Periodic solutions for predator–prey systems with Beddington–DeAngelis functional response on time scales

Mostafa Fazly*, Mahmoud Hesaaraki

Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11365-9415, Tehran, Iran

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Abstract


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1. Introduction

In this paper we prove some theorems related to the existence of periodic solutions of predator–prey dynamical systems with Beddington–DeAngelis functional response, on time scales. The theory of time scales (measure chain), which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis in (1988), in order to unify continuous and discrete analysis (see [10]). Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different in nature from their continuous counterparts.

The study of dynamic equations on time scales reveals such discrepancies and helps us to avoid proving results twice, once for differential equations and, then for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which is an arbitrary closed subset of the real numbers. By choosing a time scale, general results can be applied to ordinary differential equations, and by choosing the time scale to be the set of integers, they yield similar results for difference equations.

The time scales calculus has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, economics, neural
networks and social sciences. For more information, see the monographs of Aulbach and Hilger [2], Bohner and Peterson [5,6], Lakshmikantham et al. [11] and the references therein. We may state that unification and extension are the two main features of the time scales calculus.

In order to make an easy and convenient reading of this paper we present some definitions and notations on time scales which are common in the recent literature, as follows.

**Definition 1.** A time scale is an arbitrary nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$, the real numbers. The set $\mathbb{T}$ inherits the standard topology of $\mathbb{R}$.

**Definition 2.** For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$, by

$$
\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup \{s \in \mathbb{T} : s < t\},
$$

respectively.

In this definition we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum $t$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum $t$), where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say that $t$ is right-scattered, while if $\rho(t) < t$ we say that $t$ is left-scattered. A point that is simultaneously right-scattered and left-scattered is called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then $t$ is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then $t$ is called left-dense. A point that is simultaneously right-dense and left-dense is called dense.

**Definition 3.** A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sides limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions is shown by $C_{\text{rd}} = C_{\text{rd}}(\mathbb{T}) = C_{\text{rd}}(\mathbb{T}, \mathbb{R})$.

**Definition 4.** For $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}$ we define $f^A(t)$, the delta-derivative of $f$ at $t$, to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ in $\mathbb{T}$ such that

$$
|f(\sigma(t)) - f(s)) - f^A(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.
$$

Thus, $f$ is said to be delta-differentiable if its delta-derivative exists. The set of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ that are delta-differentiable and whose delta-derivative are rd-continuous functions is denoted by $C^1_{\text{rd}} = C^1_{\text{rd}}(\mathbb{T}) = C^1_{\text{rd}}(\mathbb{T}, \mathbb{R})$.

**Definition 5.** A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^A(t) = f(t)$, for all $t \in \mathbb{T}$. Then, we write

$$
\int_t^s f(t) \Delta t := F(s) - F(r) \quad \text{for all } s, t \in \mathbb{T}.
$$

For the usual time scale $\mathbb{T} = \mathbb{R}$, rd-continuity coincides with the usual continuity in calculus. Moreover, every rd-continuous function on $\mathbb{T}$ has a delta-antiderivative, see [5]. For more information about the above definitions and their related concepts, the reader is referred to [2,5,6,10,11].

The dynamical relationship between predators and their prey has long been one of the dominant themes in both ecology and mathematical ecology, in part due to its universal existence and importance. At first sight, these problems may appear to be mathematically simple. However, they are very challenging and complicated. The dynamics of traditional Lotka-type predator–prey model with Beddington–DeAngelis functional response that Beddington in [3] derived and DeAngelis et al. in [7] proposed, independently, has been studied extensively in both cases of discrete and continuous models in the literature [3,4,7,8,12,14].

The question of existence of periodic solutions of the nonautonomous predator–prey system with Beddington–DeAngelis functional response for the case of discrete time has been studied by Zhang et al. in [14]. The result of this work will be appearing in the following Theorem D. The case of continuous system has been investigated by Fan and Kuang in [8]. The main results of their work will be stated in Theorems B and C as follows. Bohner et al. in [4] considered this system on time scales and obtained Theorem A. This theorem covers Theorems B and D.

In this paper by dropping some of the conditions of these four theorems we extend their application.
Consider the following predator–prey system on the time scale \(\mathbb{T}\):

\[
\begin{align*}
\dot{x}_1(t) &= a(t) - b(t) \exp(x_1(t)) - \frac{c(t) \exp(x_2(t))}{\bar{x}(t) + \beta(t) \exp(x_1(t)) + \gamma(t) \exp(x_2(t))}, \\
\dot{x}_2(t) &= -d(t) + \frac{f(t) \exp(x_1(t))}{\bar{x}(t) + \beta(t) \exp(x_1(t)) + \gamma(t) \exp(x_2(t))},
\end{align*}
\]

where \(a, b, c, d, x, \beta, \gamma \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}^2)\) are positive \(\omega\)-periodic functions. The symbol \(A\) stands for the delta-derivative.

We use the following notations throughout this paper, related to the time scale \(\mathbb{T}\) and the \(\omega\)-periodic function \(f \in C_{\text{rd}}(\mathbb{T})\):

\[
k = \min([0, \infty) \cap \mathbb{T}), \quad I_\omega = [k, k + \omega] \cap \mathbb{T}, \quad f^{\omega} = \sup_{t \in I} f(t), \quad f^{\bar{\omega}} = \inf_{t \in I} f(t), \quad \bar{f} = \frac{1}{\omega} \int_{I_\omega} f(s) \Delta s.
\]

Here we state Theorem A.

