r, "\star_r"'] & \mathrm{W}^{2-\epsilon}(A,\tau,R) \ ,
\end{tikzcd}$$

where we denote the connecting homomorphism $\mathrm{W}(U)\longrightarrow W^1_{\mathfrak{m}}(R)$ also by $\partial$. Hence we have taking [17, Thm. 2.9] again into account and that the product commutes with the 4-periodicity isomorphism by [5, App. B]:

$$\partial(\mathfrak{E}_{\underline{t}}\star(x|_U)) \;=\; \partial(\mathfrak{E}_{\underline{t}})\star x \;=\; \rho^{-q}(\mathfrak{K}os(R,\underline{t}))\star x \;=\; \mathfrak{K}os(R,\underline{t})\star\rho^{-q}(x)\,,$$

proving our claim.

Consequently, if $R$ is complete the connecting homomorphism

$$\partial\,:\;\mathrm{W}_\epsilon(A|_U,\tau|_U)\;\longrightarrow\;\mathrm{W}^{2-\epsilon}(A,\tau,R)$$

in the short exact sequence (6) is a split epimorphism, and so using that by [19, Chap. II (4.6.1)] the homomorphism $\mathrm{W}_\epsilon(A,\tau)\longrightarrow \mathrm{W}_\epsilon(A_k,\tau_k)$ is an isomorphism in this case, where we have set $(A_k,\tau_k):=k\otimes_R(A,\tau)$, we conclude the following result.

**Theorem.** *If* $\dim R\equiv 1 \bmod 4$ *and $R$ is complete we have an isomorphism*

$$\mathrm{W}_\epsilon(A|_U,\tau|_U)\;\simeq\;\mathrm{W}_\epsilon(A_k,\tau_k)\oplus\mathrm{W}_\epsilon(A_k,\tau_k)$$

*for all* $\epsilon\in\{\pm 1\}$.

**Remarks.**

(i) If $R$ is a discrete valuation ring it is shown in [15] that (6) is exact even if $R$ is not complete.

(ii) The same argument shows that if $R$ is a complete discrete valuation ring with residue field $k$ and fraction field $K$ and $G$ a finite group then

$$\mathrm{W}_\epsilon(KG,\nu_g)\;\simeq\;\mathrm{W}_\epsilon(kG,\nu_G)\oplus\mathrm{W}_\epsilon(kG,\nu_G)$$

for all $\epsilon\in\{\pm1\}$, where $\nu_G\,:\,g\mapsto g^{-1}$ is the 'standard' involution on the group rings involved. (Use that $\sum\limits_{g\in G}a_g g\mapsto a_e$, $e$ the neutral element of $G$, is an involution trace to identify the functors $\overline{\mathrm{Hom}_{RG}(-, RG)}^{\nu_G}$ and $\overline{\mathrm{Hom}_R(-, R)}^{\nu_G}$.)

Note that no assumption on the characteristic of $K$ or $k$ is made except for not being 2.

**6.3. Case** $r = 2$**.** We assume now that $R$ is complete and consider the following diagram, whose right hand column arrow is again an isomorphism by Theorem 5.7. It is commutative by the same reasoning as in 6.2 above:

$$\begin{tikzcd}
\mathrm{W}_\epsilon(A|_U, \tau|_U) \arrow[r, "\simeq"] & \mathrm{W}^{1-\epsilon}(A|_U, \tau|_U, \mathcal{O}_U) \arrow[r, "\partial"] & \mathrm{W}^{2-\epsilon}_{\mathfrak{m}}(A, \tau, R) \\
& \mathrm{W}^{-1-\epsilon}(A, \tau, R) \arrow[u, "x \mapsto \mathfrak{E}_{\underline{t}} \star (x|_U)"] \arrow[r, "\rho^{-q}", "\simeq"'] & \mathrm{W}^{2-\epsilon-(4q+2+1)}(A, \tau, R) \arrow[u, "\simeq"', "y \mapsto \mathfrak{Kos}(\underline{t}) \star y"]
\end{tikzcd} .$$

Now by 4-periodicity and Balmer's [3] isomorphism between usual and derived Witt groups, see 3.1, we have $\mathrm{W}^{-1-\epsilon}(A, \tau, R) \simeq \mathrm{W}_{-\epsilon}(A, \tau) \simeq \mathrm{W}_{-\epsilon}(A_k, \tau_k)$ (the latter isomorphism by [19, Chap. II (4.6.1)]), and so get our computation of $\mathrm{W}_\epsilon(A|_U, \tau|_U)$ in the case $r = 2$.

