DIRECT SUMS OF CHOW MOTIVES AND ROST NILPOTENCE

STEFAN GILLE

ABSTRACT. We examine 'stronger' versions of the Rost nilpotence principle, which hold in almost all cases where Rost nilpotence is known, and show that a direct sum of motives for which these stronger principles hold also satisfy it.

0. Introduction

Let k be a field, R a commutative ring (with 1), and $\mathfrak{Chow}(k,R)$ the category of Chow motives over k with coefficients in R. One says that Rost nilpotence holds for a motive M in $\mathfrak{Chow}(k,R)$ if for all field extensions $E \supseteq k$ the kernel, here denoted $\mathcal{I}^R_{E/k}(M)$, of the restriction morphism

$$\mathrm{End}_{\mathfrak{Chow}(k,R)}(M) \longrightarrow \mathrm{End}_{\mathfrak{Chow}(E,R)}(E \times_k M)$$

is a nil ideal. This property has been proven for motives of smooth projective quadrics by Rost [13] and independently with different methods by Vishik [15]. Its main application is Rost's decomposition of the splitting quadric of a symbol in Milnor K-theory modulo 2, which in turn is crucial for Voevodsky's [17] proof of the Milnor conjecture.

Later Rost nilpotence has been verified for other smooth projective schemes including surfaces and projective homogeneous varieties. It is believed to hold for all Chow motives, but right now a proof of this conjecture seems to be out of reach. We observe however, that if Rost nilpotence is invariant under blow-ups then it holds for all motives. In fact, let $X \subseteq \mathbb{P}^N_k$ be a smooth projective k-scheme and Y its blow-up in $\mathbb{P}^N_k$. Then by Manin's blow-up fomula [11, §9, Cor.] we know that $X \otimes \mathbb{Z}(1)$, where $\mathbb{Z}(1)$ is the Tate-motive, is a direct summand of Y, and so if Rost nilpotence holds for Y it holds for X. On the other hand, Rost nilpotence is trivially true for projective spaces. Hence if it is invariant under blow-ups the smooth projective variety X also satisfies Rost nilpotence.

These considerations lead to the following basic problem:

Let M, N be motives in $\mathfrak{Chow}(k,R)$ satisfying Rost nilpotence. Does then Rost nilpotence hold for the direct sum $M \oplus N$ as well?

In case the Köthe conjecture is true then the answer to this question is 'yes', see Remark 2.6. However most ring theorists seem to believe that Köthe's conjecture is wrong as pointed out by Rowen in his book on ring theory [14, top of page 210].

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On the other hand, a rather elementary computation reveals that the answer to above question is 'yes' if one assumes that the kernel ideals $\mathcal{I}^R_{E/k}(M)$ and $\mathcal{I}^R_{E/k}(N)$ are not only nil but nilpotent ideals for all field extensions $E \supseteq k$, see our Theorem 2.4. We say in this case that *strong Rost nilpotence* holds for M and N.

We show here however that if the coefficient ring is noetherian a weaker property is enough to assure that direct sums satisfy Rost nilpotence if all summands do so: A motive M in $\mathfrak{Chow}(k,R)$ satisfies *Rost nilpotence with bounded exponent* if for every field extension $E \supseteq k$ there exists an integer $N \geq 1$ (depending on E), such that $x^N = 0$ for all $x \in \mathcal{I}^R_{E/k}(M)$. This implies that $\mathcal{I}^R_{E/k}(M)$ is a PI-algebra (without 1) over R, and we use then PI-theory to verify that given another motive N satisfying Rost nilpotence with bounded exponent then the direct sum $M \oplus N$ has the same property.

In the last section we show that for almost (?) all examples where 'usual' Rost nilpotence is known actually strong Rost nilpotence holds.

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1. Chow motives

1.1. We briefly recall the definition of the category of Chow motives referring to Fulton's book [6, Chap. 16] and Manin [11] for details and more information.

We fix a ground field k (of any characteristic) and denote by $\bar{k}$ its algebraic closure. For a k-scheme X we denote by $k(x)$ the residue field of $x \in X$ and by $k(X)$ the function field if X is integral. Given a field extension $E \supset k$ we set $X_E := E \times_k X$. We denote by $\mathrm{CH}_i(X)$ the Chow group (modulo rational equivalence) of dimension i cycles of X.

Let PSm_k be the category of smooth and projective k-schemes. Given two such schemes X, Y, a *correspondence of degree* 0 between X and Y is an element α in $\bigoplus\limits_{i=1}^{l} \mathrm{CH}_{\dim X_i}(X_i \times Y)$, where $X_1, \ldots, X_l$ are the connected components of X. We write then $\alpha : X \rightsquigarrow Y$. Given $\alpha : X \rightsquigarrow Y$ and $\beta : Y \rightsquigarrow Z$ with both Y and Z irreducible their composition is defined as

$$\beta \circ \alpha := p_{XZ\,*}\big(p^*_{XY}(\alpha) \cdot p^*_{YZ}(\beta)\big)\,,$$

where p_{XY}, p_{XZ}, and p_{YZ} are the respective projections from $X \times_k Y \times_k Z$ to $X \times_k Y$, $X \times_k Z$, and $Y \times_k Z$. This product is associative and the class of the image of the diagonal morphism acts as identity. Hence we have an additive category of correspondences of degree 0 over k, denoted $\mathrm{Corr}^0(k)$, whose objects are the smooth projective k-schemes and morphisms are correspondences of degree 0.