**Theorem A (Bohner et al. [4, Theorem 3.1]).** Assume that the inequalities

\[
\begin{align*}
(A1) \quad a - c/\gamma > 0, \\
(A2) \quad (\bar{f} - d\beta^{\omega})(a - c/\gamma) \exp(-2\bar{a}\omega) - \bar{b}d\bar{x}^{\omega} > 0,
\end{align*}
\]

hold. Then system (1) has at least one \(\omega\)-periodic solution.

In Theorem 1 we extend this theorem by eliminating condition (A1) and the negative term \(- (\bar{f} - d\beta^{\omega})(a - c/\gamma) \exp(-2\bar{a}\omega)\) from condition (A2).

In a particular case, if \(u_1(t) = \exp(x_1(t))\), \(u_2(t) = \exp(x_2(t))\) and \(\mathbb{T} = \mathbb{R}\), then system (1) reduces to the following system of ordinary differential equations:

\[
\begin{align*}
\dot{u}_1(t) &= (a(t) - b(t)u_1(t))u_1(t) - \frac{c(t)u_1(t)u_2(t)}{\bar{x}(t) + \beta(t)u_1(t) + \gamma(t)u_2(t)}, \\
\dot{u}_2(t) &= -d(t)u_2(t) + \frac{f(t)u_1(t)u_2(t)}{\bar{x}(t) + \beta(t)u_1(t) + \gamma(t)u_2(t)},
\end{align*}
\]

where \(a, b, c, d, x, \beta, \gamma \in C(\mathbb{R}, \mathbb{R}^2)\) are positive \(\omega\)-periodic functions.

Fan and Kuang in [8] obtained the following two theorems for the existence of positive periodic solutions of this system.

**Theorem B (Fan and Kuang [8, Theorem 3.2]).** Assume that:

\[
\begin{align*}
(B1) \quad \bar{a} > c/\gamma, \\
(B2) \quad (\bar{f} - d\beta^{\omega})(\bar{a} - c/\gamma)\bar{b}^{-1} \exp(-2\bar{a}\omega) - \bar{d}\bar{x}^{\omega} > 0.
\end{align*}
\]

Then, system (2) has at least one positive \(\omega\)-periodic solution.

**Theorem C (Fan and Kuang [8, Theorem 3.1]).** Assume that we have:

\[
\begin{align*}
(C1) \quad a^l a^l > c^u, \\
(C2) \quad (f^l - d^u\beta^{\omega})(a^l - (c^u/\gamma)) / b^u - c^u - d^u a^l > 0.
\end{align*}
\]

Then, system (2) has at least one positive \(\omega\)-periodic solution.

In Theorems 1 and 3 we reconsider these results and extend them by eliminating conditions (B1) and (C1), and the negative term \(- (\bar{f} - d\beta^{\omega})(a - c/\gamma) \exp(-2\bar{a}\omega)\) from condition (B2), and the negative term \(- (f^l - d^u\beta^{\omega})(c^u/\gamma) / b^u\) from condition (C2).
In the case of \( u_1(t) = \exp(x_1(t)) \), \( u_2(t) = \exp(x_2(t)) \) and \( \mathbb{T} = \mathbb{Z} \), system (1) reduces to the following discrete system:

\[
\begin{align*}
  u_1(k + 1) &= u_1(k) \exp \left( a(k) - b(k)u_1(k) - \frac{c(k)u_2(k)}{x(k) + \beta(k)u_1(k) + \gamma(k)u_2(k)} \right), \\
  u_2(k + 1) &= u_2(k) \exp \left( -d(k) + \frac{f(k)u_1(k)}{x(k) + \beta(k)u_1(k) + \gamma(k)u_2(k)} \right),
\end{align*}
\]

(3)

where the parameters are \( \omega \)-periodic sequence of positive real numbers with \( k, \omega \in \mathbb{Z} \) and \( \omega > 1 \).

We have the following theorem from [14] related to system (3).

**Theorem D (Zhang and Wang [14, Theorem 2.1]).** Assume that:

1. \( \tilde{a} > \frac{c}{\gamma} \).
2. \( \tilde{f}(k) + \tilde{d} \beta(k) - \tilde{c}(k) \tilde{b}^{-1} \exp(-2\tilde{a}\omega) - \tilde{d} \tilde{b} \tilde{a} > 0 \).

Then, system (3) has at least one positive \( \omega \)-periodic solution.

Similar to the above, in this theorem we eliminate condition (D1) and the negative term \( -(\tilde{f} - \tilde{d} \beta(k))(\tilde{c}(k) \tilde{b}^{-1} \exp(-2\tilde{a}\omega)) \) from (D2). The main tool in our proofs are a new estimation technique and a theorem from Gaines and Mawhin.

