

REALIZING THE LOCAL WEIL REPRESENTATION OVER A NUMBER FIELD

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ABSTRACT. We show that the Weil representation of the symplectic group $\mathrm{Sp}(2n, F)$, where F is a non-archimedean local field, can be realized over the field $K = \mathbf{Q}(\sqrt{p}, \sqrt{-p})$, where p is the residue characteristic of F .

1. INTRODUCTION

Our main result is that the Weil representation of the symplectic group $\mathrm{Sp}(2n, F)$, where F is a non-archimedean local field of residue characteristic $\neq 2$, can be realized over a number field K . We take an infinite-dimensional complex vector space \mathcal{V} such that the Weil representation is given by $\rho : \mathrm{Sp}(2n, F) \rightarrow \mathrm{PGL}(\mathcal{V})$ and we find a K -subspace \mathcal{V}_0 of \mathcal{V} such that $\rho(g)(\mathcal{V}_0) = \mathcal{V}_0$ for all $g \in \mathrm{Sp}(2n, F)$.

This answers a question raised by D. Prasad [P]. Indeed, we show that we can take $K = \mathbf{Q}(\sqrt{p}, \sqrt{-p})$ where p is the residue characteristic of F . We assume that p is odd. A consequence of this, also pointed out by Prasad, is that the local theta correspondence can be defined for representations which are realized over K .

Let W be the Weil representation of $\mathrm{Sp}(2n, F)$. The Weil representation can be defined using the Schrödinger representation of the Heisenberg group H . Let λ be a fixed complex character on the additive group of the field F . Suppose that F^{2n} is the direct sum $X \oplus Y$ of totally isotropic F -subspaces. The Schrödinger model is realized in the Bruhat-Schwartz space $\mathcal{S}(X)$ of locally constant functions $f : X \rightarrow \mathbf{C}$ of compact support. For $h \in H$, there are operators $S_\lambda(h)$ on $\mathcal{S}(X)$ such that $S_\lambda : H \rightarrow \mathrm{GL}(\mathcal{S}(X))$ is the unique smooth irreducible representation of H with central character λ . The natural action of the symplectic groups extends to an action on H , and the Weil representation is given by operators $W_\lambda(g)$ on $\mathcal{S}(X)$, $g \in \mathrm{Sp}(2n, F)$, such that

$$W_\lambda(g)^{-1} S_\lambda(h) W_\lambda(g) = S_\lambda(hg) \quad h \in H, g \in \mathrm{Sp}(2n, F).$$

Let $\mathbf{Q}(\lambda)$ be the field obtained by adjoining all the character values of λ to \mathbf{Q} , and let $E = \mathbf{Q}(\lambda)(\sqrt{-1})$. In the case that F has characteristic 0, E is the field obtained from \mathbf{Q} by adjoining $\sqrt{-1}$ and all p -power roots of unity. For a subfield L of \mathbf{C} , define $\mathcal{S}(X, L)$ to be the space of locally constant functions on X of compact support having values in L . We show that there is an explicit choice of Weil operators $W_\lambda(g)$ on $\mathcal{S}(X)$ which leave $\mathcal{S}(X, E)$ invariant.

The Galois group of E over \mathbf{Q} acts on $\mathcal{S}(X, E)$ and on $\mathrm{End}(\mathcal{S}(X, E))$. In Section 7 we define a 1-cocycle δ on $\mathrm{Gal}(E/\mathbf{Q}(\sqrt{p}, \sqrt{-p}))$ with values in $\mathrm{GL}(\mathcal{S}(X, E))$ such that

$$(I) \quad {}^\sigma W_\lambda(g) = \delta(\sigma)^{-1} W_\lambda(g) \delta(\sigma), \quad g \in \mathrm{Sp}(V).$$

Using Galois descent, we show that there exists $\alpha \in \mathrm{GL}(\mathcal{S}(X, E))$ such that $\delta(\sigma) = \alpha^{-1} \sigma \alpha$ for $\sigma \in \mathrm{Gal}(E/\mathbf{Q}(\sqrt{p}, \sqrt{-p}))$.

Main Theorem. *The operators $\alpha W_\lambda(g)\alpha^{-1}$ leave $\mathcal{S}(X, \mathbf{Q}(\sqrt{p}, \sqrt{-p}))$ invariant, and provide a form of the Weil representation realized over $\mathbf{Q}(\sqrt{p}, \sqrt{-p})$.*

To indicate how we find the 1-cocycle satisfying (I), for the rest of the introduction we assume that F has characteristic 0. The Galois group of $\mathbf{Q}(\lambda)/\mathbf{Q}$ is isomorphic to the units \mathbf{Z}_p^* of the p -adic integers. For an element s of \mathbf{Z}_p^* , we let σ_s denote the corresponding element of $\text{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$. For an element $t \in F^*$, we define the character $\lambda[t]$ of F by $\lambda[t](r) = \lambda(tr)$, $r \in F$.

For $t \in F^*$, let $g_t \in \text{Sp}(2n, F)$ be defined by $(x + y)g_t = t^{-1}x + ty$, and $f_t \in \text{GL}(2n, F)$ by $(x + y)f_t = x + ty$, where $x \in X$, $y \in Y$. Then f_t is not in general in $\text{Sp}(2n, F)$, but conjugation by f_t leaves $\text{Sp}(2n, F)$ invariant. We have

$$(II) \quad W_\lambda(g^{f_t}) = W_{[\lambda(t)]}(g), \quad g \in \text{Sp}(V).$$

Furthermore, observing f_{t^2} is the composite $tI \circ g_t$, we show

$$(III) \quad W_\lambda(g^{f_{t^2}}) = W_\lambda(g_t)^{-1}W_\lambda(g)W_\lambda(g_t).$$

For $\sigma \in \text{Gal}(E/\mathbf{Q}(\sqrt{p}, \sqrt{-p}))$, $\sigma|_{\mathbf{Q}(\lambda)}$ is a square, so $\sigma|_{\mathbf{Q}(\lambda)} = \sigma_{\epsilon^{2i}s}$ where ϵ is a primitive $p-1$ root of unity, s is a principal unit of \mathcal{O} , and i is an integer, $1 \leq i \leq (p-1)/2$. We note

$$(IV) \quad {}^\sigma W_\lambda(g) = W_{\lambda[\epsilon^{2i}s^2]}(g).$$

In light of (II) and (III), we deduce

$${}^\sigma W_\lambda(g) = W_\lambda(g_{\epsilon^i s})^{-1}W_\lambda(g)W_\lambda(g_{\epsilon^i s}).$$

The last equation is used to show that $\delta(\sigma) = W_\lambda(g_{\epsilon^i s})$ satisfies (I) and almost satisfies the one-cocycle condition.

Equations (II), (III), and (IV) are proved using an integral formula for Weil operators due to Ranga Rao [RR]; see equation (3) in Section 3. This formula is also used to show that conjugation by the Weil operators $W_\lambda(g)$ leaves $\mathcal{S}(X, E)$ invariant.

2. PRELIMINARY REMARKS ON LOCAL FIELDS, CHARACTERS AND MEASURES

We fix some notation and recall some elementary facts about the characters of the additive group of a local field. Further details can be found in the first two chapters of [W].

Let F be a non-Archimedean local field, \mathcal{O} its ring of integers, and \mathfrak{m} the maximal ideal of \mathcal{O} . The order of the residue class field $\kappa = \mathcal{O}/\mathfrak{m}$ shall be denoted q ; we note that q is power of $p = \text{char } \kappa$. We assume throughout that p is different from 2; in particular, 2 is a unit of \mathcal{O} .

Given a fractional \mathcal{O} -ideal \mathfrak{a} , there exists a unique integer $v(\mathfrak{a})$, the *valuation of \mathfrak{a}* , such that

$$\mathfrak{a} = \mathfrak{m}^{v(\mathfrak{a})}.$$

If $s \in F$ is non-zero, the valuation of the ideal $s\mathcal{O}$ is referred to as the valuation of s , denoted $v(s)$. The absolute value on F is related to the valuation v on F by

$$|s| = q^{-v(s)}, \quad s \in F, s \neq 0.$$

Let λ be a non-trivial, continuous, complex linear (unitary) character of F^+ . The continuity of λ ensures that its kernel contains a fractional \mathcal{O} -ideal. The fact that λ is non-trivial allows one to deduce that the set of all such fractional \mathcal{O} -ideals has a unique maximal element $\mathfrak{i} = \mathfrak{i}_\lambda$, the *conductor of λ* . The *level of λ* is defined to be the valuation of \mathfrak{i}_λ .