1.2. The idempotent completion of $\mathrm{Corr}^0(k)$ is the category of (effective) Chow motives over k, denoted $\mathfrak{Chow}(k)$. The objects of $\mathfrak{Chow}(k)$ are pairs (X, p), where $p : X \rightsquigarrow X$ is a correspondence of degree 0 satisfying $p \circ p = p$, *i.e.* p is an idempotent morphism in $\mathrm{Corr}^0(k)$, and the morphisms are given by

$$\mathrm{Hom}_k\big((X,p),(Y,q)\big) := q \circ \mathrm{Mor}_{\mathrm{Corr}^0(k)}(X,Y) \circ p \ \subseteq \mathrm{Mor}_{\mathrm{Corr}^0(k)}(X,Y)\,.$$

If X is a smooth projective k-scheme we denote by the same symbol X its motive in $\mathfrak{Chow}(k)$. We also set $\mathrm{End}_k(X,p) := \mathrm{Hom}_k((X,p),(X,p))$. The cartesian product induces a 'tensor product' on $\mathfrak{Chow}(k)$, denoted $(X,p)\otimes(Y,q)$.

There is a covariant functor $\mathrm{PSm}_k \longrightarrow \mathfrak{Chow}(k)$, which is the identity on objects and sends a morphism $f : X \longrightarrow Y$ to the class of its graph Γ_f.

The *Tate-motive* and its (positive) twists are denoted by $\underline{\mathbb{Z}}(i) := (\underline{\mathbb{Z}}(1))^{\otimes i}$, $i \in \mathbb{N}\cup\{0\}$, and if X is a smooth projective k-scheme we set $X(i) := X \otimes \underline{\mathbb{Z}}(i)$. Recall that the Tate motive is the complement of the motive of the point in the projective line: $\mathbb{P}^1_k \simeq \underline{\mathbb{Z}} \oplus \underline{\mathbb{Z}}(1)$ in $\mathfrak{Chow}(k)$. A motive is called *split* if it is a direct sum of twists of Tate motives, and *geometrically split* if this is the case over the algebraic closure of the base field. Examples of geometrically split motives are the motives of projective quadrics, or more generally of projective homogeneous varieties, see Köck [10], and the motives of geometrically rational surfaces.

Replacing $\mathrm{CH}_i(-)$ by $\mathrm{CH}_i(-)_R := R \otimes_{\mathbb{Z}} \mathrm{CH}_i(-)$ for a commutative ring R (with 1) we get Chow motives with coefficients, denoted $\mathfrak{Chow}(k,R)$. In this case we denote the homomorphism and endomorphism groups by $\mathrm{Hom}_k((X,p),(Y,q))_R$ and $\mathrm{End}_k(X,p)_R$, respectively. The Tate motives and its twists will then be denoted by $\underline{R}$ and $\underline{R}(i)$, and we set $X(i) = X \otimes \underline{R}(i)$ for $i \geq 0$.

We fix in the following a commutative coefficient ring R with 1.

1.3. Let $E \supseteq k$ be a field extension. Then $X \mapsto X_E$ and $\alpha \mapsto \alpha_E$ induces a restriction morphism

$$\mathrm{res}_{E/k} \,:\, \mathrm{Hom}_k(M,N)_R \longrightarrow \mathrm{Hom}_E(M_E,N_E)_R$$

for $M,N \in \mathfrak{Chow}(k,R)$. This defines a contravariant functor $\mathfrak{Chow}(k,R) \longrightarrow \mathfrak{Chow}(E,R)$, called *restriction*, mapping a motive M in $\mathfrak{Chow}(k,R)$ onto $M_E \in \mathfrak{Chow}(E,R)$.

As in the introduction we denote the kernel of

$$\mathrm{res}_{E/k} \,:\, \mathrm{End}_k(M)_R \longrightarrow \mathrm{End}_E(M_E)_R$$

by $\mathcal{I}^R_{E/k}(M)$ for all $M \in \mathfrak{Chow}(k,R)$ and all field extensions $E \supseteq k$.

2. Rost nilpotence

2.1. Definitions. Let k be a field, R a commutative ring, and $M \in \mathfrak{Chow}(k,R)$.

(a) We say that *Rost nilpotence* (respectively *Rost nilpotence with bounded exponent*) holds M in $\mathfrak{Chow}(k,R)$ if for all field extensions $E \supseteq k$ the kernel $\mathcal{I}^R_{E/k}(M)$ of $\mathrm{res}_{E/k}$ is a nil ideal (respectively a nil ideal with bounded exponent).

(b) We say that *strong Rost nilpotence* holds for M in $\mathfrak{Chow}(k,R)$ if for every field extension $E \supseteq k$ the ideal $\mathcal{I}^R_{E/k}(M)$ is nilpotent.

2.2. Remark. If $\mathrm{End}_k(M)_R$ is $\mathbb{Z}$-torsion free, *e.g.* if R is a field of characteristic 0, or if M is a split motive and $R = \mathbb{Z}$, then $\mathcal{I}^R_{E/k}(M)$ is trivial. This follows since for a purely transcendental extension E/k the base change morphism $\mathrm{CH}_i(Y) \longrightarrow \mathrm{CH}_i(Y_E)$ is an isomorphism, see *e.g.* [5, Prop. 2.1.8], and for E/k a finite extension the kernel of $\mathrm{CH}_i(Y) \longrightarrow \mathrm{CH}_i(Y_E)$ is annihilated by the degree $[E : k]$.