Let \( X \) and \( Y \) be two Banach spaces, \( L: \text{dom} \ L \cap X \to Y \) be a linear mapping and \( N: X \to Y \) be a continuous mapping. The mapping \( L \) is called a Fredholm mapping of index zero if there exist continuous projections \( P: X \to X \) and \( Q: Y \to Y \) such that \( \text{Im} \ P = \text{Ker} \ L \) and \( \text{Im} \ L = \text{Ker} \ Q = \text{Im}(I - Q) \). It follows that the mapping \( L|_{\text{dom} \ L \cap \text{Ker} \ P} : (I - P)X \to \text{Im} \ L \) has an inverse mapping, denoted by \( K_P \). For an open bounded subset \( \Omega \) of \( X \), the mapping \( N \) is called \( L \)-compact on \( \Omega \) if \( QN(\Omega) \) is bounded and \( K_P(I - Q)N : \Omega \to X \) is compact. Since \( \text{Im} \ Q \) is isomorphic to \( \text{Ker} \ L \), there exists an isomorphism \( J : \text{Im} \ Q \to \text{Ker} \ L \).

Here we state the Gaines–Mawhin theorem from [9, p. 40], which is a main tool in the proofs of our theorems.

**Theorem E (Continuation theorem).** Let \( L \) be a Fredholm mapping of index zero. Assume that \( N: \hat{\Omega} \to Z \) is \( L \)-compact on \( \hat{\Omega} \) with \( \hat{\Omega} \) open bounded in \( X \). Furthermore, assume:

(a) For each \( \lambda \in (0, 1) \), every solution \( x \) of \( Lx = \lambda Nx \) is such that \( x \notin \partial \Omega \),
(b) \( QN \neq 0 \) for each \( x \in \partial \Omega \cap \text{Ker} \ L \) and the Brouwer degree,

\[ \deg\{QN, \Omega \cap \text{Ker} \ L, 0\} \neq 0. \]

Then, the operator equation \( Lx = Nx \) has at least one solution in \( \text{dom} \ L \cap \bar{\Omega} \).

In Section 2, by taking advantage from Theorem E, we shall prove three existence theorems. In Section 3, we will see some applications of these theorems. Finally, in Section 4, we will discuss the weakness and superiority of these theorems.

## 2. Existence of periodic solutions

In this section, we shall prove some theorems related to system (1), which extend Theorems A, B, C and D with fewer conditions.

The proofs of the following lemmas can be found in [4] and [1,13], respectively.

**Lemma 1.** Let \( t_1, t_2 \in I_\omega \) and \( t \in \mathbb{T} \). If \( f: \mathbb{T} \to \mathbb{R} \) is \( \omega \)-periodic, then

\[ f(t) \leq f(t_1) + \int_{I_\omega} |f^A(s)| \Delta s \quad \text{and} \quad f(t) \geq f(t_2) - \int_{I_\omega} |f^A(s)| \Delta s. \]
Theorem 1. Assume that in system (1)
\[ (\ddot{f} - \ddot{\beta}a^{\mu})\exp(-2\bar{a}\omega) - \ddot{\alpha} z^{\mu} > 0. \] (4)

Then the system has at least one \( \omega \)-periodic solution.

Proof. By considering (4), there exists \( n_0 \in \mathbb{N} \) such that
\[ \forall n \geq n_0: \quad (\ddot{f} - \ddot{\beta}a^{\mu})\exp(-2\bar{a}\omega) - \ddot{\alpha} z^{\mu} > 0. \]

In order to apply Theorem E to system (1), let
\[ X = Z = (x_1, x_2)^T | x_1 \in C_{rd}, \text{ and } x_1(t + \omega) = x_1(t); \quad i = 1, 2, \]
\[ \| (x_1, x_2) \| = \max_{t \in [a, b]} |x_1(t)| + \max_{t \in [a, b]} |x_2(t)|, \quad (x_1, x_2) \in X \text{ or } Z, \]
\[ \text{dom } L = \{ x = (x_1, x_2)^T \in X | x_1 \in C_{rd}^1, i = 1, 2 \}. \]

It is easy to see that \( X \) and \( Z \) are both Banach spaces, if they are endowed with the above norm \( \| . \| \).

For \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in X \), we define
\[ N \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} N_1(t) \\ N_2(t) \end{bmatrix} = \begin{bmatrix} a(t) - b(t) \exp(x_1(t)) - \frac{c(t) \exp(x_2(t))}{x(t) + \beta(t) \exp(x_1(t)) + \gamma(t) \exp(x_2(t))} \\ -d(t) + \frac{f(t) \exp(x_1(t))}{x(t) + \beta(t) \exp(x_1(t)) + \gamma(t) \exp(x_2(t))} \end{bmatrix}, \]
\[ L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^A \\ x_2^A \end{bmatrix}, \quad P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\alpha \int_{a}^{t} x_1(t) dt \\ 1/\alpha \int_{a}^{t} x_2(t) dt \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}. \]

Then \( \text{dim Ker } L = 2 = \text{codim } \text{Im } L \). Since \( \text{Im } L \) is closed in \( Z \), \( L \) is a Fredholm mapping of index zero. It is easy to show that \( P \) and \( Q \) are continuous projections and \( \text{Im } P = \text{Ker } L, \text{ Im } L = \text{Ker } Q = \text{Im } (I - Q) \). Obviously, \( \text{QN} \) and \( K_{\beta}(I - Q)N \) are continuous. It can be shown that \( N \) is \( L \)-compact on \( \bar{\Omega} \), for every open bounded set, \( \Omega \subset X \).