**Theorem.** *If* $\dim R \equiv 3 \bmod 4$ *and* $R$ *is complete we have an isomorphism*

$$\mathrm{W}_\epsilon(A|_U, \tau|_U) \simeq \mathrm{W}_\epsilon(A_k, \tau_k) \oplus \mathrm{W}_{-\epsilon}(A_k, \tau_k)$$

*for all* $\epsilon \in \{\pm 1\}$

## 7. Symmetric Witt groups of the punctured spectrum

**7.1.** Let $R$ be a regular semilocal ring with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_l$, $l \geq 1$. We assume that

$$1 \leq n = \dim R = \mathrm{ht}\, \mathfrak{m}_i$$

for all $1 \leq i \leq l$. Let $\kappa : U := \mathrm{Spec}\, R \setminus \{\mathfrak{m}_1, \ldots, \mathfrak{m}_l\} \hookrightarrow \mathrm{Spec}\, R$ be the punctured spectrum of $R$, and $\kappa_i : U_i := \mathrm{Spec}\, R_{\mathfrak{m}_i} \setminus \{\mathfrak{m}_i R_{\mathfrak{m}_i}\} \hookrightarrow \mathrm{Spec}\, R_{\mathfrak{m}_i}$ the punctured spectrum of the localized ring $R_{\mathfrak{m}_i}$, $i = 1, \ldots, l$. We denote $\ell_i : R \longrightarrow R_{\mathfrak{m}_i}$ the localization homomorphism and $\alpha_i : U_i \hookrightarrow U$ the morphism of schemes induced by $\ell_i$ for all $1 \leq i \leq l$.

Balmer [4, Thm. 3.3] has shown that if $n \not\equiv 1 \bmod 4$ then the pull-back $\kappa^* : \mathrm{W}(R) \longrightarrow \mathrm{W}(U)$ is an isomorphism, and if $n = 4q + 1$, where $q$ is an integer $\geq 0$, then there is an exact sequence

$$0 \longrightarrow \mathrm{W}(R) \xrightarrow{\kappa^*} \mathrm{W}(U) \xrightarrow{\partial} \mathrm{W}^1_{\mathfrak{J}(R)}(R) \longrightarrow 0\,, \tag{8}$$

where $\partial$ is the connecting homomorphism in Balmer's [2] localization sequence associated with the open embedding $\kappa : U \hookrightarrow \mathrm{Spec}\, R$, and $\mathfrak{J}(R) = \bigcap_{i=1}^{l} \mathfrak{m}_i$ is the Jacobson radical of $R$.

The localization morphisms $\ell_i$ induce a $\mathrm{W}(R)$-linear isomorphism

$$(\ell_i^*)_{i=1}^l \,:\, \mathrm{W}^1_{\mathfrak{J}(R)}(R) \xrightarrow{\simeq} \bigoplus_{i=1}^{l} \mathrm{W}^1_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i})\,, \tag{9}$$

see Balmer and Walter [7, Sect. 7] (and [17, Sect. 3] for the $\mathrm{W}(R)$-linearity).

**7.2.** From now on we assume that $n = 4q+1$ with $q \geq 1$.

We choose elements $x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}$, such that $x_{1\,\mathfrak{m}_i}^{(i)}, x_{2\,\mathfrak{m}_i}^{(i)}, \ldots, x_{n\,\mathfrak{m}_i}^{(i)}$ is a regular system of parameters for $R_{\mathfrak{m}_i}$ for all $1 \leq i \leq l$. By the Chinese remainder theorem there are for all $1 \leq i \leq l$ sequences of elements

$$\underline{t}^{(i)} \;=\; (t_1^{(i)}, t_2^{(i)}, \ldots, t_n^{(i)})$$

satisfying

$$t_j^{(i)} \;\equiv\; x_j^{(i)} \bmod \mathfrak{m}_i^2 \quad \text{and} \quad t_j^{(i)} \;\equiv\; 1 \bmod \bigcap_{r\neq i} \mathfrak{m}_r$$

for all $1 \leq j \leq n$. Then $(t_{1\,\mathfrak{m}_i}^{(i)}, t_{2\,\mathfrak{m}_i}^{(i)}, \ldots, t_{n\,\mathfrak{m}_i}^{(i)})$ is a regular system of parameters for $R_{\mathfrak{m}_i}$ for all $1 \leq i \leq l$, and we have $t_j^{(i)} \not\in \mathfrak{m}_r$ for all $1 \leq i \neq r \leq l$ and all $1 \leq j \leq n$. In particular, we have an open embedding

$$\varsigma_i \,:\, U \hookrightarrow V_i \,:=\, \bigcup_{j=1}^{n} \operatorname{Spec} R[(t_j^{(i)})^{-1}]$$

for all $1 \leq i \leq l$.