Most proofs of Rost nilpotence rely on a lemma due to Rost [13, Prop. 1], which we recall here in a seemingly more general version. However this is what is actually proven in *loc.cit.* as we explain now.

2.3. Rost's Lemma. *Let X and Y be smooth and projective k-schemes with Y connected, and $\alpha_0,\ldots,\alpha_d \in \operatorname{End}_k(X)_R$, where $d=\dim Y$. Assume that*

$$(\alpha_{i\,k(y)})_*\big(\operatorname{CH}_i(k(y)\times_k X)_R\big) \;=\; 0$$

for all $y\in Y$ and $0\le i\le \dim X$. Then

$$(\alpha_0\circ\alpha_1\circ\,\ldots\,\circ\alpha_{d-1}\circ\alpha_d)_*\big(\operatorname{CH}_j(Y\times_k X)_R\big) \;=\; 0$$

for all $0\le j\le \dim X+\dim Y$. In particular, if $X=Y$ we have

$$\alpha_0\circ\alpha_1\circ\,\ldots\,\circ\alpha_{d-1}\circ\alpha_d \;=\; 0\,.$$

Proof. Let $\pi : Y\times_k X \longrightarrow Y$ be the projection onto the first factor. Define for $0\le j\le \dim X+\dim Y$ a filtration on the Chow group $\operatorname{CH}_j(Y\times_k X)_R$ by setting $F_{-1}\operatorname{CH}_j(Y\times_k X)_R=0$ and letting $F_p\operatorname{CH}_j(Y\times_k X)_R$ be the subgroup generated by the classes of j-dimensional subvarieties V of $Y\times_k X$ with $\dim\pi(V)\le p$ for $0\le p\le d=\dim Y$.

Now Rost [13, Proof of Prop. 1], or Brosnan [1] for a more geometric argument, show that if $\alpha\in\operatorname{End}_k(X)_R$ satisfies

$$\alpha_{k(y)\,*}\big(\operatorname{CH}_i(k(y)\times_k X)_R\big) \;=\; 0$$

for all $y\in Y$ and all $0\le i\le\dim X$ then we have

$$\alpha_*\big(F_p\operatorname{CH}_j(Y\times_k X)_R\big) \;\subseteq\; F_{p-1}\operatorname{CH}_j(Y\times_k X)_R$$

for all $d\ge p\ge 0$. Hence the assumption on the α_i's implies that

$$\alpha_{0\,*}\Big(\alpha_{1\,*}\big(\,\ldots\,\alpha_{d\,*}(CH_j(Y\times_k X)_R)\,\ldots\,\big)\Big) \;\subseteq\; F_{-1}\operatorname{CH}_j(Y\times_k X)_R \;=\; 0$$

as claimed. ∎

The stronger version of Rost nilpotence is 'additive'. More precisely, we have the following fact.

2.4. Theorem. *Let $M_1,\ldots,M_d$ be motives in $\mathfrak{Chow}(k,R)$ satisfying strong Rost nilpotence. Then the direct sum $\bigoplus_{i=1}^{d} M_i$ satisfies strong Rost nilpotence.*

Proof. By induction it is enough to show this for $d=2$. Let for this $M,N\in\mathfrak{Chow}(k,R)$ satisfy strong Rost nilpotence, and $E\supseteq k$ a field extension. We set for brevity of notation $\mathcal{I}_M:=\mathcal{I}^R_{E/k}(M)$ and $\mathcal{I}_N:=\mathcal{I}^R_{E/k}(N)$. Let further B and C be the kernels of $\operatorname{res}_{E/k} : \operatorname{Hom}_k(N,M)_R \longrightarrow \operatorname{Hom}_E(N_E,M_E)_R$ and of $\operatorname{res}_{E/k} : \operatorname{Hom}_k(M,N)_R \longrightarrow \operatorname{Hom}_E(M_E,N_E)_R$, respectively.

Identifying

$$\operatorname{End}_k(M\oplus N)_R \simeq \begin{pmatrix} \operatorname{End}_k(M)_R & \operatorname{Hom}_k(N,M)_R \\ \operatorname{Hom}_k(M,N)_R & \operatorname{End}_k(N)_R \end{pmatrix}$$

we have

$$\mathcal{I}^R_{E/k}(M \oplus N) = \begin{pmatrix} \mathcal{I}_M & B \\ C & \mathcal{I}_N \end{pmatrix} \subseteq \begin{pmatrix} \mathrm{End}_k(M)_R & \mathrm{Hom}_k(N,M)_R \\ \mathrm{Hom}_k(M,N)_R & \mathrm{End}_k(N)_R \end{pmatrix},$$

and so the following assertion proves the theorem.