Now we are in the position to build up the suitable open bounded subset \( \Omega \) of \( X \), and apply Theorem E to system (1) for the existence of periodic solutions. Consider the operator equation \( Lx = \lambda Nx, \lambda \in (0, 1) \):
\[ \begin{cases} x_1^A(t) = \lambda \left[ a(t) - b(t) \exp(x_1(t)) - \frac{c(t) \exp(x_2(t))}{x(t) + \beta(t) \exp(x_1(t)) + \gamma(t) \exp(x_2(t))} \right], \\ x_2^A(t) = \lambda \left[ -d(t) + \frac{f(t) \exp(x_1(t))}{x(t) + \beta(t) \exp(x_1(t)) + \gamma(t) \exp(x_2(t))} \right]. \] (5)
Suppose that \((x_1(t), x_2(t))^T \in X\) is a solution of system (5) for a certain \(\lambda \in (0, 1)\). By integrating (5) over the set \(I_\omega\), we obtain

\[
\bar{a} \omega = \int_{I_\omega} \left[ b(t) \exp(x_1(t)) + \frac{c(t) \exp(x_2(t))}{\alpha(t) + \beta(t) \exp(x_1(t)) + \gamma(t) \exp(x_2(t))} \right] \Delta t, \tag{6}
\]

\[
\bar{d} \omega = \int_{I_\omega} \left[ \frac{f(t) \exp(x_1(t))}{\alpha(t) + \beta(t) \exp(x_1(t)) + \gamma(t) \exp(x_2(t))} \right] \Delta t. \tag{7}
\]

It follows from (5)–(7) that

\[
\int_{I_\omega} |x_1^A(t)| \Delta t < \int_{I_\omega} a(t) \Delta t + \int_{I_\omega} \left[ b(t) \exp(x_1(t)) + \frac{c(t) \exp(x_2(t))}{\alpha(t) + \beta(t) \exp(x_1(t)) + \gamma(t) \exp(x_2(t))} \right] \Delta t = 2\bar{a} \omega, \tag{8}
\]

\[
\int_{I_\omega} |x_2^A(t)| \Delta t < \int_{I_\omega} d(t) \Delta t + \int_{I_\omega} \left[ \frac{f(t) \exp(x_1(t))}{\alpha(t) + \beta(t) \exp(x_1(t)) + \gamma(t) \exp(x_2(t))} \right] \Delta t = 2\bar{d} \omega. \tag{9}
\]

Since \((x_1(t), x_2(t))^T \in X\), there exist \(\xi_i, \eta_i \in I_\omega\), \(i = 1, 2\), such that

\[
x_i(\xi_i) = \min_{t \in I_\omega} x_i(t), \quad x_i(\eta_i) = \max_{t \in I_\omega} x_i(t). \tag{10}
\]

From (6) and (10) we obtain

\[
\bar{a} \omega \geq \int_{I_\omega} b(t) \exp(x_1(t)) \Delta t \geq b \omega \exp(x_1(\xi_1)).
\]

Thus \(x_1(\xi_1) \leq \ln(\bar{a} / \bar{b}) := M_1\). By considering Lemma 1, we have

\[
x_1(t) \leq x_1(\xi_1) + \int_{I_\omega} |x_1^A(t)| \Delta t \leq \ln(\bar{a} / \bar{b}) + 2\bar{a} \omega. \tag{11}
\]

From (7) and (10), we also have

\[
\bar{d} \omega \leq \int_{I_\omega} \frac{f(t) \exp(x_1(t))}{\alpha(t)} \Delta t \leq \exp(x_1(\eta_1))(f/2) \omega.
\]

Hence \(x_1(\eta_1) \geq \ln(\bar{d} / (f/2)) := M_2\). Again from Lemma 1, we get

\[
x_1(t) \geq x_1(\eta_1) - \int_{I_\omega} |x_1^A(t)| \Delta t \geq M_2 - 2\bar{d} \omega. \tag{12}
\]

Therefore, from (11) and (12) we have

\[
\max_{t \in I_\omega} |x_1(t)| \leq \max\{|M_1|, |M_2|\} := R_1.
\]

On the other hand, from (7), (10) and (11) we also have

\[
\bar{d} \omega \leq \int_{I_\omega} \frac{f(t) \exp(x_1(t))}{\gamma(t) \exp(x_2(t))} \Delta t \leq \frac{\exp(R_1)}{\exp(x_2(\xi_2))} \frac{(f/\gamma) \omega}{(f/\gamma) \omega}.
\]
Thus \(x_2(\xi_2) \leq R_1 + \ln(\frac{f(\xi_2)}{d}) := M_3\). By using Lemma 1, we obtain
\[
x_2(t) \leq x_2(\xi_2) + \int_{t_0}^t |x_2^2(t)|\Delta t \leq M_3 + 2\tilde{d}\omega. \tag{13}
\]

We need to obtain \(M_4\) such that \(x_2(t) \geq M_4\). Suppose there exist \(s, t \in I_0\) such that
\[
x_2(s) \geq x_1(t) - \ln(n_0),
\]
then from (10) and (12) we have
\[
x_2(\eta_2) \geq x_2(s) \geq x_1(t) - \ln(n_0) \geq x_1(\xi_1) - \ln(n_0) \geq M_2 - 2\tilde{a}\omega - \ln(n_0) := M_4^1.
\]