Given $n \geq 1$, let

$$\nu_{p^n} = \{ z \in \mathbf{C} : z^{p^n} = 1 \}, \quad \nu_{p^\infty} = \bigcup_{n=1}^{\infty} \nu_{p^n}.$$

(The more customary symbol μ will be used to denote a measure.)

Lemma 1. *We have*

$$\text{im } \lambda = \begin{cases} \nu_p, & \text{if } \text{char } F = p; \\ \nu_{p^\infty}, & \text{if } \text{char } F = 0. \end{cases}$$

Proof. Take $x \in F$. If $\text{char } F = p$ then

$$1 = \lambda(0) = \lambda(px) = \lambda(x)^p.$$

This shows $\text{im } \lambda \subseteq \nu_p$. Equality follows from the fact $\text{im } \lambda$ is a non-trivial subgroup of the simple abelian group ν_p .

If $\text{char } F = 0$ then, since $p \in \mathfrak{m}$, there exists an $n \geq 0$ such that $p^n x \in \mathfrak{i}_\lambda$. For such n ,

$$1 = \lambda(p^n x) = \lambda(x)^{p^n}.$$

Then $\text{im } \lambda \subseteq \nu_{p^\infty}$. If the inclusion were proper then there would exist $m \geq 0$ such that $\text{im } \lambda = \nu_{p^m}$. In this case, if $x \in F$ then

$$\lambda(x) = \lambda\left(p^m \cdot \frac{x}{p^m}\right) = \lambda\left(\frac{x}{p^m}\right)^{p^m} = 1$$

since $\lambda(x/p^m)$ is a p^m -th root of unity. As this would contradict the non-triviality of λ , $\text{im } \lambda = \nu_{p^\infty}$. \square

Define $\mathbf{Q}(\lambda)$ to be the field obtained by adjoining to \mathbf{Q} all the character values $\lambda(x)$, $x \in F$. Define

$$\mathcal{P} \simeq \begin{cases} \mathbf{Z}/p\mathbf{Z}, & \text{if } \text{char } F = p; \\ \mathbf{Z}_p, & \text{if } \text{char } F = 0. \end{cases}$$

Note that \mathcal{P} is the topological closure of the prime ring of F .

Lemma 2. *There is a canonical topological isomorphism*

$$\text{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q}) \simeq \mathcal{P}^*.$$

Proof. The preceding lemma ensures that $\text{im } \lambda$ is invariant under the action of Galois, hence restriction yields a homomorphism

$$\text{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q}) \rightarrow \text{Aut}(\text{im } \lambda) \simeq \begin{cases} (\mathbf{Z}/p\mathbf{Z})^*, & \text{if } \text{char } F = p; \\ \mathbf{Z}_p^*, & \text{if } \text{char } F = 0. \end{cases}$$

It is readily checked that this map is an isomorphism of topological groups. The proof is completed by appealing to the description of \mathcal{P} given above. \square

The pairing

$$(s, t) \rightarrow \lambda(st), \quad s, t \in F,$$

is non-degenerate and leads to an identification of F^+ with its Pontryagin dual [W, II.5]. The image of $s \in F$ in the dual shall be denoted $\lambda[s]$:

$$\lambda[s](t) = \lambda(st), \quad t \in F.$$

Let $\mu = dt$ be a Haar measure on F^+ . If ϕ is a locally constant, complex valued function on F of compact support, the Fourier transform $\mathcal{F}_\lambda\phi$ is the complex valued function on F defined by

$$\mathcal{F}_\lambda\phi(s) = \int_F \lambda[s](t)\phi(t) dt, \quad s \in F.$$

It can be shown that $\mathcal{F}_\lambda\phi$ is locally constant and has compact support. Furthermore, the general theory of Fourier transforms asserts the existence of a positive constant c , depending only on the Haar measure dt , such that

$$(\mathcal{F}_\lambda\mathcal{F}_\lambda\phi)(t) = c\phi(-t), \quad t \in F.$$

There is a unique Haar measure on F^+ for which $c = 1$; it shall be denoted $d_\lambda t$ and will be referred to as the *self-dual Haar measure associated with λ* . [W, VII.2]

Lemma 3. *If λ has level l then the associated self-dual Haar measure is characterized by the condition*

$$(1) \quad \int_{\mathcal{O}} d_\lambda t = q^{l/2}.$$

Proof. This follows from [W, Corollary 3, VII.2]. □

Corollary. *If $s \in F^*$ then*

$$d_{\lambda[s]}t = |s|^{1/2}d_\lambda t.$$

Proof. Since $\mathfrak{i}_\lambda = s\mathfrak{i}_{\lambda[s]}$, the levels l_1 of λ and l_2 of $\lambda[s]$ satisfy the relation $l_1 = v(s) + l_2$. Therefore, Lemma 3 yields

$$\int_{\mathcal{O}} d_{\lambda[s]}t = q^{l_2/2} = q^{-v(s)/2}q^{l_1/2} = |s|^{1/2} \int_{\mathcal{O}} d_\lambda t.$$

This completes the proof of the corollary. □

3. THE SCHRÖDINGER AND WEIL REPRESENTATIONS

Let \langle , \rangle be a non-degenerate, alternating, F -bilinear form on a finite dimensional F -vector space V . The *Heisenberg group* H is the group on $V \times F$ having multiplication

$$(v, t)(v', t') = (v + v', t + t' + \langle v, v' \rangle / 2), \quad t, t' \in F, v, v' \in V.$$

Let λ be a non-trivial, continuous, complex linear character of F^+ . Since $Z(H) = 0 \times F \simeq F^+$, it may be viewed as a character of the center of the Heisenberg group H .

Stone-von Neumann Theorem. *There exists a smooth, irreducible representation of H having central character λ . Such a representation is necessarily admissible, and is unique up to isomorphism.*

A proof of the Stone-von Neumann Theorem can be found in [MVW, 2.I]. The representation provided by the Stone-von Neumann Theorem is referred to as *the Schrödinger representation of type λ* .

The symplectic group

$$\mathrm{Sp}(V) = \{ g \in \mathrm{GL}(V) : \langle vg, wg \rangle = \langle v, w \rangle, v, w \in V \}$$

acts on the Heisenberg group H as a group of automorphisms as follows: if $g \in \mathrm{Sp}(V)$ and $(t, v) \in H$ then

$$(t, v)g = (t, vg).$$

Given a Schrödinger representation S_λ of type λ and $g \in \mathrm{Sp}(V)$, consider the representation S_λ^g of H defined by

$$S_\lambda^g(h) = S_\lambda(hg), \quad h \in H.$$

It is readily verified that S_λ^g is a smooth, irreducible representation of H . Furthermore, observing that g acts trivially on $Z(H)$, S_λ^g has central character λ . The Stone-von Neumann Theorem allows us to conclude that the representation S_λ and S_λ^g are equivalent, hence the ambient space affording S_λ admits an operator $W_\lambda(g)$ for which

$$S_\lambda^g(h) = W_\lambda(g)^{-1} S_\lambda(h) W_\lambda(g), \quad h \in H$$

In light of Schur's Lemma, the operator $W_\lambda(g)$ is uniquely defined up to multiplication by a non-zero constant. As a result, the map

$$g \mapsto W_\lambda(g), \quad g \in \mathrm{Sp}(V),$$

is a projective representation of $\mathrm{Sp}(V)$, called a *Weil representation of type λ* .