Claim. *We have*

$$\begin{pmatrix} \mathcal{I}_M & B \\ C & \mathcal{I}_N \end{pmatrix}^{2^l} \subseteq \begin{pmatrix} \mathcal{I}^l_M + B\mathcal{I}^l_N C & \mathcal{I}^l_M B + B\mathcal{I}^l_N + \mathcal{I}^{l-1}_M B\mathcal{I}^{l-1}_N \\ C\mathcal{I}^l_M + \mathcal{I}^l_N C + \mathcal{I}^{l-1}_N C\mathcal{I}^{l-1}_M & C\mathcal{I}^l_M B + \mathcal{I}^l_N \end{pmatrix}$$

for all $l \geq 2$.

We prove this by induction. Let first $l = 1$. Then

$$\begin{pmatrix} \mathcal{I}_M & B \\ C & \mathcal{I}_N \end{pmatrix} \cdot \begin{pmatrix} \mathcal{I}_M & B \\ C & \mathcal{I}_N \end{pmatrix} = \begin{pmatrix} \mathcal{I}^2_M + BC & \mathcal{I}_M B + B\mathcal{I}_N \\ C\mathcal{I}_M + \mathcal{I}_N C & CB + \mathcal{I}^2_N \end{pmatrix},$$

and so we get

$$\begin{pmatrix} \mathcal{I}_M & B \\ C & \mathcal{I}_N \end{pmatrix}^4 = \begin{pmatrix} \mathcal{I}^2_M + BC & \mathcal{I}_M B + B\mathcal{I}_N \\ C\mathcal{I}_M + \mathcal{I}_N C & CB + \mathcal{I}^2_N \end{pmatrix}^2$$

$$\subseteq \begin{pmatrix} \mathcal{I}^2_M + B\mathcal{I}^2_N C & \mathcal{I}^2_M B + B\mathcal{I}^2_N + \mathcal{I}_M B\mathcal{I}_N \\ C\mathcal{I}^2_M + \mathcal{I}^2_N C + \mathcal{I}_N C\mathcal{I}_M & C\mathcal{I}^2_M B + \mathcal{I}^2_N \end{pmatrix}.$$

Let now $l \geq 2$. Then by the induction assumption we have

$$\begin{pmatrix} \mathcal{I}_M & B \\ C & \mathcal{I}_N \end{pmatrix}^{2^{l+1}}$$

$$\subseteq \begin{pmatrix} \mathcal{I}^l_M + B\mathcal{I}^l_N C & \mathcal{I}^l_M B + B\mathcal{I}^l_N + \mathcal{I}^{l-1}_M B\mathcal{I}^{l-1}_N \\ C\mathcal{I}^l_M + \mathcal{I}^l_N C + \mathcal{I}^{l-1}_N C\mathcal{I}^{l-1}_M & C\mathcal{I}^l_M B + \mathcal{I}^l_N \end{pmatrix}^2,$$

and a short computation, where we use that $B\mathcal{I}^r_N C \subseteq \mathcal{I}_M$ and $C\mathcal{I}^r_M B \subseteq \mathcal{I}_N$ for all integers $r \geq 0$, as well as that $\mathcal{I}_N C, C\mathcal{I}_M \subseteq C$ and $\mathcal{I}_M B, B\mathcal{I}_N \subseteq B$, shows that the latter is contained in

$$\begin{pmatrix} \mathcal{I}^{l+1}_M + B\mathcal{I}^{l+1}_N C & \mathcal{I}^{l+1}_M B + B\mathcal{I}^{l+1}_N + \mathcal{I}^l_M B\mathcal{I}^l_N \\ C\mathcal{I}^{l+1}_M + \mathcal{I}^{l+1}_N C + \mathcal{I}^l_N C\mathcal{I}^l_M & C\mathcal{I}^{l+1}_M B + \mathcal{I}^{l+1}_N \end{pmatrix}.$$

□

By the blow-up formula the theorem implies:

2.5. Corollary. *Let Y be a smooth closed subscheme of $X \in \mathrm{PSm}_k$ of pure codimension r, and Z the blow-up of X along Y. Then in $\mathfrak{Chow}(k,R)$ strong Rost nilpotence holds for Z if and only if it holds for X and Y.*

Proof. By the blow-up formula, see Manin [11, §9, Cor.] or [6, Chap. 6], we have

$$Z \simeq X \oplus \bigoplus_{i=1}^{r-1} Y(i)$$

in $\mathfrak{Chow}(k,R)$ and so the corollary follows from Theorem 2.4 above. □

2.6. Remark. If Köthe's conjecture is true (for all rings) 'usual' Rost nilpotence is additive as well. In fact, using the notation of the proof of Theorem 2.4 above we have

$$\mathcal{I}^R_{E/k}(M\oplus N) = \begin{pmatrix} \mathcal{I}_M & 0 \\ C & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & \mathcal{I}_N \end{pmatrix}.$$

Both left ideals $\begin{pmatrix} \mathcal{I}_M & 0 \\ C & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & B \\ 0 & \mathcal{I}_N \end{pmatrix}$ are nil and so by (an equivalent form of) Köthe's conjecture, see *e.g.* [14, 2.6.35 (ii)] their sum $\mathcal{I}^R_{E/k}(M\oplus N)$ is nil as well.

2.7. Assume now that R is a noetherian ring. In this case a weaker version of Rost nilpotence, namely Rost nilpotence with bounded exponent, is enough for additivity.