**7.3. Lemma.** *The* $\mathrm{W}(R)$*-module* $\mathrm{W}^1_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i})$ *is generated by* $\rho^{-q}(\mathfrak{Kos}(R_{\mathfrak{m}_i}, \underline{t}_{\mathfrak{m}_i}^{(i)}))$ *for all integers* $1 \leq i \leq l$.

*Proof.* We fix $i \in \{1, \ldots, l\}$ and set for ease of notation $\underline{t} := \underline{t}_{\mathfrak{m}_i}^{(i)}$. By Theorem 5.7 we know that every element in $\mathrm{W}^n_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i})$ is equal

$$\mathfrak{Kos}(R_{\mathfrak{m}_i}, \underline{t}) \star x$$

for some $x \in \mathrm{W}(R_{\mathfrak{m}_i})$ as $\mathrm{W}(R_{\mathfrak{m}_i}) \longrightarrow \mathrm{W}(R/\mathfrak{m}_i)$ is onto. Since $\frac{1}{2} \in R_{\mathfrak{m}_i}$ every symmetric bilinear space over $R_{\mathfrak{m}_i}$ has an orthogonal basis, see *e.g.* Baeza [1, Chap. I, Prop. (3.4)], and so $\mathrm{W}^n_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i})$ is generated by the classes of forms $\mathfrak{Kos}(R_{\mathfrak{m}_i}, \underline{t}) \star < x >$, $x \in R_{\mathfrak{m}_i}^{\times}$. After multiplying by a square we can assume that $x$ is in the image of $R \longrightarrow R_{\mathfrak{m}_i}$.

Given $x \in R$ with $x$ a unit in $R_{\mathfrak{m}_i}$ there exists by the Chinese remainder theorem a unit $\tilde{x} \in R$ with $\tilde{x} \equiv x \mod \mathfrak{m}_i$. By (4) we have

$$\mathfrak{Kos}(R_{\mathfrak{m}_i}, \underline{t}) \star < x > = \mathfrak{Kos}(R_{\mathfrak{m}_i}, \underline{t}) \star < \tilde{x} > \,.$$

The lemma follows from this since by [5, App. B] we have

$$\rho^{-q}\big(\, \mathfrak{Kos}(R_{\mathfrak{m}_i}, \underline{t}) \star < \tilde{x} > \,\big) \,=\, \rho^{-q}\big(\, \mathfrak{Kos}(R_{\mathfrak{m}_i}, \underline{t})\,\big) \star < \tilde{x} > \,.$$

□

**Remark.** Note that by the dévissage Theorem and 4-periodicity

$$\mathrm{W}^1_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i}) \,\simeq\, \mathrm{W}^n_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i}) \,\simeq\, \mathrm{W}(R/\mathfrak{m}_i) \,\neq\, 0\,,$$

and so in particular $\rho^{-q}(\mathfrak{Kos}(R_{\mathfrak{m}_i}, \underline{t}_{\mathfrak{m}_i}^{(i)})) \neq 0$.

**7.4.** We give generators for the $\mathrm{W}(R)$-algebra $\mathrm{W}(U)$. Let for this $\tilde{\gamma}_i : \operatorname{Spec} R \longrightarrow \mathbb{A}^n_{\mathbb{Z}[\frac{1}{2}]}$ be the morphism of schemes induced by the morphism of rings

$$f_i \,:\, S_n = \mathbb{Z}[\frac{1}{2}][T_1,\ldots,T_n] \longrightarrow R\,,\; T_j \longmapsto t_j^{(i)}\,,$$