In this paper we will consider the Schrödinger models of S_λ and W_λ ([K, Lemma 2.2, Proposition 2.3], [MVW, 2.I.4(a), 2.II.6], [RR, §3]). Let

$$V = X + Y$$

where X and Y are maximal, totally isotropic subspaces. The Schrödinger model is realized in the Bruhat-Schwartz space $\mathcal{S}(X)$ of locally constant functions $f : X \rightarrow \mathbf{C}$ of compact support: if $x \in X$, $y \in Y$ and $t \in F$ then $S_\lambda((x + y, t))$ is the operator defined by

$$[S_\lambda((x + y, t))\phi](x') = \lambda \left(t + \frac{\langle x, y \rangle}{2} + \langle x', y \rangle \right) \phi(x + x'), \quad \phi \in \mathcal{S}(X), x' \in X.$$

The description of the Weil representation requires some additional notation. Viewing $x + y \in V$ as a row vector (x, y) , each $g \in \mathrm{Sp}(V)$ can be expressed in the matrix form

$$(2) \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a : X \rightarrow X$, $b : X \rightarrow Y$, $c : Y \rightarrow X$, and $d : Y \rightarrow Y$. With this notation, set

$$Y_g = Y / \ker c.$$

If μ_g is a Haar measure on Y_g then the action of $W_\lambda(g)$ on $\mathcal{S}(X)$ is given by

$$(3) \quad [W_\lambda(g)\phi](x) = \int_{Y_g} \left[\lambda \left(\frac{\langle xa, xb \rangle - 2\langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \right] d\mu_g y, \quad \phi \in \mathcal{S}(X), \quad x \in X.$$

Note that the integral appearing in (3) is well-defined, for the integrand is constant on the cosets of $\ker c$, hence can be viewed as a function on Y_g . The fact $\phi \in \mathcal{S}(X)$ can be used to show that the integrand belongs to $\mathcal{S}(Y_g)$, hence the integral converges, and that the resulting function $W_\lambda(g)\phi$ belongs to $\mathcal{S}(X)$.

We now recall a particular choice of Haar measures $\mu_{\lambda, g}$ on Y_g , $g \in \mathrm{Sp}(V)$ [RR, §3.3]. Fix a basis x_1, \dots, x_n of X and let y_1, \dots, y_n be the dual basis of Y defined by the conditions

$$\langle x_i, y_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Let τ_i , $0 \leq i \leq n$, be the element of $\mathrm{Sp}(V)$ defined by

$$x_j \tau_i = \begin{cases} -y_j, & \text{if } j \leq i; \\ x_j, & \text{if } i < j. \end{cases} \quad \text{and} \quad y_j \tau_i = \begin{cases} x_j, & \text{if } j \leq i; \\ y_j, & \text{if } i < j. \end{cases}$$

We note that Y_{τ_i} can be identified with the subspace of Y spanned by the elements y_1, \dots, y_i . We define

$$(4) \quad d\mu_{\lambda, \tau_i} y = \prod_{k=1}^i d_\lambda y_k.$$

where $d_\lambda y_k$ is the self-dual Haar measure associated with λ .

Let

$$P = \{g \in \mathrm{Sp}(V) : Yg = g\},$$

the parabolic subgroup that leaves Y invariant. If $\dim Y_g = i$ then [RR, Theorem 2.14] ensures the existence of elements p_1 and p_2 of P such that

$$(5) \quad g = p_1 \tau_i p_2.$$

Observing that the operator p_1 induces an isomorphism $\bar{p}_1 : Y_g \rightarrow Y_{\tau_i}$, we set

$$(6) \quad \mu_{\lambda, g} = |\det(p_1 p_2|_Y)|^{-1/2} \bar{p}_1 \cdot \mu_{\lambda, \tau_i}.$$

Here, $\bar{p}_1 \cdot \mu_{\lambda, \tau_i}$ denotes the pullback of the Haar measure μ_{λ, τ_i} to Y_g via \bar{p}_1 : if E is a measurable subset of Y_g then

$$\bar{p}_1 \cdot \mu_{\lambda, \tau_i}(O) = \mu_{\lambda, \tau_i}(O\bar{p}_1).$$

Theorem 4. *The measures $\mu_{\lambda, g}$, $g \in \mathrm{Sp}(V)$, are well-defined. Furthermore, the projective representation W_λ of $\mathrm{Sp}(V)$ defined by (3) with the Haar measures $\mu_g = \mu_{\lambda, g}$ has the following properties.*

(i) *If $g \in \mathrm{Sp}(V)$ and $p_1, p_2 \in P$ then $W_\lambda(p_1 g p_2) = W_\lambda(p_1) W_\lambda(g) W_\lambda(p_2)$; in particular W_λ restricts to an ordinary representation of P .*

(ii) *If $\phi \in \mathcal{S}(X)$ and $p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in P$ then*

$$[W_\lambda(p)\phi](x) = |\det a|^{1/2} \lambda \left(\frac{\langle xa, xb \rangle}{2} \right) \phi(xa), \quad x \in X.$$

Proof. This follows from [RR, Theorem 3.5] □

Lemma 5. *If $s \in F^*$ and $g \in \mathrm{Sp}(V)$ then $\mu_{\lambda[s], g} = |s|_{Y_g}^{1/2} \mu_{\lambda, g}$.*

Proof. In light of the Corollary to Lemma 3, (4) yields

$$d\mu_{\lambda[s], \tau_i} y = \prod_{k=1}^i d_{\lambda[s]} y_k = \prod_{k=1}^i [|s|^{1/2} d_\lambda y_k] = |s|^{i/2} \left[\prod_{k=1}^i d_\lambda y_k \right] = |s|^{i/2} d\mu_{\lambda, \tau_i} y.$$

Therefore, (6) gives

$$\mu_{\lambda[s], g} = |\det(p_1 p_2|_Y)|^{-1/2} \bar{p}_1 \cdot \mu_{\lambda[s], \tau_i} = |s|^{i/2} |\det(p_1 p_2|_Y)|^{-1/2} \bar{p}_1 \cdot \mu_{\lambda, \tau_i} = |s|^{i/2} \mu_{\lambda, g} = |s|_{Y_g}^{1/2} \mu_{\lambda, g},$$

since Y_g has dimension i over F . □

4. ACTION OF SYMPLECTIC SIMILITUDES

Given $s \in F^*$, let f_s be the element of $\mathrm{GL}(V)$ defined by

$$(x + y)f_s = x + sy, \quad x \in X, y \in Y.$$

Conjugation by f_s leaves the symplectic group $\mathrm{Sp}(V)$ invariant. In fact, if $g \in \mathrm{Sp}(V)$ is expressed in the matrix form (2) then

$$(7) \quad g^{f_s} = \begin{pmatrix} a & sb \\ s^{-1}c & d \end{pmatrix}.$$

In particular, we note that the spaces Y_g and $Y_{g^{f_s}}$ are equal, since $\ker c = \ker s^{-1}c$.

Lemma 6. *If $s \in F^*$ then $\mu_{\lambda, g^{f_s}} = |s|_{Y_g}^{-1/2} \mu_{\lambda, g}$.*

Proof. Let $p_{i,s}$, $0 \leq i \leq n$, be the elements of $\mathrm{Sp}(V)$ defined by

$$x_j p_{i,s} = \begin{cases} s^{-1}x_j, & \text{if } j \leq i; \\ x_j, & \text{if } i < j. \end{cases} \quad \text{and} \quad y_j p_{i,s} = \begin{cases} sy_j, & \text{if } j \leq i; \\ y_j, & \text{if } i < j. \end{cases}$$

Note that $p_{i,s} \in P$ and

$$\det(p_{i,s}|_Y) = s^i.$$

Moreover, one readily verifies that

$$\tau_i^{f_s} = \tau_i p_{i,s}.$$

Let $g \in G$. If $g = p_1 \tau_i p_2$, $p_1, p_2 \in P$, then

$$g^{f_s} = (p_1 \tau_i p_2)^{f_s} = p_1^{f_s} \tau_i^{f_s} p_2^{f_s} = p_1^{f_s} \tau_i (p_{i,s} p_2^{f_s}).$$

Observing that both $p_1^{f_s}$ and $p_{i,s} p_2^{f_s}$ belong to P , (6) yields

$$\mu_{\lambda, g^{f_s}} = |\det(p_1^{f_s} p_{i,s} p_2^{f_s}|_Y)|^{-1/2} \overline{p_1^{f_s}} \cdot \mu_{\lambda, \tau_i}.$$