Let M,N be motives in $\mathfrak{Chow}(k,R)$, which satisfy Rost nilpotence with bounded exponent, and $E\supseteq k$ a field extension. Let further $r=r(E)$ and $s=s(E)$ be integers ≥ 1, such that $x^{r+1}=0$ and $y^{s+1}=0$ for all $x\in\mathcal{I}^R_{E/k}(M)$ and $y\in\mathcal{I}^R_{E/k}(N)$, respectively. In particular, this means that the R-algebras (without 1) $\mathcal{I}^R_{E/k}(M)$ and $\mathcal{I}^R_{E/k}(N)$ are PI-algebras.

We aim to show that then $\mathcal{I}^R_{E/k}(M\oplus N)$ is also nil ideal with bounded nilpotence index, *i.e.* there is $t\geq 1$, such that $\alpha^t=0$ for all α in $\mathcal{I}^R_{E/k}(M\oplus N)$.

Given $\alpha\in\mathcal{I}^R_{E/k}(M\oplus N)$ we can write

$$\alpha = \begin{pmatrix} x & b \\ c & y \end{pmatrix} \tag{1}$$

with $x\in\mathcal{I}^R_{E/k}(M)$, $y\in\mathcal{I}^R_{E/k}(N)$, and $b\in\mathrm{Hom}_k(N,M)_R$ and $c\in\mathrm{Hom}_k(M,N)_R$ satisfying $b_E=c_E=0$.

We let $\mathcal{I}_M$ be the $\mathbb{Z}$-subalgebra (without 1!) of $\mathcal{I}^R_{E/k}(M)$ generated by the elements $x,x^2,\ldots,x^r$, $bc,byc,by^2c,\ldots,by^sc$ and $\mathcal{I}_N$ be the $\mathbb{Z}$-subalgebra (without 1!) of $\mathcal{I}^R_{E/k}(N)$ generated by the elements $y,y^2,\ldots,y^s$, $cb,cxb,cx^2b,\ldots,cx^rb$. Both $\mathcal{I}_M$ and $\mathcal{I}_N$ are finitely generated sub algebras of the PI-algebras $\mathcal{I}^R_{E/k}(M)$ and $\mathcal{I}^R_{E/k}(N)$, respectively, and therefore by [12, Chap. VI, Thm. 2.13] there exist integers $u,v\geq 0$, such that $\mathcal{I}_M^u=0$ and $\mathcal{I}_N^v=0$ (here we use that R is a noetherian ring). The integers u and v do not depend on x,y,b, and c but only on (a) the number of generators of $\mathcal{I}_M$ and $\mathcal{I}_N$, respectively, (b) the identities of $\mathcal{I}^R_{E/k}(M)$ and $\mathcal{I}^R_{E/k}(N)$, respectively, as well as (c) the nilpotence exponents r and s, respectively. (Hence in the end on r and s only.)

We define now two abelian groups. Let for this b, c be as in (1). The first abelian group, denoted B, is the subgroup of $\mathrm{Hom}_k(N,M)_R$ generated by the elements b, fb, bg, and fbg, where $f \in \mathcal{I}_M$ and $g \in \mathcal{I}_N$, and the second one, denoted C, is the subgroup of $\mathrm{Hom}_k(M,N)_R$ generated by the elements c, cf, gc, and gcf, where (again) $f \in \mathcal{I}_M$ and $g \in \mathcal{I}_N$.

2.8. Sublemma. *We have $BC \subseteq \mathcal{I}_M$, $CB \in \mathcal{I}_N$, and for $h \geq 1$ an integer:*

(i) $C\mathcal{I}_M^h B \subseteq C\mathcal{I}_M B \subseteq \mathcal{I}_N$*; and*
(ii) $B\mathcal{I}_N^h C \subseteq B\mathcal{I}_N C \subseteq \mathcal{I}_M$.

Proof. We observe first that an element of $\mathcal{I}_M$ is a sum of elements of the form

$$x^{i_1} \cdot (\prod_{l=1}^{m_1} by^{i_{1l}}c) \cdot x^{i_2} \cdot (\prod_{l=1}^{m_2} by^{i_{2l}}c) \cdot \ldots \cdot (\prod_{l=1}^{m_m} by^{i_{ml}}c) \cdot x^{i_{m+1}}, \tag{2}$$

where $m, m_l \geq 1$ are integers and i_l and i_{hl} are integers ≥ 0. Analogous an element of $\mathcal{I}_N$ is a sum of elements of the form

$$y^{j_1} \cdot (\prod_{l=1}^{n_1} cx^{j_{1l}}b) \cdot y^{j_2} \cdot (\prod_{l=1}^{n_2} cx^{j_{2l}}b) \cdot \ldots \cdot (\prod_{l=1}^{n_n} cx^{j_{nl}}b) \cdot y^{i_{n+1}}, \tag{3}$$

where $n, n_l \geq 1$ are integers and j_l and j_{hl} are integers ≥ 0.