If \(s\) and \(t\) do not exist, then
\[
\forall s, t \in I_0: \quad x_2(s) < x_1(t) - \ln(n_0).
\]

It then follows from (6) and (10) that
\[
\tilde{a}\omega \leq \bar{b}\omega \exp(x_1(\eta_1)) + \frac{(c/\bar{x})\omega \exp(x_2(\eta_2))}{x_2(t)} - \ln(n_0) \geq \frac{1/n_0(c/\bar{x})\omega \exp(x_1(\eta_1))}{x_2(t)},
\]
and from Lemma 1, we have
\[
\exp(x_1(t)) \geq (\bar{a}/(\bar{b} + 1/n_0(c/\bar{x}))) \exp(-2\tilde{a}\omega). \tag{14}
\]
Then, from (7) and (9), we get
\[
\tilde{d}\omega \geq \frac{\tilde{f}(\tilde{x} + \tilde{a}/(\bar{b} + 1/n_0(c/\bar{x}))) \exp(-2\tilde{a}\omega)}{\tilde{a}\mu + \tilde{b}^\alpha(\tilde{a}/(\bar{b} + 1/n_0(c/\bar{x}))) \exp(-2\tilde{a}\omega) + \gamma^\mu \exp(x_2(\eta_2))},
\]
which implies
\[
x_2(\eta_2) \geq \ln((\tilde{f} - \tilde{a}\tilde{\beta})\tilde{x} + \tilde{a}/(\bar{b} + 1/n_0(c/\bar{x}))) \exp(-2\tilde{a}\omega) - \tilde{d}\tilde{x}^\mu - \ln(\tilde{d}\tilde{x}^\mu) := M_4^2.
\]

Hence, according to the above discussion we have
\[
x_2(\eta_2) \geq M_4 := \min\{M_4^1, M_4^2\}.
\]

This inequality, together with (13), leads to
\[
\max_{t \in I_0}\ |x_2(t)| \leq \max\{|M_3|, |M_4|\} + 2\tilde{d}\omega := R_2.
\]

Clearly \(R_1\) and \(R_2\) are independent of \(\lambda\). Consider the algebraic equations
\[
\begin{align*}
\tilde{a} \tilde{b} \exp(x_1) - \frac{1}{\omega} \int_{t_0}^t \frac{\mu \exp(x_2)}{\tilde{x}(t) + \tilde{b}(t) \exp(x_1) + \gamma(t) \exp(x_2)} \Delta t &= 0, \\
\tilde{d} \tilde{b} \exp(x_1) - \frac{1}{\omega} \int_{t_0}^t \frac{\tilde{f}(\tilde{x}(t) + \tilde{b}(t) \exp(x_1) + \gamma(t) \exp(x_2)) \exp(x_1)}{\tilde{x}(t) + \tilde{b}(t) \exp(x_1) + \gamma(t) \exp(x_2)} \Delta t &= 0,
\end{align*}
\tag{15}
\]

where \(\mu \in [0, 1]\) is a parameter and \((x_1, x_2) \in \mathbb{R}^2\). By similar arguments as above, we can easily show that each solution \((x_1^*, x_2^*)\) of (15) satisfies \(M_1 \leq x_1^* \leq M_2\) and \(M_4 \leq x_2^* \leq M_3\).

Let \(R = R_1 + R_2 + R_3\), where \(R_1\) and \(R_2\) are as defined above and \(R_3\) is sufficiently large such that, \(R_3 > |M_1| + |M_2| + |M_3| + |M_4|\). Now, we take \(\Omega = \{x = (x_1, x_2)^T \in X ||x|| < R\}\). This open bounded subset of \(X\) satisfies condition (a) of Theorem E. Let \(x \in \partial \Omega \cap \text{Ker } L = \partial \Omega \cap \mathbb{R}^2\), then \(x\) is a constant vector in \(\mathbb{R}^2\) with \(\sum_{j=1}^2 |x_j| = R\). From the definition of \(R\) we see that, \(QN(x) \neq 0\).
Finally, we prove that condition (b) of Theorem E is also satisfied. To this end, we define \( \phi: \text{dom } L \times [0, 1] \rightarrow X \) by

\[
\phi(x_1, x_2, \mu) = \begin{bmatrix}
\hat{a} - \hat{b} \exp(x_1) \\
\int \frac{\hat{a} - \hat{b} \exp(x_1)}{\alpha(t) + \beta(t) \exp(x_1) + \gamma(t) \exp(x_2)} f(t) \, dt \\
\mu \left[ -\int \frac{\hat{a} - \hat{b} \exp(x_1) + \gamma(t) \exp(x_2)}{\alpha(t) + \beta(t) \exp(x_1) + \gamma(t) \exp(x_2)} \, dt \right]
\end{bmatrix},
\]

where \( \mu \in [0, 1] \), is a parameter. By comparing \( \phi \) with (15), we see that \( \phi(x_1, x_2, \mu) \neq 0 \) on \( \partial \Omega \cap \text{Ker } L \). By using the invariance property under a homotopy of topological degree, and some direct calculations, we get

\[
\deg(J Q N u, \Omega \cap \text{Ker } L, (0, 0)^T) = 0,
\]

where \( \deg(\ldots) \) is the Brouwer degree and the isomorphism \( J \) of \( \text{Im } Q \) onto \( \text{Ker } L \) can be chosen to be the identity mapping, since \( \text{Im } Q = \text{Ker } L \). Hence, condition (b) of Theorem E holds, too. Thus, system (1) has at least one \( \omega \)-periodic solution. \( \square \)

**Remark 1.** Notice that both of systems (2) and (3) are special cases of system (1), Theorems B and D are special cases of Theorem A and conditions (B1) and (D1) are special cases of condition (A1). Thus, Theorems B and D follow without using conditions (B1) and (D1), respectively. Also, conditions (B2) and (D2) can be modified similar to condition (A2).