Using (7), if $p \in P$ then $p^{f_s}|_Y = p|_Y$. As a consequence,

$$\overline{p_1^{f_s}} = \overline{p_1} : Y_g \rightarrow Y_{\tau_i}.$$

In light of these observations,

$$\det(p_1^{f_s} p_{i,s} p_2^{f_s}|_Y) = \det(p_1 p_{i,s} p_2|_Y) = \det(p_{i,s}|_Y) \cdot \det(p_1 p_2|_Y) = s^i \det(p_1 p_2|_Y),$$

hence

$$\mu_{\lambda, g^{f_s}} = |s^i \det(p_1 p_2|_Y)|^{-1/2} \overline{p_1} \cdot \mu_{\lambda, \tau_i} = |s|^{-i/2} \mu_{\lambda, g} = |s|_{Y_g}^{-1/2} \mu_{\lambda, g},$$

since Y_g has dimension i over F . □

Let $W_\lambda^{f_s}$ be the projective representation of $\mathrm{Sp}(V)$ defined by

$$W_\lambda^{f_s}(g) = W_\lambda(g^{f_s}).$$

For the proof of the next result, let $|\alpha|_V$ denote the module of an automorphism α of an F -vector space V [W, I.2]. We have

$$|\alpha|_V = |\det \alpha|.$$

In particular, the module of left multiplication by $s \in F^*$ on V satisfies

$$|s|_V = |s|^{\dim V}.$$

Proposition 7. *If $s \in F^*$ then $W_\lambda^{f_s} = W_{\lambda[s]}$.*

Proof. Let $g \in \mathrm{Sp}(V)$. We assume that g has the matrix representation (2), hence that of g^{fs} is given by (7). If $\phi \in \mathcal{S}(X)$ and $x \in X$ then the integral formula (3) and Lemma 6 yield

$$\begin{aligned} [W_\lambda(g^{fs})\phi](x) &= \int_{Y_{g^{fs}}} \left[\lambda \left(\frac{\langle xa, sxb \rangle - 2 \langle sxb, s^{-1}yc \rangle + \langle s^{-1}yc, yd \rangle}{2} \right) \phi(xa + s^{-1}yc) \right] d\mu_{\lambda, g^{fs}y} \\ &= |s|_{Y_g}^{-1/2} \int_{Y_g} \left[\lambda \left(\frac{\langle xa, sxb \rangle - 2 \langle sxb, s^{-1}yc \rangle + \langle s^{-1}yc, yd \rangle}{2} \right) \phi(xa + s^{-1}yc) \right] d\mu_{\lambda, gy}. \end{aligned}$$

Replacing y by sy , the definition of $|s|_{Y_g}$ and Lemma 4 yield

$$\begin{aligned} [W_\lambda(g^{fs})\phi](x) &= |s|_{Y_g}^{-1/2} |s|_{Y_g} \int_{Y_g} \left[\lambda \left(\frac{\langle xa, sxb \rangle - 2 \langle sxb, yc \rangle + \langle yc, syd \rangle}{2} \right) \phi(xa + yc) \right] d\mu_{\lambda gy} \\ &= |s|_{Y_g}^{1/2} \int_{Y_g} \left[\lambda \left(s \cdot \frac{\langle xa, xb \rangle - 2 \langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \right] d\mu_{\lambda, gy} \\ &= |s|_{Y_g}^{1/2} \int_{Y_g} \left[\lambda[s] \left(\frac{\langle xa, xb \rangle - 2 \langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \right] d\mu_{\lambda, gy} \\ &= \int_{Y_g} \left[\lambda[s] \left(\frac{\langle xa, xb \rangle - 2 \langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \right] d\mu_{\lambda[s], gy} \\ &= [W_{\lambda[s]}(g)\phi](x). \end{aligned}$$

This completes the proof of the proposition. \square

5. ACTION OF GALOIS

Let μ be a Haar measure on a totally disconnected topological group A . If O_1 and O_2 are non-empty compact open sets in A then the ratio

$$(O_1 : O_2) = \frac{\mu(O_1)}{\mu(O_2)}$$

is a rational number [C, I.1.1.]. Hence, if $\mu(O)$ lies in a subfield L of \mathbf{C} for some non-empty compact open set O then the same is true for all non-empty compact open sets. The measure μ is said to be L -rational if this is the case.

Lemma 8. *The measures $\mu_{\lambda, g}$, $g \in \mathrm{Sp}(V)$, are $\mathbf{Q}(\sqrt{q})$ -rational.*

Proof. If $t \in F^*$ then $|t|$ is a power of q . Therefore, (6) shows that it is sufficient to verify that the measures μ_{λ, τ_i} are $\mathbf{Q}(\sqrt{q})$ -rational. Formulas (1) and (4) ensure that this is indeed the case: if $\mathcal{Y}_i = \sum_{k=1}^i \mathcal{O}y_k$ then

$$\int_{\mathcal{Y}_i} d\mu_{\lambda, \tau_i} y = q^{i/2}.$$

This completes the proof of the lemma. \square

Let A be a totally disconnected topological group. If L is a subfield of \mathbf{C} , let $\mathcal{S}(A, L)$ denote the space of locally constant, L -valued functions on A of compact support.

Lemma 9. *Let A be a totally disconnected topological group, $L \subseteq K$ an extension of fields, and μ a L -rational Haar measure on A . If $\phi \in \mathcal{S}(A, K)$ then $\int_A \phi d\mu$ belongs to K .*

Proof. Since $\phi \in \mathcal{S}(A, K)$, there exists compact open subsets A_1, \dots, A_k of A and scalars c_1, \dots, c_k in K such that

$$\phi = \sum_{i=1}^k c_i \chi_{A_i}.$$

Here, χ_{A_i} denotes the characteristic function of A_i . Since $\mu(A_i) \in L \subset K$, it follows that

$$\int_A \phi d\mu = \sum_{i=1}^k c_i \mu(A_i)$$

lies in K . □

Let $\mathbf{Q}(\lambda)$ be the character field of λ and set

$$E = \mathbf{Q}(\lambda)(\sqrt{-1}).$$

Observe that Lemma 1 ensures that $\mathbf{Q}(\sqrt{q})$ is a subfield of E .

Proposition 10. *The operators $W_\lambda(g)$, $g \in \mathrm{Sp}(V)$, leave the subspace $\mathcal{S}(X, E)$ invariant.*

Proof. If $\phi \in \mathcal{S}(X, E)$ then the integrand in (3) lies in $\mathcal{S}(Y_g, E)$, since $\mathbf{Q}(\lambda) \subseteq E$. In light of Lemma 8, Lemma 9 applied in the case $A = Y_g$, $K = E$, $L = \mathbf{Q}(\sqrt{q})$, and $\mu = \mu_{\lambda, g}$ allows us to deduce that the integral (3) lies in E . It follows immediately that $W_\lambda(g)\phi \in \mathcal{S}(X, E)$. □

By Lemma 1, E is a Galois extension of \mathbf{Q} . Its Galois group acts on $\mathcal{S}(X, E)$: if $\sigma \in \mathrm{Gal}(E/\mathbf{Q})$ and $\phi \in \mathcal{S}(X, E)$ then

$$(8) \quad (\sigma(\phi))(x) = \sigma(\phi(x)), \quad x \in X.$$

There is an associated Galois action on $\mathrm{End} \mathcal{S}(X, E)$: if $\sigma \in G$ and $T \in \mathrm{End} \mathcal{S}(X, E)$ then

$$(9) \quad {}^\sigma T(\phi) = \sigma [T(\sigma^{-1}(\phi))], \quad \phi \in \mathcal{S}(X, E).$$

The Galois group also permutes the unitary characters of F^+ : if $\sigma \in \mathrm{Gal}(E/\mathbf{Q})$ and λ is a unitary character of F^+ then ${}^\sigma \lambda$ is the character defined by

$${}^\sigma \lambda(t) = \sigma(\lambda(t)), \quad t \in F^+.$$

Let \mathcal{P} be the topological closure of the prime ring of F . The image of $s \in \mathcal{P}^*$ in $\mathrm{Gal}(\mathbf{Q}(\lambda)/\lambda)$ under the canonical isomorphism of Lemma 2 will be denoted σ_s .