If now $a \in \mathcal{I}_M$ is of the form (2) then $c \cdot a \cdot b$ is an element of the form (3) and so in $\mathcal{I}_N$, and analogous if $d \in \mathcal{I}_N$ is an element of the form (3) then $b \cdot d \cdot c$ is of the form (2) and so in $\mathcal{I}_M$. From these two observations the sublemma follows. □

We come back to the element α in $\mathcal{I}_{E/k}^R(M \oplus N)$, see (1). We have

$$\alpha^2 = \begin{pmatrix} x & b \\ c & y \end{pmatrix}^2 = \begin{pmatrix} x^2+bc & xb+by \\ cx+yc & cb+y^2 \end{pmatrix},$$

from which we conclude that

$$\alpha^4 \in \begin{pmatrix} \mathcal{I}_M^2 + B\mathcal{I}_N^2 C & \mathcal{I}_M^2 B + B\mathcal{I}_N^2 + \mathcal{I}_M B \mathcal{I}_N \\ C\mathcal{I}_M^2 + \mathcal{I}_N^2 C + \mathcal{I}_N C \mathcal{I}_M & C\mathcal{I}_M^2 B + \mathcal{I}_N^2 \end{pmatrix}.$$

As in the proof of Theorem 2.4 it follows now by induction on $l \geq 2$ that

$$\alpha^{2^l} \in \begin{pmatrix} \mathcal{I}_M^l + B\mathcal{I}_N^l C & \mathcal{I}_M^l B + B\mathcal{I}_N^l + \mathcal{I}_M^{l-1} B \mathcal{I}_N^{l-1} \\ C\mathcal{I}_M^l + \mathcal{I}_N^l C + \mathcal{I}_N^{l-1} C \mathcal{I}_M^{l-1} & C\mathcal{I}_M^l B + \mathcal{I}_N^l \end{pmatrix},$$

and therefore if $l \geq 1 + \max\{u, v\}$ we have $\alpha^{2^l} = 0$. By another induction on the number of summands this gives the next result.

2.9. Theorem. *Let R be a noetherian ring and $M_1, \ldots, M_d$ be motives in the category $\mathfrak{Chow}(k,R)$ satisfying Rost nilpotence with bounded exponent. Then Rost nilpotence with bounded exponent holds also for the direct sum $\bigoplus_{i=1}^{d} M_i$.*

2.10. Remarks. Let k and R be as above.

(a) If M is geometrically split then $\mathcal{I}_{E/k}^R(M)$ is nil, nil with bounded exponent, respectively nilpotent for all field extensions $E \supseteq k$ if and only if $\mathcal{I}_{k/k}^R(M)$ is so.

(b) The endomorphism ring over the algebraic closure of a geometrically split motive M in $\mathfrak{Chow}(k,R)$ is a product of matrix rings over the coefficient ring and so a PI-algebra. Hence if a geometrically split motive satisfies Rost nilpotence with bounded exponent its endomorphism ring is a PI-algebra over R, which is moreover integral of bounded degree over R. In particular, the endomorphism ring $\mathrm{End}_k(M)_R$ is locally finite if R is noetherian by [12, Chap. VI, Cor. 2.8 (1)].

We close this section with a technical lemma needed in the next one.

2.11. Lemma. *Let R be a noetherian ring. Assume that strong Rost nilpotence holds for the geometrically split motive M in $\mathfrak{Chow}(k)$, and that Rost nilpotence holds for M in $\mathfrak{Chow}(k,R)$. Then strong Rost nilpotence holds for M in $\mathfrak{Chow}(k,R)$.*

Proof. By Remarks 2.10 (a) above it is enough to show that $\mathcal{I}^R_{\bar{k}/k}(M)$ is a nilpotent ideal.

By assumption we have $\mathcal{I}^{\mathbb{Z}}_{\bar{k}/k}(M)^t = 0$ for some $t \geq 1$ and there is a short exact sequence

$$0 \longrightarrow \mathcal{I}^{\mathbb{Z}}_{\bar{k}/k}(M) \longrightarrow \mathrm{End}_k(M) \xrightarrow{\iota} \overline{\mathrm{End}_k(M)} \longrightarrow 0\,,$$

where $\overline{\mathrm{End}_k(M)} \subseteq \mathrm{End}_{\bar{k}}(M_{\bar{k}})$ denotes the image of $\mathrm{res}_{\bar{k}/k} : \mathrm{End}_k(M) \longrightarrow \mathrm{End}_{\bar{k}}(M_{\bar{k}})$. Tensoring above exact sequence with R we get an exact sequence

$$R \otimes_{\mathbb{Z}} \mathcal{I}^{\mathbb{Z}}_{\bar{k}/k}(M) \longrightarrow R \otimes_{\mathbb{Z}} \mathrm{End}_k(M) \xrightarrow{\mathrm{id}_R \otimes \iota} R \otimes_{\mathbb{Z}} \overline{\mathrm{End}_k(M)} \longrightarrow 0\,, \qquad (4)$$

and consequently the kernel of $\mathrm{id}_R \otimes \iota$ is a nilpotent ideal. We are reduced to prove that the kernel of

$$\mathrm{id}_R \otimes j \;:\; R \otimes_{\mathbb{Z}} \overline{\mathrm{End}_k(M)} \longrightarrow R \otimes_{\mathbb{Z}} \mathrm{End}_{\bar{k}}(M_{\bar{k}})$$

is a nilpotent ideal, where j denotes the inclusion $\overline{\mathrm{End}_k(M)} \hookrightarrow \mathrm{End}_{\bar{k}}(M_{\bar{k}})$.