In the following, we have some more criteria for the existence of periodic solutions of system (1).

**Theorem 2.** Assume that in system (1) the following inequality holds:

\[
(1 - \hat{d}(\hat{b}/f))(\hat{a}/\hat{b}) \exp(-2\hat{a} \omega) - \hat{d}(\hat{a}/\hat{b}) \exp(-2\hat{a} \omega) > 0.
\]

Then, the system has at least one \( \omega \)-periodic solution.

**Proof.** By considering (16), there exists \( n_0 \in \mathbb{N} \) such that

\[
\forall n \geq n_0: (1 - \hat{d}(\hat{b}/f))(\hat{a}/\hat{b} + 1/n(c/\hat{x})) \exp(-2\hat{a} \omega) - \hat{d}(\hat{a}/\hat{b}) \exp(-2\hat{a} \omega) > 0.
\]

From (7), (10) and (14) we have

\[
\hat{d}(\hat{a}/\hat{b}) \geq \int_{t_0}^{\omega} (\hat{a}/\hat{b} + 1/n_0(c/\hat{x})) \exp(-2\hat{a} \omega) - \hat{d}(\hat{a}/\hat{b} + 1/n_0(c/\hat{x})) \exp(-2\hat{a} \omega) \exp(x_2(\eta_2)) \Delta t.
\]

By using Lemma 2, for an rd-continuous and positive function \( g \) we have

\[
\int_{t_0}^{\omega} \frac{\Delta t}{g(t)} \geq \frac{\omega^2}{\int_{t_0}^{\omega} g(t) \Delta t}.
\]

It follows from this inequality and (17) that

\[
\hat{d} \geq \frac{(\hat{a}/\hat{b} + 1/n_0(c/\hat{x})) \exp(-2\hat{a} \omega)}{(\hat{a}/\hat{b} + 1/n_0(c/\hat{x})) \exp(-2\hat{a} \omega) + (c/\hat{f}) \exp(x_2(\eta_2))},
\]

which implies

\[
x_2(\eta_2) \geq \ln((1 - \hat{d}(\hat{b}/f))(\hat{a}/\hat{b} + 1/n_0(c/\hat{x})) \exp(-2\hat{a} \omega) - \hat{d}(\hat{a}/\hat{b}) \exp(-2\hat{a} \omega) - \ln(\hat{d}(\hat{a}/\hat{b}))) + M_4^2.
\]

By using \( n_0 \) and \( M_4^2 \) as in the proof of Theorem 1, the theorem is proven. \( \square \)
For system (1) with $\mathbb{T} = \mathbb{R}$ we have another criterion as follows.

**Theorem 3.** Assume that in system (1) with $\mathbb{T} = \mathbb{R}$ we have:

$$ (\tilde{f} - \tilde{d} \beta^\mu)(a^l/b^\mu) - \tilde{d} x^u > 0. $$

Then the system has at least one $\omega$-periodic solution.

**Proof.** By considering (18), there exists $n_0 \in \mathbb{N}$ such that

$$ \forall n \geq n_0: (\tilde{f} - \tilde{d} \beta^\mu)(a^l/b^\mu + 1/n_0(c^u/x^l)) - \tilde{d} x^u > 0. $$

Suppose there exist $s, t \in I_\omega$ such that

$$ x_2(s) \geq x_1(t) - \ln(n_0). $$

Similar to the proof of Theorem 1 we can introduce $M^1_4$. If $x_2(s) < x_1(t) - \ln(n_0)$ for all $s, t \in [0, \omega]$, then we can obtain $M^2_4$ as follows. From (10) we see that $x_1(\xi_1) = 0$. Then, from the first equation of (5) we have

$$ a^l \leq a(\xi_1) = b(\xi_1) \exp(x_1(\xi_1)) + \frac{c(\xi_1) \exp(x_2(\xi_1))}{\alpha(\xi_1) + \beta(\xi_1) \exp(x_1(\xi_1)) + \gamma(\xi_1) \exp(x_2(\xi_1))}, $$

which gives

$$ a^l \leq b^u \exp(x_1(\xi_1)) + 1/n_0 \left(c^u/x^l\right) \exp(x_1(\xi_1)). $$

Thus,

$$ \exp(x_1(\xi_1)) \geq a^l/b^u + 1/n_0(c^u/x^l). $$

(19)