Lemma 11. *Let $\sigma \in \mathrm{Gal}(E/\mathbf{Q})$. If $\sigma|_{\mathbf{Q}(\lambda)} = \sigma_s$ then ${}^\sigma \lambda = \lambda[s]$.*

Proof. ($\mathrm{char} F = 0$) Let \mathfrak{i} be the conductor of λ . Given $t \in F$, fix $n \geq 1$ such that $t \in p^{-n}\mathfrak{i}$. Since $p^{nt} \in \mathfrak{i}$,

$$1 = \lambda(p^{nt}) = \lambda(t)^{p^n},$$

thus $\lambda(t) \in \nu_{p^n}$. Fixing $r \in \mathbf{Z}$ such that $s \equiv r \pmod{p^n \mathcal{P}}$,

$$({}^\sigma \lambda)(t) = \sigma(\lambda(t)) = \lambda(t)^r = \lambda(rt) = \lambda(st),$$

the last equality following from the fact $rt \equiv st \pmod{\mathfrak{i}}$. □

Given $\sigma \in \mathrm{Gal}(E/\mathbf{Q})$, let ${}^\sigma W_\lambda$ be the projective representation defined by

$$({}^\sigma W_\lambda)(g) = \sigma(W_\lambda(g)), \quad g \in \mathrm{Sp}(V).$$

Proposition 12. *Let $\sigma \in \mathrm{Gal}(E/\mathbf{Q}(\sqrt{q}))$. If $\sigma|_{\mathbf{Q}(\lambda)} = \sigma_s$ then ${}^\sigma W_\lambda(g) = W_{\lambda[s]}(g)$.*

The proof of Proposition 12 is based on the integral formula (3) and the following

Lemma 13. *Let A be a totally disconnected topological group, $L \subseteq K$ an extension of fields, and μ a L -rational Haar measure on A . If σ is an L -automorphism of K then, for all $\phi \in \mathcal{S}(A, K)$,*

$$\int_A \sigma(\phi) d\mu = \sigma \left(\int_A \phi d\mu \right).$$

Proof. Using the notation introduced in the proof of Lemma 5, if

$$\phi = \sum_{i=1}^k c_i \chi_{A_i}$$

then

$$\sigma(\phi) = \sum_{i=1}^k \sigma(c_i) \chi_{A_i}.$$

Therefore, since $\mu(A_i) \in L$ is fixed by σ ,

$$\int \sigma(\phi) d\mu = \sum_{i=1}^k \sigma(c_i) \mu(A_i) = \sum_{i=1}^k \sigma(c_i) \sigma(\mu(A_i)) = \sigma \left(\sum_{i=1}^k c_i \mu(A_i) \right) = \sigma \left(\int_A \phi d\mu \right).$$

This completes the proof of the lemma. \square

Proof of Proposition 12. Let $g \in \mathrm{Sp}(V)$, $\phi \in \mathcal{S}(X, E)$, and $x \in X$. We assume g has the matrix representation (2). Lemma 8 asserts that the measure $\mu_{\lambda, g}$ is $\mathbf{Q}(\sqrt{q})$ -rational. Applying Lemma 13 to the case $A = Y_g$, $L = \mathbf{Q}(\sqrt{q})$, $K = E$, and $\mu = \mu_{\lambda, g}$, the definition of ${}^\sigma W_\lambda$, the formula (3), and Lemma 11 yield

$$\begin{aligned} [{}^\sigma W_\lambda(g)\phi](x) &= \sigma [W_\lambda(g)(\sigma^{-1}\phi)(x)] \\ &= \sigma \left[\int_{Y_g} \left[\lambda \left(\frac{\langle xa, xb \rangle - 2\langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) (\sigma^{-1}\phi)(xa + yc) \right] d\mu_{\lambda, g} y \right] \\ &= \int_{Y_g} \left[\sigma \lambda \left(\frac{\langle xa, xb \rangle - 2\langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \right] d\mu_{\lambda, g} y \\ &= \int_{Y_g} \left[\lambda[s] \left(\frac{\langle xa, xb \rangle - 2\langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \right] d\mu_{\lambda, g} y. \end{aligned}$$

Observing $s \in \mathcal{P}^* \subseteq \mathcal{O}^*$, Lemma 5 implies that $\mu_{\lambda[s], g} = \mu_{\lambda, g}$. The preceding calculation thus gives

$$\begin{aligned} [{}^\sigma W_\lambda(g)\phi](x) &= \int_{Y_g} \left[\lambda[s] \left(\frac{\langle xa, xb \rangle - 2\langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \right] d\mu_{\lambda[s], g} y \\ &= [W_{\lambda[s]}(g)\phi](x). \end{aligned}$$

This completes the proof. \square

6. THE FUNDAMENTAL IDENTITY

Let

$$\mathfrak{G} = \{ \sigma \in \mathrm{Gal}(E/\mathbf{Q}(\sqrt{q})) : \exists s \in \mathcal{O}^* \text{ such that } \sigma|_{\mathbf{Q}(\lambda)} = \sigma_{s^2} \}.$$

Note that \mathfrak{G} is a subgroup of $\text{Gal}(E/\mathbf{Q}(\sqrt{q}))$. Given $s \in F^*$, let $g_s \in \text{Sp}(V)$ be the map defined by

$$(x + y)g_s = s^{-1}x + sy, \quad x \in X, y \in Y.$$

We observe that g_s lies in the parabolic subgroup P that leaves Y invariant and is related to the operator f_{s^2} defined earlier by the identity

$$f_{s^2} = sI \circ g_s.$$

Proposition 14. *Let $\sigma \in \mathfrak{G}$ and $g \in \text{Sp}(V)$. If $\sigma|_{\mathbf{Q}(\lambda)} = \sigma_{s^2}$, $s \in \mathcal{O}^*$, then*

$${}^\sigma W_\lambda(g) = W_\lambda(g_s)^{-1} W_\lambda(g) W_\lambda(g_s).$$

Proof. In light of Propositions 7 and 12,

$${}^\sigma W_\lambda(g) = W_{\lambda[s^2]}(g) = W_\lambda^{f_{s^2}}(g) = W_\lambda(g^{f_{s^2}}) = W_\lambda(g^{g_s}).$$

Applying Theorem 4(i) with $p_1^{-1} = p_2 = g_s$,

$$W_\lambda(g^{g_s}) = W_\lambda(g_s^{-1}) W_\lambda(g) W_\lambda(g_s) = W_\lambda(g_s)^{-1} W_\lambda(g) W_\lambda(g_s).$$

This completes the proof of the proposition. \square

Corollary. *If $t \in F^*$ and $\sigma \in \mathfrak{G}$ then ${}^\sigma W_\lambda(g_t) = W_\lambda(g_t)$.*

Proof. Fix $s \in \mathcal{O}^*$ such that $\sigma|_{\mathbf{Q}(\lambda)} = \sigma_{s^2}$. Observing that g_s and g_t are commuting elements of P , the preceding proposition combines with Theorem 4(i) to yield

$${}^\sigma W_\lambda(g_t) = W_\lambda(g_s)^{-1} W_\lambda(g_t) W_\lambda(g_s) = W_\lambda(g_s^{-1} g_t g_s) = W_\lambda(g_t),$$

as required. \square

7. THE COCYCLE

Define

$$\mathfrak{H} = \text{Gal}(E/\mathbf{Q}(\sqrt{p}, \sqrt{-p})).$$

Our aim in this section is the construction of a 1-cocycle $\delta : \mathfrak{H} \rightarrow \text{GL}(\mathcal{S}(X, E))$ such that

$$(10) \quad {}^\sigma W_\lambda(g) = \delta(\sigma)^{-1} W_\lambda(g) \delta(\sigma), \quad g \in \text{Sp}(V), \sigma \in \mathfrak{H}.$$