$1 \leq i \leq l$. By restriction to $V_i$ the morphism $\tilde{\gamma}_i$ induces a morphism of schemes

$$\gamma_i \,:\, U \xrightarrow{\varsigma_i} V_i \xrightarrow{(\tilde{\gamma}_i)|_{V_i}} \mathbb{U}^n_{\mathbb{Z}[\frac{1}{2}]}$$

for all $1 \leq i \leq l$. Set

$$\mathfrak{E}_i \,:=\, \gamma_i^*(\mathfrak{E}_n) \qquad \text{and} \qquad \mathfrak{e}_i \,:=\, \alpha_i^*(\mathfrak{E}_i) \tag{10}$$

for $i = 1,\ldots,l$. These are symmetric spaces over $U$ and $U_i$, respectively. By Lemma 4.3 we have

$$(\tilde{\gamma}_i)|_{V_i}(\mathfrak{E}_n) \star (\tilde{\gamma}_i)|_{V_i}(\mathfrak{E}_n) \,=\, (\tilde{\gamma}_i)|_{V_i}(\mathfrak{E}_n \star \mathfrak{E}_n) \,=\, 0\,,$$

as well as

$$(\vartheta_j^{(i)})^*\Big(\big((\tilde{\gamma}_i)|_{V_i}\big)^*(\mathfrak{E}_n)\Big) \,=\, 0$$

for all $1 \leq i \leq l$ and $1 \leq j \leq n$, where $\vartheta_j^{(i)} : \operatorname{Spec} R[(t_j^{(i)})^{-1}] \hookrightarrow V_i$. It follows that

$$\mathfrak{E}_i \star \mathfrak{E}_i \,=\, 0$$

for all $1 \leq i \leq l$, and

$$\mathfrak{E}_i\big|_{U\cap \operatorname{Spec} R[(t_j^{(i)})^{-1}]} \,=\, 0 \tag{11}$$

for all $1 \leq i \leq l$ and $1 \leq j \leq n$. The latter implies that $\mathfrak{E}_i$ is in the image of

$$\mathrm{W}^0_{Z_j^{(i)}\cap U}(U) \longrightarrow \mathrm{W}(U)$$

for all $1 \leq i \leq l$ and $1 \leq j \leq n$, where $Z_j^{(i)} = \operatorname{Spec} R/Rt_j^{(i)} \subset \operatorname{Spec} R$. Since $Z_j^{(i)} \cap Z_s^{(r)} = 0$ for all $1 \leq r \neq i \leq l$ and any $1 \leq j,s \leq n$ this implies that the product $\mathfrak{E}_i \star \mathfrak{E}_r$ is trivial for all $1 \leq i \neq r \leq l$. We have shown the following result.

**7.5. Lemma.** *We have*

$$\mathfrak{E}_i \star \mathfrak{E}_r \,=\, 0$$

*in* $\mathrm{W}(U)$ *for all* $1 \leq i,r \leq l$.

**7.6.** By construction, see the proof of Lemma 4.3, we have

$$\partial_i(\mathfrak{e}_i) \,=\, \rho^{-q}(\mathfrak{Kos}(\underline{t}^{(i)}_{\mathfrak{m}_i})) \,\neq\, 0$$

in $\mathrm{W}^1_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i})$ for all $1 \leq i \leq l$, where $\partial_i : \mathrm{W}(U_i) \longrightarrow \mathrm{W}^1_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i})$ is the connecting homomorphism in Balmer's [2] localization sequence associated with the open embedding $\kappa_i : U_i \hookrightarrow \operatorname{Spec} R_{\mathfrak{m}_i}$. In particular, we have $\mathfrak{e}_i \neq 0$ and so by definition, see (10), also $\mathfrak{E}_i \neq 0$ for all $1 \leq i \leq l$.

**Lemma.** *We have*

$$\ell_j^*\big(\partial(\mathfrak{E}_i)\big) \,=\, 0$$

*for all* $1 \leq i \neq j \leq l$.

*Proof.* By construction the 1-symmetric space $\partial(\mathfrak{E}_i)$ is Witt equivalent to a space which lives on the shifted Koszul complex $T^{-2q}(\mathrm{K}_*(\underline{t}^{(i)}))$. Since $\underline{t}^{(i)} \not\subset \mathfrak{m}_j$ for $1 \leq i \neq j \leq l$ by construction of the sequences $\underline{t}^{(s)}$ the pull-back $\ell_j^*(\mathrm{K}_*(\underline{t}^{(i)}))$ has trivial homology for all $1 \leq i \neq j \leq l$. The lemma follows. □

**7.7. Theorem.** *The (classes of the) symmetric spaces $\mathfrak{E}_1, \dots, \mathfrak{E}_l$ are non trivial and generate* $\mathrm{W}(U)$ *as* $\mathrm{W}(R)$*-algebra. These generators satisfy*

(i) $\mathfrak{E}_i \star \mathfrak{E}_j = 0$ *in* $\mathrm{W}(U)$ *for all* $1 \le i, j \le l$*; and*

