Mathematical Modelling of Spotting

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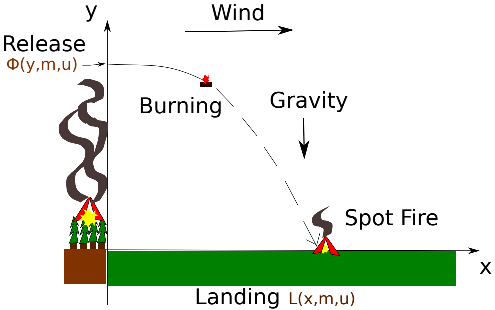
Spotting

- launching of fire brands
- flight of fire brands
- burning during flight
- landing distribution
- likelihood of starting a new fire



Two Models

- Model 1: Balance law for spotting
  spotting distribution ?

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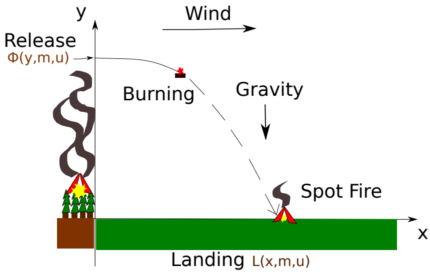
  spotting distribution ?
- Model 2: The effect of spotting on the propagation of a fire front
  acceleration ?

Model 1: Spotting

t: time
x: space
y: height
m: mass of fire brand
u: temperature of fire brand



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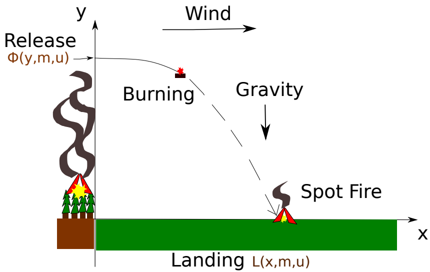

$p(t, x, y, m, u)$: Probability density of a fire brand of mass m burning with temperature u at location (x, y) at time t.

Balance law

$$\frac{\partial p}{\partial t}$$

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Each of the four processes can be studies separately.

Transport by wind

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Three cases:

- (w.1) $w(y) = \text{const.}$

Wind

- (w.2) (Okubo, Levin 2001)

$$w(y) = A \ln\left(\frac{y-d}{y_0}\right), \quad \text{for} \quad y > H$$

H: canopy height, y_0: roughness parameter, d: zero plane displacement.

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- (w.4) Q1: Realistic wind profile ?

Gravity

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- (v.2) Q2: Realistic vertical velocity ?

Burning mass

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$$m(t) = \frac{m}{1+\eta t^2}$$

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$$f_m = -\kappa\ m\ w$$

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- (m.3) Q3: Realistic burning law for flying brands?

Temperature

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- $f_u = \alpha\ m\ r(u)\ H(u - u_{ign})$
  α: constant, $r(u)$ reaction rate, H heavyside function.
- (u.1) $r(u) =$ const.
- (u.2) (Arrhenius law)

$$r(u) = \exp\left(-\frac{E}{\hat{R}u}\right)$$

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- (u.3) Q4: Realistic temperature law?

Initial and Boundary Conditions

- Initial condition

$$p(0, x, y, m, u) = p_0(x, y, m, u)$$

- Influx at $x = 0$:

$$p(t, 0, y, m, u) = \varphi(t, y, m, u)$$

- Absorption at the bottom (landed branches)

$$L(t, x, m, u) = \int_0^t p(t, x, 0, m, u)dt$$

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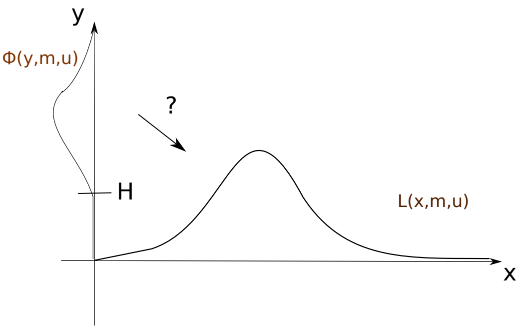
- Total accumulation of branches

$$\mathcal{L}(x, m, u) = \lim_{t\to\infty} L(t, x, m, u)$$

One time release

- One release at time $t = 0$:

$$\varphi(t, y, m, u) = \delta_0(t)\phi(y, m, u)$$



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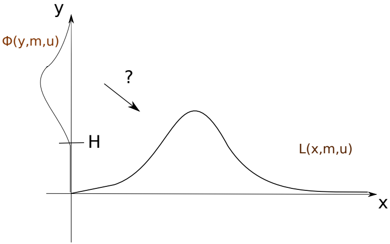
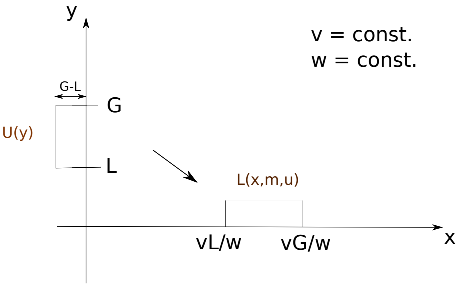
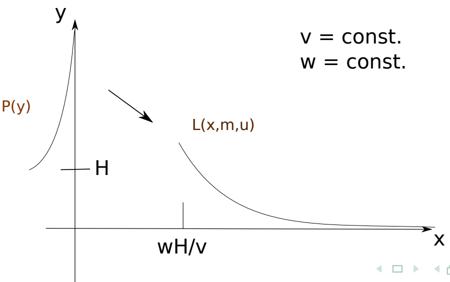
The x-distribution of $\mathcal{L}$ is a scaled version of the y-distribution of ϕ.




y
v = const.
w = const.
G-L
G
U(y)
L
L(x,m,u)
vL/w
vG/w
x


y
v = const.
w = const.
P(y)
L(x,m,u)
H
wH/v
x

Example 2: Fat tail

- One release at time $t = 0$:

$$\varphi(t, y, m, u) = \delta_0(t) P(y) U(m) \Psi(u)$$

$$P(y) = \begin{cases} ae^{-a(y-H)} & y \geq H \\ 0 & y < H \end{cases}$$

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Then for large x:

$$\mathcal{L}(x, m, u) \sim A \exp\left(-\alpha(\beta x + \gamma)^{\frac{1}{\beta+1}} \right) \psi(\tilde{u})$$

Example (2a)

Special choice of parameters leads to

$$\mathcal{L}(x, m, u) \sim A \exp\left(-\sqrt{x}\right) \; \psi(\tilde{u})$$

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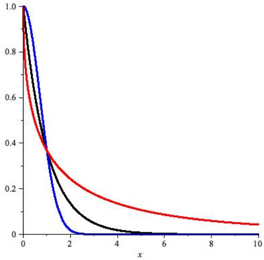
e^{-x}, e^{-x^2}, $e^{-\sqrt{x}}$



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$$m_t = -r(u)mH(u - u_{ign})$$

$J(m,u)$: probability that a brand of mass m and temperature u starts a new fire.

Acceleration

- If $\mathcal{L} = 0$, i.e. no spotting, then Babak, Bourlioux and H' showed the existence of travelling waves, i.e. fire fronts with constant speed.

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- If $\mathcal{L} \neq 0$:

Question: Can spotting accelerate fire propagation ?

Example, Kot 1996

$$N_{t+1} = \int_{-\infty}^{+\infty} k(x-y) f(N_t(y)) dy$$

$$k(x) = \frac{1}{4} e^{-\sqrt{x}}$$

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$$
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\end{aligned}
$$

Mean spread speed

$$\langle X(t) \rangle = (\alpha t + \beta)^2 \sim \alpha^2 t^2$$

Hence mean velocity

$$V = \frac{\langle X(t) \rangle}{t} \sim \alpha^2 t$$

Summary

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- The spotting model needs realistic input on wind profile, burning laws and branch release distribution.
- The fire propagation model can be used to understand when spotting might accelerate a fire front and when not.

MITACS team

MITACS Project on

Forest Fires and Spread in Heterogeneous Landscapes

- A. Bourlioux (U Montreal)
- C. Bose (U Victoria)
- J. Braun (U Western Ontario)
- D. Martell (U Toronto)
- C. Tymstra (Alberta Government)
- P. Babak (postdoc, UofA)

www.math.ualberta.ca/∼forestfire/