Let $p^* = (-1)^{(p-1)/2} p$. When combined with restriction to $\mathbf{Q}(\lambda)$, the canonical isomorphism of Lemma 2 yields

$$(11) \quad \mathfrak{H} \simeq \text{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q}(\sqrt{p^*})) \simeq (\mathcal{P}^*)^2.$$

If $\epsilon \in \mathcal{P}^*$ is a primitive $p-1$ root of unity then

$$\mathcal{P}^* = \langle \epsilon \rangle \times U_1$$

with

$$U_1 = \begin{cases} \{1\}, & \text{if char } F = p; \\ \{r \in \mathcal{P} : r \equiv 1 \pmod{p}\}, & \text{if char } F = 0, \end{cases}$$

a pro- p group. As p is assumed to be odd, the map $r \mapsto r^2$ is an automorphism of U_1 , hence

$$\mathcal{P}^{*2} = \langle \epsilon^2 \rangle \times U_1.$$

The isomorphism (11) identifies U_1 with $\text{Gal}(E/\mathbf{Q}(\nu_p, \sqrt{-1}))$, where ν_p is the group of complex p -th roots of unity. This in turn leads to an identification of $\langle \epsilon^2 \rangle$ with

$$\mathfrak{H} / \text{Gal}(E/\mathbf{Q}(\nu_p, \sqrt{-1})) \simeq \text{Gal}(\mathbf{Q}(\nu_p, \sqrt{-1})/\mathbf{Q}(\sqrt{p}, \sqrt{-p})).$$

In particular, the element η of \mathfrak{H} characterized by

$$(12) \quad \eta|_{\mathbf{Q}(\lambda)} = \sigma_{\epsilon^2}$$

has order $(p-1)/2$ and restricts to a generator of $\text{Gal}(\mathbf{Q}(\nu_p, \sqrt{-1})/\mathbf{Q}(\sqrt{p}, \sqrt{-p}))$.

Given $\sigma \in \mathfrak{H}$, there is a unique integer i , $1 \leq i \leq (p-1)/2$, and a unique element $s \in U_1$, such that

$$\sigma|_{\mathbf{Q}(\lambda)} = \sigma_{\epsilon^{2i}s^2}.$$

If τ is a second element of \mathfrak{H} , say

$$\tau|_{\mathbf{Q}(\lambda)} = \sigma_{\epsilon^{2j}t^2}, \quad 1 \leq j \leq (p-1)/2, \quad t \in U_1.$$

then

$$\sigma\tau|_{\mathbf{Q}(\lambda)} = \sigma_{\epsilon^{2k}(st)^2},$$

where $st \in U_1$ and

$$k = \begin{cases} i+j, & \text{if } i+j \leq (p-1)/2; \\ i+j - \frac{p-1}{2}, & \text{if } i+j > (p-1)/2. \end{cases}$$

Our initial attempt at the construction of the cocycle is to define

$$D(\sigma) = W_\lambda(g_{\epsilon^i s}), \quad \sigma|_{\mathbf{Q}(\lambda)} = \sigma_{\epsilon^{2i}s^2}, \quad 1 \leq i \leq (p-1)/2, \quad s \in U_1.$$

Proposition 14 ensures that

$$(13) \quad {}^\sigma W_\lambda(g) = D(\sigma)^{-1} W_\lambda(g) D(\sigma), \quad g \in \text{Sp}(V), \sigma \in \mathfrak{H}.$$

Assuming σ and τ are as above, the definition of D yields

$$D(\sigma\tau) = W_\lambda(g_{\epsilon^k st}).$$

On the other hand, the Corollary to Proposition 14 gives

$${}^\sigma D(\tau) = {}^\sigma W_\lambda(g_{\epsilon^j t}) = W_\lambda(g_{\epsilon^j t}),$$

hence Theorem 4(i) yields

$$D(\sigma) {}^\sigma D(\tau) = W_\lambda(g_{\epsilon^i s}) W_\lambda(g_{\epsilon^j t}) = W_\lambda(g_{\epsilon^{i+j} st}).$$

If $i+j \leq (p-1)/2$ then

$$W_\lambda(g_{\epsilon^{i+j} st}) = W_\lambda(g_{\epsilon^k}).$$

If $i+j > (p-1)/2$ then, since $\epsilon^{(p-1)/2} = -1$, Theorem 4(i) yields

$$W_\lambda(g_{\epsilon^{i+j} st}) = W_\lambda(g_{-\epsilon^k st}) = W_\lambda(\iota) W_\lambda(g_{\epsilon^k st}),$$

where $\iota = g_{-1}$ is the central involution of $\text{Sp}(V)$ that maps $v \in V$ to $-v$. In summary,

$$(14) \quad D(\sigma) {}^\sigma D(\tau) = \begin{cases} D(\sigma\tau), & \text{if } i+j \leq (p-1)/2; \\ W_\lambda(\iota) D(\sigma\tau), & \text{if } i+j > (p-1)/2. \end{cases}$$

In particular, D is not a 1-cocycle; to get one we must account for the factor $W_\lambda(\iota)$.

Since $\iota \in P$, Theorem 4(i) implies that if ϕ belongs to $\mathcal{S}(X, E)$ then

$$[W_\lambda(\iota)\phi](x) = \phi(-x), \quad x \in X.$$

In particular, $W_\lambda(\iota)$ is an involution, hence the operators

$$\rho_e = \frac{1}{2}(I + W_\lambda(\iota)) \quad \text{and} \quad \rho_o = \frac{1}{2}(I - W_\lambda(\iota))$$

are orthogonal idempotents. Furthermore, recalling $\iota = g_{-1}$, the Corollary to Proposition 14 shows that both ρ_e and ρ_o are fixed by the action of Galois. Finally, since $I = \rho_e + \rho_o$, it is easily verified that the operators

$$\rho_e + c\rho_o, \quad c \in E, c \neq 0,$$

are invertible.

Lemma 15. *The norm equation*

$$N(u) = -1, \quad N : \mathbf{Q}(\nu_p, \sqrt{-1}) \rightarrow \mathbf{Q}(\sqrt{p}, \sqrt{-p})$$

has a solution.

Proof. The case $p \equiv 1 \pmod{4}$ is covered by [CMS, Lemma 24(2)]. In the case $p \equiv 3 \pmod{4}$ one has

$$N(-1) = (-1)^{[\mathbf{Q}(\nu_p, \sqrt{-1}) : \mathbf{Q}(\sqrt{p}, \sqrt{-p})]} = (-1)^{(p-1)/2} = -1,$$

since $(p-1)/2$ is odd. □

Let u be a solution of the norm equation of the preceding lemma. Given $\sigma \in \mathfrak{H}$, set

$$A(\sigma) = \rho_e + \left(\prod_{l=0}^{k-1} \eta^l(u) \right) \rho_0, \quad \text{where } \sigma|_{\mathbf{Q}(\lambda)} = \sigma_{\epsilon^{2i}s^2}, \quad 1 \leq i \leq (p-1)/2, \quad s \in U_1$$

where η satisfies (12). The remarks preceding Lemma 15 ensure that $A(\sigma) \in \text{GL}(\mathcal{S}(X, E))$. With the notation introduced earlier, if σ and τ belong to \mathfrak{H} then

$$A(\sigma\tau) = \rho_e + \left(\prod_{l=0}^{k-1} \eta^l(u) \right) \rho_0.$$

On the other hand, observing

$$\sigma\eta^{-i}|_{\mathbf{Q}(\lambda)} = \sigma_{\epsilon^{2i}s^2}\sigma_{\epsilon^{-2i}} = \sigma_{\epsilon^{2i}s^2}\sigma_{\epsilon^{-2i}} = \sigma_{s^2},$$

the fact (11) identifies U_1 with $\text{Gal}(E/\mathbf{Q}(\nu_p, \sqrt{-1}))$ allows us to deduce that the restrictions of σ and η^i to $\mathbf{Q}(\nu_p, \sqrt{-1})$ coincide. Therefore,

$$\begin{aligned} \sigma A(\tau) &= \sigma \left[\rho_e + \left(\prod_{l=0}^{j-1} \eta^l(u) \right) \rho_0 \right] = \rho_e + \sigma \left(\prod_{l=0}^{j-1} \eta^l(u) \right) \rho_0 \\ &= \rho_e + \eta^i \left(\prod_{l=0}^{j-1} \eta^l(u) \right) \rho_0 = \rho_e + \left(\prod_{l=i}^{i+j-1} \eta^l(u) \right) \rho_0, \end{aligned}$$

hence

$$\begin{aligned} A(\sigma)^\sigma A(\tau) &= \left[\rho_e + \left(\prod_{l=0}^{i-1} \eta^l(u) \right) \rho_0 \right] \left[\rho_e + \left(\prod_{l=i}^{i+j-1} \eta^l(u) \right) \rho_0 \right] \\ &= \left[\rho_e + \left(\prod_{l=0}^{i+j-1} \eta^l(u) \right) \rho_0 \right]. \end{aligned}$$