Since M is geometrically split the ring $\mathrm{End}_{\bar{k}}(M_{\bar{k}})$ is a free abelian group of finite rank and so the subgroup $\overline{\mathrm{End}_k(M)}$ is free abelian of finite rank as well. Hence $R \otimes_{\mathbb{Z}} \overline{\mathrm{End}_k(M)}$ is a free R-module of finite rank and therefore a noetherian R-algebra. Hence by Levitzki's theorem, see *e.g.* [14, Thm. 2.6.23], it is enough to show that the kernel of $\mathrm{id}_R \otimes j$ is a nil ideal. To see this let $\alpha \in \mathrm{Ker}(\mathrm{id}_R \otimes j)$. We have by the exact sequence (4) that $\alpha = (\mathrm{id}_R \otimes \iota)(\beta)$ for some β in the endomorphism ring $\mathrm{End}_k(M)_R = R \otimes_{\mathbb{Z}} \mathrm{End}_k(M)$.

But β is in the kernel of the restriction homomorphism

$$R \otimes_{\mathbb{Z}} \mathrm{End}_k(M) \longrightarrow R \otimes_{\mathbb{Z}} \mathrm{End}_{\bar{k}}(M_{\bar{k}})\,,$$

and so nilpotent by our assumption. It follows that α is nilpotent as well. □

3. Examples

3.1. Projective homogenous varieties. Let X be a projective homogenous variety over k. In [3], see also Brosnan [2], it is shown that Rost nilpotence with bounded exponent holds for the motive of X in $\mathfrak{Chow}(k,R)$ (any coefficient ring R). But the argument there shows that in fact strong Rost nilpotence holds for X. For the sake of completeness we recall the details.

Note that since the motive of X is geometrically split it is enough to show that the ideal $\mathcal{I}^R_{\bar{k}/k}(X)$ is nilpotent, see Remark 2.10 (i).

By the main result of *loc.cit.* we have

$$X \simeq \bigoplus_{i=1}^{l} Y_i(n_i) \tag{5}$$

in $\mathfrak{Chow}(k,R)$ for anisotropic projective homogeneous varieties Y_i of dimension strictly less than $\dim X$ if $l \geq 2$, and integers $n_i \geq 0$. We claim by descending induction on d that $\mathcal{I}^R_{\bar{k}/k}(X)$ is a nilpotent ideal with nilpotence exponent only depending on $\dim X$ and the number of summands in (5).

If l is maximal then X is split and so a sum of twists of Tate motives. Otherwise let $y \in Y_i$ for some $1 \leq i \leq l$ in above decomposition (5). Then $Y_{i,k(y)}$ is isotropic and therefore splits by the main theorem of *loc.cit.* into a direct sum of at least two twists of motives of projective homogeneous varieties. The induction assumption gives then

$$\mathcal{I}^R_{\overline{k(y)}/k(y)}(X_{k(y)})^{\,t} \;=\; 0\,,$$

where $\overline{k(y)}$ is an algebraic closure of the residue field $k(y)$, for some integer $t \geq 0$ which depends only on $\dim X$ and the number of summands in the motivic decomposition of $X_{k(y)}$ into twists of motives of projective homogeneous varieties. In particular, we can find $t_i > 0$ which works for all $y \in Y_i$.

Then Rost's Lemma 2.3 implies

$$\mathcal{I}^R_{\bar{k}/k}(X)_*^{t_i\cdot(1+\dim Y_i)}\big(\,\mathrm{CH}_j(Y_i \times X)_R\big) \;=\; 0$$

for all $0 \leq j \leq \dim X + \dim Y_i$.

Setting now $\ell := \max\limits_{1\leq i\leq d} \big[t_i \cdot (1+\dim Y_i)\big]$ we get $\mathcal{I}^R_{\bar{k}/k}(X)^{\ell} = 0$ by (5).

3.2. Geometrically rational surfaces. Let S be a geometrically rational surface over k. We claim that strong Rost nilpotence holds for S in $\mathfrak{Chow}(k,R)$ for R either the ring of integers $\mathbb{Z}$, or integers modulo $m \geq 2$, denoted $\mathbb{Z}/m$. Note that the motive of S is geometrically split.

We assume first that $R = \mathbb{Z}$. By the Hochschild-Serre spectral sequence we know that $\mathrm{Pic}(S_{k(s)}) \simeq \mathrm{CH}_1(S_{k(s)})$ is torsion free for all $s \in S$ and the same holds for $\mathrm{CH}_2(S_{k(s)})$ since S is geometrically integral. Hence if α is in $\mathcal{I}^{\mathbb{Z}}_{\bar{k}/k}(S)$ then $\alpha_{k(s)}$ acts trivial on both $\mathrm{CH}_1(S_{k(s)})$ and $\mathrm{CH}_2(S_{k(s)})$ for all $s \in S$. Hence to apply Rost's lemma we are left to study the action of $\alpha_{k(s)}$ on $\mathrm{CH}_0(S_{k(s)})$ for all $\alpha \in \mathcal{I}^{\mathbb{Z}}_{\bar{k}/k}(S)$ and all $s \in S$.

To this end we denote for a field extension $E \supseteq k$ by $\mathrm{A}_0(S_E)$ the kernel of the degree map $\deg : \mathrm{CH}_0(S_E) \longrightarrow \mathbb{Z}$. By the main result of Coombes [4] the surface S is rational over the separable closure k_{sep} of k and so its motive is split in $\mathfrak{Chow}(k_{\mathrm{sep}})$. Hence there exists for a field extension $E \supseteq k$ a Galois extension $L \supseteq E$, such that $(\alpha_E)_L = 0$. If $S(E) \neq \emptyset$ the subgroup $\mathrm{A}_0(S_E)$ is the torsion part of $\mathrm{CH}_0(S_E)$.