(ii) *are locally trivial, i.e. the class of the localized form* $\mathfrak{E}_{i\,P}$ *is zero in* $\mathrm{W}(R_P)$ *for all* $1 \le i \le l$ *and all prime ideals* $P$ *in* $U$*.*

*In particular, if $R$ is an integral domain with fraction field $K$ then*

$$\theta^*(\mathfrak{E}_i) \;=\; 0$$

*for all* $1 \le i \le l$*, where* $\theta : \operatorname{Spec} K \longrightarrow U$ *is the generic point, and so* $\theta^* : \mathrm{W}(U) \longrightarrow \mathrm{W}(K)$ *is not one-to-one.*

*Proof.* To show that the spaces $\mathfrak{E}_1, \dots, \mathfrak{E}_l$ are non trivial and generate $\mathrm{W}(U)$ as $\mathrm{W}(R)$-algebra we consider the following commutative diagram whose upper row is exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{W}(R) \arrow[r, "\kappa^*"] & \mathrm{W}(U) \arrow[r, "\partial"] \arrow[d, "(\alpha_i^*)_{i=1}^l"'] & \mathrm{W}^1_{\mathfrak{J}(R)}(R) \arrow[r] \arrow[d, "(\ell_i^*)_{i=1}^l", "\simeq"'] & 0 \\
 & & \bigoplus\limits_{i=1}^{l} \mathrm{W}(U_i) \arrow[r, "(\partial_i)_{i=1}^l"'] & \bigoplus\limits_{i=1}^{l} \mathrm{W}^1_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i}) & 
\end{tikzcd}$$

Note that the right hand column arrow of this diagram is an isomorphism by (9).

By Lemma 7.3 the 1-symmetric space $\rho^{-q}(\mathfrak{K}os(R_{\mathfrak{m}}, \underline{t}^{(i)}_{\mathfrak{m}_i}))$ generates the $\mathrm{W}(R)$-module $\mathrm{W}^1_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i})$, and we have $\rho^{-q}(\mathfrak{K}os(R_{\mathfrak{m}}, \underline{t}^{(i)}_{\mathfrak{m}_i})) = \partial_i(\mathfrak{e}_i)$ for all $1 \le i \le l$, see 7.6 for the last equation. Now be the very definition $\mathfrak{e}_i = \alpha_i^*(\mathfrak{E}_i)$, see (10), and therefore by the commutative diagram above we have

$$\partial_i(\mathfrak{e}_i) \;=\; \partial_i(\alpha_i^*(\mathfrak{E}_i)) \;=\; \ell_i^*(\partial(\mathfrak{E}_i))$$

for all $1 \le i \le l$. Since $\mathrm{W}^1_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i}) \neq 0$ this implies $\partial(\mathfrak{E}_i) \neq 0$ and so also $\mathfrak{E}_i \neq 0$ for all $1 \le i \le l$.

On the other hand, the lemma in 7.6 tells us $\ell_j^*(\partial(\mathfrak{E}_i)) = 0$ for all $1 \le i \neq j \le l$, and so since the right hand column arrow in above diagram is an isomorphism we get $\mathrm{W}^1_{\mathfrak{J}(R)}(R) \simeq \sum\limits_{i=1}^{l} \mathrm{W}(R) \cdot \partial(\mathfrak{E}_i)$ as $\mathrm{W}(R)$-module. Hence by the exactness of the upper row of the diagram above $\mathfrak{E}_1, \dots, \mathfrak{E}_l$ generate the $\mathrm{W}(R)$-algebra $\mathrm{W}(U)$.

We are left to show (i) and (ii). The first assertion is Lemma 7.5, and the second follows since for every $P \in U$ the morphism $\operatorname{Spec} R_P \longrightarrow U$ factors via $U \cap \operatorname{Spec} R[(t_j^{(i)})^{-1}]$ for some $1 \le i \le l$ and some $1 \le j \le n$, and therefore this is a consequence of (11). □

**Remarks.**

(i) By [5, App. A] the underlying vector bundles of the forms $\mathfrak{e}_i$ over $U_i$ can not be extended to $\operatorname{Spec} R_{\mathfrak{m}_i}$ and so are in particular not free for all $1 \le i \le l$. This implies that also the underlying vector bundles of the generators $\mathfrak{E}_i$, $1 \le i \le l$, of $\mathrm{W}(U)$ are not free.

(ii) The proof shows also that no proper subset of $\{\mathfrak{E}_1, \dots, \mathfrak{E}_l\}$ can generated $\mathrm{W}(U)$ as $\mathrm{W}(R)$-algebra.

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