If $i+j \leq (p-1)/2$ then

$$\prod_{l=0}^{i+j-1} \eta^l(u) = \prod_{l=0}^{k-1} \eta^l(u),$$

hence

$$A(\sigma)^\sigma A(\tau) = A(\sigma\tau).$$

If $i + j > (p - 1)/2$ then the choice of η and u yield

$$\prod_{l=0}^{i+j-1} \eta^l(u) = \left(\prod_{l=0}^{(p-3)/2} \eta^l(u) \right) \left(\prod_{l=(p-1)/2}^{i+j-1} \eta^l(u) \right) = N(u) \prod_{l=0}^{k-1} \eta^l(u) = - \prod_{l=0}^{k-1} \eta^l(u).$$

Observing that $\rho_e = \rho_e W_\lambda(\iota)$ and $-\rho_o = \rho_o W_\lambda(\iota)$,

$$A(\sigma)^\sigma A(\tau) = \rho_e - \left(\prod_{l=0}^{k-1} \eta^l(u) \right) \rho_o = \left[\rho_e + \left(\prod_{l=0}^{k-1} \eta^l(u) \right) \rho_o \right] W_\lambda(\iota) = A(\sigma\tau) W_\lambda(\iota).$$

In summary,

$$(15) \quad A(\sigma)^\sigma A(\tau) = \begin{cases} A(\sigma\tau), & \text{if } i + j \leq (p - 1)/2; \\ A(\sigma\tau) W_\lambda(\iota), & \text{if } i + j > (p - 1)/2. \end{cases}$$

Consider the map $\delta : \mathfrak{H} \rightarrow \text{GL}(\mathcal{S}(X, E))$ given by

$$\delta(\sigma) = A(\sigma)D(\sigma).$$

If $\sigma, \tau \in \mathfrak{H}$ are as above

$$\delta(\sigma)^\sigma \delta(\tau) = (A(\sigma)D(\sigma))^\sigma (A(\tau)D(\tau)) = A(\sigma)D(\sigma)^\sigma A(\tau)^\sigma D(\tau).$$

By Theorem 4(1), ${}^\sigma A(\tau) \in E[W_\lambda(i)]$ commutes with $D(\sigma) = W_\lambda(g_{e^{i_s}})$, hence

$$A(\sigma)D(\sigma)^\sigma A(\tau)^\sigma D(\tau) = A(\sigma)^\sigma A(\tau)D(\sigma)^\sigma D(\tau).$$

If $i + j > (p - 1)/2$ then (14) and (15) yield

$$A(\sigma)^\sigma A(\tau)D(\sigma)^\sigma D(\tau) = A(\sigma\tau)W_\lambda(\iota)W_\lambda(\iota)D(\sigma\tau) = A(\sigma\tau)D(\sigma\tau).$$

Since this is trivially true if $i + j \leq (p - 1)/2$, we conclude

$$\delta(\sigma)^\sigma \delta(\tau) = A(\sigma\tau)D(\sigma\tau) = \delta(\sigma\tau).$$

This shows that δ is a 1-cocycle. Furthermore, if $g \in \text{Sp}(V)$ then Theorem 4(i) shows that $A(\sigma) \in E[W_\lambda(\iota)]$ commutes with $W_\lambda(g)$, hence (12) yields

$$\begin{aligned} \delta(\sigma)^{-1} W_\lambda(g) \delta(\sigma) &= (A(\sigma)D(\sigma))^{-1} W_\lambda(g) A(\sigma)D(\sigma) \\ &= D(\sigma)^{-1} A(\sigma)^{-1} W_\lambda(g) A(\sigma)D(\sigma) \\ &= D(\sigma)^{-1} W_\lambda(g) D(\sigma) \\ &= {}^\sigma W_\lambda(g) \end{aligned}$$

which verifies that (10) is satisfied.

8. THE TRIVIALITY OF THE COCYCLE

Let $\delta : \mathfrak{H} \rightarrow \mathrm{GL}(\mathcal{S}(X, E))$ be the 1-cocycle satisfying (10) constructed above.

Lemma 16. *If $\phi \in \mathcal{S}(X, E)$ then there exists an open subgroup \mathfrak{K} of \mathfrak{H} such that*

$$\delta(\sigma)\phi = \phi, \quad \sigma \in \mathfrak{K}.$$

Proof. If $\mathrm{char} F = p$ then \mathfrak{H} is a finite discrete group, so one may take \mathfrak{K} to be the trivial subgroup.

Assume $\mathrm{char} F = 0$. If \mathfrak{X} is a lattice in X then the subgroups

$$p^k \mathfrak{X}, \quad k \in \mathbf{Z},$$

form a local base at the origin. Therefore, given $x \in X$, there exist $i_x \in \mathbf{Z}$ such that ϕ is constant on the coset $x + p^{i_x} \mathfrak{X}$. As the family $\{x + p^{i_x} \mathfrak{X} : x \in X\}$ is an open cover of X , there exists x_1, \dots, x_m in X such that

$$\mathrm{supp} \phi \subseteq \bigcup_{j=1}^m x_j + p^{i_{x_j}} \mathfrak{X}.$$

Set

$$i = \max \{i_{x_1}, \dots, i_{x_m}\}$$

and consider $x + p^i \mathfrak{X} \cap \mathrm{supp} \phi$, $x \in X$. If it is empty then the restriction of ϕ to the coset $x + p^i \mathfrak{X}$ is identically 0. If not, there exists j such that $x + p^i \mathfrak{X} \cap x_j + p^{i_{x_j}} \mathfrak{X}$ is non-empty, hence

$$x + p^i \mathfrak{X} \subseteq x_j + p^{i_{x_j}} \mathfrak{X}$$

by choice of i . The choice of i_{x_j} thus ensures that the restriction of ϕ to $x + p^i \mathfrak{X}$ is the constant function with value $\phi(x_j)$. We conclude that ϕ is constant on the $p^i \mathfrak{X}$ -cosets of X .

Let $\sigma \in \mathfrak{H}$. If $\sigma|_{\mathbf{Q}(\lambda)} = \sigma_{r,2}$, $r \in U_1$, then by construction $\delta(\sigma) = W_\lambda(g_r)$. Observing

$$g_r = \begin{pmatrix} r^{-1} \cdot 1_X & 0 \\ 0 & r \cdot 1_Y \end{pmatrix} \in P,$$

if $x \in X$ then Theorem 4(i) yields

$$(\delta(\sigma)\phi)(x) = (W_\lambda(g_r)\phi)(x) = |r|^{-\dim X/2} \lambda \left(\frac{\langle r^{-1}x, rx \rangle}{2} \right) \phi(r^{-1}x) = \phi(r^{-1}x),$$

since r is a unit and $\langle \cdot, \cdot \rangle$ is F -bilinear and alternating. Fix $j \in \mathbf{Z}$ such that $i > j$ and

$$\mathrm{supp} \phi \subseteq p^j \mathfrak{X}.$$

If $x \notin p^j \mathfrak{X}$ then neither is $r^{-1}x$, so the choice of j ensures that

$$(\delta(\sigma)\phi)(x) = \phi(r^{-1}x) = 0 = \phi(x).$$

On the other hand, suppose $x \in p^j \mathfrak{X}$. In this case, if $r \equiv 1 \pmod{p^{i-j}}$ then