It follows from (7), (10) and (19) that

$$ \tilde{d} \geq \frac{\tilde{f}\exp(x_1(\xi_1))}{a^u + \beta^u \exp(x_1(\xi_1)) + \gamma^u \exp(x_2(\eta_2))} \geq \frac{\tilde{f} a^l/b^u + 1/n_0(c^u/x^l)}{a^u + \beta^u (a^l/b^u + 1/n_0(c^u/x^l)) + \gamma^u \exp(x_2(\eta_2))}. $$

Hence,

$$ x_2(\eta_2) \geq \ln((\tilde{f} - \tilde{d} \beta^\mu)(a^l/b^\mu + 1/n_0(c^u/x^l)) - \tilde{d} x^u) - \ln(\tilde{d}^\mu) := M^2_4. $$

By considering $n_0$ and $M^2_4$ as in the proof of Theorem 1, the proof is completed. □

**Remark 2.** Notice that the exponential term, $\exp(-2\tilde{d} \omega)$, disappeared in assumption (18), if we compare this with assumptions (4) and (16). This enables us to apply Theorem 3 for cases when $a(t)$ and $\omega$ are large and Theorems A, B, 1 and 2 are not applicable.

3. Applications

There are many discrete and continuous predator–prey systems of form (1) which can be investigated by our theorems for the existence of $\omega$-periodic solutions. Here, we give some examples which cannot be studied by the previous results stated in Theorems A, B, C and D.

The following theorem from [5] is helpful for computation of the delta-antiderivative of an rd-continuous function.

**Theorem F (Bohner and Peterson [5, Theorem 1.79]).** Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.

(i) If $\mathbb{T} = \mathbb{R}$, then

$$ \int_a^b f(t) \Delta t = \int_a^b f(t) dt, $$

where the integral on the right-hand side is the Riemann integral.
(ii) If \([a, b]\) consists of only isolated points, then
\[
\int_a^b f(t) \Delta t = \begin{cases} 
\sum_{t \in (a, b]} \mu(t) f(t) & \text{if } a < b, \\
0 & \text{if } a = b, \\
-\sum_{t \in [b, a)} \mu(t) f(t) & \text{if } b < a,
\end{cases}
\]
where \(\mu(t) := \sigma(t) - t\).

For example, in the case of \(\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}\), where \(h > 0\), according to the above theorem we have
\[
f^l = \min_{0 \leq k \leq \omega/h-1} f(hk), \quad f^u = \max_{0 \leq k \leq \omega/h-1} f(hk), \quad \bar{f} = \frac{1}{\omega} \sum_{k=0}^{\omega/h-1} f(hk)h,
\]
for every \(\omega\)-periodic sequence of positive real numbers, \(f\) with \(\omega > 1\).

**Example 1.** Let us consider system (1) with the following coefficient functions on the time scale \(\mathbb{T}\), \(a(t) = 1 + 0.1 \sin(\pi t), b(t)=0.2(2+\sin(\pi t)), c(t)=2t+(h/(1+n)) \cos(\pi t), d(t)=1+0.5 \sin(\pi t), f(t)=m(15+\cos(\pi t)), \gamma(t)=1 + 1.5 \sin(\pi t),\) 
\(\beta(t) = 1 + 0.5 \sin(\pi t), \gamma(t) = h \in \mathbb{R}^+\).

**Case** \(\mathbb{T} = \mathbb{R}\): For all \(h, n \in \mathbb{R}^+ \) and \(m > 1.84, m > 1.53 \) and \(m > 0.16\), by Theorems 1–3, the system exhibits at least one 2-periodic solution.

As we see from this example, Theorem 3 gives better results than Theorems 1 and 2. The reason is the elimination of the exponential term \(\exp(-4)(=\exp(-2\bar{\omega}))\), in the assumption of this theorem, compared with the corresponding assumptions of Theorems 1 and 2.

**Case** \(\mathbb{T} = \mathbb{Z}\): By Theorems 1 and 2, for all \(h, n \in \mathbb{R}^+ \) and \(m > 1.53\), discrete system (3) has at least one positive 2-periodic solution.

**Case** \(\mathbb{T} = 2\mathbb{Z}\): For all \(h, n \in \mathbb{R}^+ \) and \(m > 1.43\), by Theorems 1 and 2, system (1) has at least one 2-periodic solution.

The interesting point is that if we choose any three positive real numbers \(h, n, m\), then from Theorem 3.1 in [4] for all cases, Theorems 3.1 and 3.2 in [8] for the case \(\mathbb{T} = \mathbb{R}\), and Theorem 2.1 in [14] for the case \(\mathbb{T} = \mathbb{Z}\), no results can be obtained about the existence of periodic solutions of system (1).

**Example 2.** Consider system (1) with the coefficient functions on the time scale \(\mathbb{T}\), \(a(t) = 0.1 + 0.09 \sin(\pi t), b(t) = 0.1 + 0.09 \cos(\pi t), c(t)=h+h/(1+n) \sin(\pi t), d(t)=1+0.1 \sin(\pi t), f(t)=m(6 + \cos(\pi t)), \gamma(t)=1+0.5 \sin(\pi t), \beta(t) = 1 + 0.01 \cos(\pi t), \gamma(t) = h \in \mathbb{R}^+\).

**Case** \(\mathbb{T} = \mathbb{R}\): For all \(h, n \in \mathbb{R}^+ \), by Theorem 1 this system has at least one positive 2-periodic solution. However, for all \(h, n \in \mathbb{R}^+ \), Theorems 2 and 3 are not applicable.