This implies that given $\alpha \in \mathcal{I}^{\mathbb{Z}}_{\bar{k}/k}(S)$ we have $\alpha_{k(s)\,*}\big(\,\mathrm{CH}_0(S_{k(s)})\big) \subseteq \mathrm{A}_0(S_{k(s)})$. and by [7, Cor. 4.9] (see [8, Sect. 2.4] if the base field is not perfect), we have $\alpha_{k(s)\,*}\big(\,\mathrm{A}_0(S_{k(s)})\big) = 0$.

Given now $\alpha, \beta \in \mathcal{I}^{\mathbb{Z}}_{\bar{k}/k}(S)$ then $(\alpha \circ \beta)_{k(s)\,*}\big(\mathrm{CH}_0(S_{k(s)})\big) = 0$ and so by Rost's Lemma 2.3 we have

$$\alpha_1 \circ \alpha_2 \circ \alpha_3 \circ \alpha_4 \circ \alpha_5 \circ \alpha_6 \;=\; 0$$

for all $\alpha_1, \ldots, \alpha_6 \in \mathcal{I}^{\mathbb{Z}}_{\bar{k}/k}(S)$. Hence $\mathcal{I}^{\mathbb{Z}}_{\bar{k}/k}(S)^6 = 0$.

Let now $R = \mathbb{Z}/m$ for some integer $m \geq 2$. In [8, Thm. 15] it is shown that Rost nilpotence holds for S in $\mathfrak{Chow}(k, \mathbb{Z}/m)$. Hence by Lemma 2.11 above we get that strong Rost nilpotence holds for S in $\mathfrak{Chow}(k, \mathbb{Z}/m)$.

3.3. Surfaces. Assume that $\mathrm{char}\, k = 0$ and $R = \mathbb{Z}$. Let S be a k-surface. We claim that strong Rost nilpotence holds for S in $\mathfrak{Chow}(k)$. Again the proof is only a slight modification of the verification of Rost nilpotence for surfaces in [8, Sect. 2.5]. We briefly sketch the details.

Let for this $E \supseteq k$ a field extension and $\alpha_i \in \mathcal{I}_{E/k}(S)$, $i = 1, 2, 3$. We have a tower of fields $E \supseteq F \supseteq k$ with F a purely transcendental extension of k and E an algebraic extension of F. Since $\mathrm{res}_{E/k} : \mathrm{End}_k(S) \longrightarrow \mathrm{End}_F(S_F)$ is an isomorphism, see *e.g.* [5, Prop. 2.1.8], replacing k by F we can assume that $E \supseteq k$ is algebraic.

Let $s \in S$. Since $\alpha_{i\, E(s)} = 0$ and $\mathrm{char}\, k = 0$ there is a Galois extension $L \supseteq k(s)$, such that $\alpha_{i\, L} = 0$ for all $1 \leq i \leq 3$. It follows from [8, Thm. 4] that we have then

$$(\alpha_1 \circ \alpha_2 \circ \alpha_3)_{k(s)} \;=\; 0\,.$$

Taking into account Rost's Lemma 2.3 this implies

$$\alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_8 \circ \alpha_9 \;=\; 0$$

for all $\alpha_1, \ldots, \alpha_9 \in \mathcal{I}_{E/k}(S)$, and so $\mathcal{I}_{E/k}(S)^9 = 0$.

3.4. Threefolds. We continue assuming $R = \mathbb{Z}$.

Let X be a threefold over k, *i.e.* a smooth projective and integral k-scheme of dimension three, where k is a field of characteristic 0. The arguments in [9] show that strong Rost nilpotence holds for X in $\mathfrak{Chow}(k)$ if and only if some power of the ideal $\mathcal{I}_{E/k}(X)$ acts trivial on $\mathrm{CH}_0(X_{k(X)})$ for all field extensions $E \supseteq k$ We leave the details to the reader.

3.5. Varieties whose motives split generically. Let k be again an arbitrary field and X a smooth projective and integral k-scheme. We say that the motive of X is *generically split* in $\mathfrak{Chow}(k, R)$ if it is split in $\mathfrak{Chow}(k(X), R)$. Vishik and Zainoulline [16] have shown that for such schemes Rost nilpotence holds (any coefficient ring R). But their argument shows that actually strong Rost nilpotence holds in this case. In [16, Proof of Lem 3.2], instead of considering only one element ϕ in the kernel of $\mathrm{res}_{U/X}$ one takes d elements $\rho_1, \ldots, \rho_d$, and sets $\phi_i := \pi^*_{i,i+1}(\rho_i)$. Following then word by word the rest of the proof gives $\rho_1 \circ \ldots \circ \rho_d = 0$, and so $(\mathrm{Ker}(\mathrm{res}_{U/X}))^d = 0$. This implies in particular the following remarkable fact:

Theorem (Vishik-Zainoulline). *Let X be an integral scheme in PSm_k. Then $\mathcal{I}^R_{k(X)/k}(X)$ is a nilpotent ideal for all coefficient rings R.*

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E-mail address: `gille@ualberta.ca`

DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA, EDMONTON T6G 2G1, CANADA