$$r^{-1}x + p^i \mathfrak{X} = x + p^{i-j} p^j \mathfrak{X} + p^i \mathfrak{X} = x + p^i \mathfrak{X},$$

hence the choice of i ensures that

$$(\delta(\sigma)\phi)(x) = \phi(r^{-1}x) = \phi(x).$$

In light of the preceding discussion,

$$\mathfrak{K} = \{ \sigma \in \mathfrak{H} : \sigma|_{\mathbf{Q}(\lambda)} = \sigma_{r,2}, r \equiv 1 \pmod{p^{i-j}} \} = \mathrm{Gal}(E/\mathbf{Q}(\nu_{p^{i-j}}, \sqrt{-1}))$$

has the required properties. \square

Let K/k be a Galois extension and M a K -vector space equipped with an semi-linear action of the Galois group $\text{Gal}(K/k) : \text{if } \sigma \in \text{Gal}(K/k), m \in M \text{ and } e \in K \text{ then}$

$$\sigma(em) = \sigma(e)\sigma(m).$$

For such an action, the fixed-point set

$$M^{\text{Gal}(K/k)} = \{m \in M : m = \sigma(m) \forall \sigma \in \text{Gal}(K/k)\}$$

is a k -vector space. The canonical action of $\text{Gal}(K/k)$ on K yields a semi-linear action on the tensor product $K \otimes_k M^{\text{Gal}(K/k)} :$

$$\sigma(e \otimes m) = \sigma(e) \otimes m, \quad \sigma \in \text{Gal}(K/k), e \in E, m \in M^{\text{Gal}(K/k)}.$$

The action of Galois on M is said to be *smooth* if the stabilizer of each $m \in M$ is open in $\text{Gal}(K/k)$.

Proposition 17 (Galois Descent). *If M is a K -vector space equipped with a semi-linear, smooth action of $\text{Gal}(K/k)$ then the canonical map*

$$\psi : K \otimes_k M_k \rightarrow M$$

is a K -linear isomorphism of $\text{Gal}(K/k)$ -modules.

Proof. The case $K = k_s$, the separable closure of k , is proved in [B, AG.14.2]. The general case is proved using the same argument, *mutatis mutandis*. \square

Proposition 18. *There exists $\alpha \in \text{GL}(\mathcal{S}(X, E))$ such that*

$$(16) \quad \delta(\sigma) = \alpha^{-1}\sigma\alpha, \quad \sigma \in \mathfrak{H}.$$

Proof. The canonical action (8) of \mathfrak{H} on $\mathcal{S}(X, E)$ is clearly semi-linear. It is furthermore smooth, since each element of $\mathcal{S}(X, E)$ takes only finitely many values in E .

On the other hand, since δ is a 1-cocycle, then

$$(\sigma, \phi) \mapsto \delta(\sigma)\sigma(\phi), \quad \sigma \in \mathfrak{H}, \phi \in \mathcal{S}(X, E),$$

is also an action of \mathfrak{H} on $\mathcal{S}(X, E)$, referred to as the twisted action by δ . It is semi-linear, since δ takes values in $\text{GL}(\mathcal{S}(X, E))$. Since the original action is smooth, if $\phi \in \mathcal{S}(X, E)$ then there exists an open subgroup \mathfrak{H}_1 such that

$$\sigma(\phi) = \phi, \quad \sigma \in \mathfrak{H}_1.$$

Furthermore, Lemma 16 asserts that there is an open subgroup \mathfrak{K} of \mathfrak{H} such that

$$\delta(\sigma)\phi = \phi, \quad \sigma \in \mathfrak{K}.$$

Therefore, if $\sigma \in \mathfrak{H}_1 \cap \mathfrak{K}$ then

$$\delta(\sigma)\sigma(\phi) = \delta(\sigma)\phi = \phi.$$

This shows that the stabilizer of ϕ under the twisted action contains the open subgroup $\mathfrak{H}_1 \cap \mathfrak{K}$. Since it is the union of its $\mathfrak{H}_1 \cap \mathfrak{K}$ -cosets, it follows that the stabilizer of ϕ under the twisted action is open. We conclude that the twisted action is smooth.

Using $\mathcal{S}(X, E)$ and ${}_\delta\mathcal{S}(X, E)$ to denote the \mathfrak{H} -modules defined by the natural and twisted actions, respectively, Galois Descent asserts the existence of E -linear, \mathfrak{H} -equivariant isomorphisms

$${}_\delta\mathcal{S}(X, E) \simeq E \otimes_{\mathbf{Q}(\sqrt{p}, \sqrt{-p})} {}_\delta\mathcal{S}(X, E)^{\mathfrak{H}} \quad \text{and} \quad E \otimes_{\mathbf{Q}(\sqrt{p}, \sqrt{-p})} \mathcal{S}(X, E)^{\mathfrak{H}} \simeq \mathcal{S}(X, E).$$

In particular,

$$\dim_{\mathbf{Q}(\sqrt{p}, \sqrt{-p})} \delta \mathcal{S}(X, E)^{\mathfrak{H}} = \dim_E \mathcal{S}(X, E) = \dim_{\mathbf{Q}(\sqrt{p}, \sqrt{-p})} \mathcal{S}(X, E)^{\mathfrak{H}},$$

so $\delta \mathcal{S}(X, E)^{\mathfrak{H}}$ and $\mathcal{S}(X, E)^{\mathfrak{H}}$ are $\mathbf{Q}(\sqrt{p}, \sqrt{-p})$ -isomorphic. As any such isomorphism extends by scalars to a E -linear, \mathfrak{H} -equivariant isomorphism

$$E \otimes_{\mathbf{Q}(\sqrt{p}, \sqrt{-p})} \delta \mathcal{S}(X, E)^{\mathfrak{H}} \simeq E \otimes_{\mathbf{Q}(\sqrt{p}, \sqrt{-p})} \mathcal{S}(X, E)^{\mathfrak{H}},$$

we conclude

$$\delta \mathcal{S}(X, E) \simeq \mathcal{S}(X, E).$$

Let $\alpha \in \mathrm{GL}(\mathcal{S}(X, E))$ be a \mathfrak{H} -equivariant isomorphism $\delta \mathcal{S}(X, E) \rightarrow \mathcal{S}(X, E)$. If $\sigma \in \mathfrak{H}$ and $\phi \in \mathfrak{H}$ then the definition of the twisted action ensures that

$$\alpha \delta(\sigma) \sigma(\phi) = \sigma(\alpha \phi),$$

hence

$$\delta(\sigma) \phi = \alpha^{-1} \alpha \delta(\sigma) \sigma(\sigma^{-1}(\phi)) = \alpha^{-1} \sigma(\alpha(\sigma^{-1}(\phi))) = \alpha^{-1 \sigma} \alpha(\phi).$$

This completes the proof of the proposition. \square

9. PROOF OF THE MAIN THEOREM

Fix $\alpha \in \mathrm{GL}(\mathcal{S}(X, E))$ satisfying the conclusion of Proposition 18. In light of (11) and (16), if $\sigma \in \mathfrak{H}$ and $g \in \mathrm{Sp}(V)$ then

$$\sigma(\alpha W_\lambda(g) \alpha^{-1}) = \sigma \alpha^\sigma W_\lambda(g) (\sigma \alpha)^{-1} = \sigma \alpha \delta(\sigma)^{-1} W_\lambda(g) \delta(\sigma) (\sigma \alpha)^{-1} = \alpha W_\lambda(g) \alpha^{-1}.$$

The compatibility of the Galois actions (8) and (9) allows us to deduce that the operators

$$\alpha W_\lambda(g) \alpha^{-1}, \quad g \in \mathrm{Sp}(V),$$

leave

$$\mathcal{S}(X, E)^{\mathfrak{H}} = \mathcal{S}(X, E^{\mathfrak{H}}) = \mathcal{S}(X, \mathbf{Q}(\sqrt{p}, \sqrt{-p}))$$

invariant, hence provide a projective Weil representation realized over $\mathbf{Q}(\sqrt{p}, \sqrt{-p})$.

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