**Case** \(\mathbb{T} = \mathbb{Z}\) and \(\mathbb{T} = 2\mathbb{Z}\): It follows from Theorems 1 and 2 that for all \(h, n \in \mathbb{R}^+ \), the system exhibits at least one 2-periodic solution.

Notice that for any two positive real numbers \(h\) and \(n\), Theorem 3.1 in [4] for all cases, Theorems 3.1 and 3.2 in [8] for the case \(\mathbb{T} = \mathbb{R}\), and Theorem 2.1 in [14] for the case \(\mathbb{T} = \mathbb{Z}\), may not imply any results for the existence of periodic solutions for system (1).

**Example 3.** We consider system (1) with the parameters on the time scale \(\mathbb{T}\), \(a(t) = 0.1 + 0.05 \sin(\pi t/3), b(t) = 0.02 + 0.01 \cos(\pi t/3), c(t) = n + (n/2) \cos(\pi t/3), d(t) = 1 + 0.1 \sin(\pi t/3), f(t) = m(6 + \cos(\pi t/3)), \gamma(t) = 0.3 + 0.1 \sin(\pi t/3), \beta(t) = 0.2 + 0.1 \cos(\pi t/3)\).

**Case** \(\mathbb{T} = \mathbb{R}\): If \(h, n \in \mathbb{R}^+ \) and \(m > 0.095, m > 0.067 \) and \(m > 0.09\), then by Theorems 1–3, system (1) has at least one 6-periodic solution, respectively.

**Case** \(\mathbb{T} = \mathbb{Z}\): By Theorems 1 and 2, for all \(h, n \in \mathbb{R}^+ \) and \(m > 0.149 \) and \(m > 0.067\) discrete system (3) has at least one positive 6-periodic solution, respectively.

**Case** \(\mathbb{T} = 2\mathbb{Z}\): In this case, for all \(h, n \in \mathbb{R}^+ \) and \(m > 0.149 \) and \(m > 0.067\), again by Theorems 1 and 2, at least one 6-periodic solution exists for system (1), respectively.
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by Theorems 1 and 2.

upon prey. The parameters densities of predators and preys, and the stability of enriched predator–prey systems.
The feeding rate describes the dynamic coupling between the predator and prey abundance, the densities of predators and preys, and the stability of enriched predator–prey systems.

In system (1), the parameter functions $a(t)$ and $f(t)$ play a positive role in the value of the predator rate of feeding upon prey. The parameters $c(t)$ and $\gamma(t)$ as well as $x(t)$ and $\beta(t)$, play a negative role in this value. On the other hand, the existence of periodic solutions denote the distribution of the predator and prey in the space, which is desirable from an ecological point of view.

Here, we discuss the effect of the above parameter functions on the existence of periodic solutions of system (1). First of all, from the predator equation in (1) we see that $\tilde{d} < \frac{f}{\bar{\beta}}$ is a necessary condition for the existence of $\omega$-periodic solutions of system (1) and conditions (A1), (B1), (C1) and (D1) seem to be unnecessary. The predator–prey system in Example (b) of [8] is a predator–prey system with Beddington–DeAngelis functional response on real numbers. For this system $\tilde{d} < \frac{f}{\bar{\beta}}$, but the system has no $\omega$-periodic solution. This means that $\tilde{d} < \frac{f}{\bar{\beta}}$ cannot be a sufficient condition for the existence of periodic solutions. In this paper, we have established the sufficient conditions closer to this necessary condition by eliminating some negative terms from the conditions (A2), (B2), (C2) and (D2), and removing the unnecessary conditions (A1), (B1), (C1) and (D1). In Theorems 1–3 we give some criteria for the existence of periodic solutions for system (1). These criteria are independent of each other, as we see from the examples in Section 3.

It should be noted that the value of $\tilde{a}$ is crucial in the assumptions of Theorems A, B and D. By conditions (A1), (B1) and (D1), it should be very large when $c(t)$ is large and $\gamma(t)$ is small. On the other hand, by conditions (A2), (B2) and (D2), it should be very small, because of the exponential term $\exp(-2\tilde{a}\omega)$. Thus, a narrow range of the parameter functions $c(t)$ and $\gamma(t)$ may be found for the existence of periodic solutions. However, Theorems 1–3, can be applied instead of Theorems A, B, C and D in a wider range, as we mentioned in Section 3.

In order to have periodic solutions by Theorems A, B and D, when $a(t)$, $c(t)$ and $\gamma(t)$ are given, we need to have $f(t)$ very large and $x(t)$, $\beta(t)$, $b(t)$ and $d(t)$ very small. By eliminating the negative term $-(\tilde{f} - \tilde{d} \bar{\beta} a) (c/\gamma) \exp(-2\tilde{a}\omega)$ from conditions (A2), (B2) and (D2), we make a wider range for these functions.

Finally, notice that Theorem 3 is obtained by eliminating condition (C1) and the negative term $-(\tilde{f} - \tilde{d} \bar{\beta} a) (c/\gamma) / b^\mu$ from condition (C2). As the exponential term $\exp(-2\tilde{a}\omega)$ is eliminated in Theorem 3, this theorem is also applicable to the large values of $\omega$ and $\tilde{a}$.

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References