PERIODIC SOLUTIONS FOR A SEMI-RATIO-DEPENDENT PREDATOR-PREY DYNAMICAL SYSTEM WITH A CLASS OF FUNCTIONAL RESPONSES ON TIME SCALES

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Abstract. In this paper we explore the existence of periodic solutions of a nonautonomous semi-ratio-dependent predator-prey dynamical system with functional responses on time scales. To illustrate the utility of this work, we should mention that, in our results this system with a large class of monotone functional responses, always has at least one periodic solution. For instance, this system with some celebrated functional responses such as Holling type-II (or Michaelis-Menten), Holling type-III, Ivlev, $mx$ (Holling type I), sigmoidal [e.g., Real and $mx^2/((A + x)(B + x))]$ and some other monotone functions, has always at least one $\omega$-periodic solution. Besides, for some well-known functional responses which are not monotone such as Monod-Haldane or Holling type-IV, the existence of periodic solutions is proved. Our results extend and improve previous results presented in [4], [10], [22] and [38].

1. Introduction. In this paper we investigate the existence of periodic solutions of a nonautonomous semi-ratio-dependent predator-prey dynamical system with some well-known functional responses, on time scales. The study of dynamic equations on time scales is an area of mathematics that tries to unify the study of differential and difference equations, and provides new powerful tools for exploring connections between the traditionally separated fields. It goes back to its founder Stefan Hilger [18] in 1988 but, in the past few years, it has found a considerable amount of interest and attracted many researchers' attention.

The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which may be an arbitrary closed subset of the real numbers. In this way, not only are the results related to the set of real numbers or to the set of integers but also those pertaining to more general time scales are obtained. In a short sentence, we may state that unification and extension are the two main features of the time scales calculus.

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The time scales calculus has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, economics, neural networks and social sciences. For the details, see the monographs of Aulbach and Hilger [2], Bohner and Peterson [5, 6], Lakshmikantham et al. [26] and the references therein.

For the convenience of the reader, we list some definitions and notations on the time scale calculus, as follows. These definitions and notations are common in the related literature [2, 4, 5, 6, 11, 26].

**Definition 1.** A time scale is an arbitrary nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$, the real numbers. The set $\mathbb{T}$ inherits the standard topology of $\mathbb{R}$. Let $\omega > 0$. Throughout this paper, the time scale $\mathbb{T}$ is assumed to be $\omega$-periodic, i.e., $t \in \mathbb{T}$ implies $t+\omega \in \mathbb{T}$.

Some examples of such time scales are

$$\mathbb{R}, \quad \mathbb{Z}, \quad \bigcup_{k \in \mathbb{Z}} [2k, 2k+1], \quad \bigcup_{k \in \mathbb{Z}} \bigcup_{n \in \mathbb{N}} \left\{ k + \frac{1}{n} \right\},$$

whose periods are any real number, any integer, any even integer, and any integer, respectively.

**Definition 2.** For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively.

In this definition we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum $t$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum $t$), where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say that $t$ is right-scattered, while if $\rho(t) < t$ we say that $t$ is left-scattered. A point that is simultaneously right-scattered and left-scattered is called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then $t$ is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then $t$ is called left-dense. A point that is simultaneously right-dense and left-dense is called dense.

**Definition 3.** A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sides limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions is shown by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

**Definition 4.** For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$ we define $f^\Delta(t)$, the delta-derivative of $f$ at $t$, to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a neighborhood $U$ of $t$ in $\mathbb{T}$ such that

$$\left| |f(\sigma(t)) - f(s)| - f^\Delta(t)|\sigma(t) - s| \right| \leq \epsilon |\sigma(t) - s|, \quad \text{for all} \quad s \in U.$$

Thus, $f$ is said to be delta-differentiable if its delta-derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are delta-differentiable and whose delta-derivative are rd-continuous functions is denoted by $C_{rd}^1 = C_{rd}^1(\mathbb{T}) = C_{rd}^1(\mathbb{T}, \mathbb{R})$.

**Definition 5.** A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$, for all $t \in \mathbb{T}$. Then we write

$$\int_t^s f(t) \Delta t := F(s) - F(r), \quad \text{for all} \quad s, t \in \mathbb{T}.$$

Thus, for the usual time scale $\mathbb{T} = \mathbb{R}$, rd-continuity coincides with the usual continuity in calculus. Moreover, every rd-continuous function on $\mathbb{T}$ has a delta-antiderivative. For more information about the above definitions and their related concepts, the reader is referred to [5, 6, 26].
The dynamical relationship between predators and their prey has long been one of the dominant themes in both ecology and mathematical ecology, in part due to its universal existence and importance. At first sight, these problems may appear to be mathematically simple. However, they are very challenging and complicated.

From the ecological point of view, the functional response of a predator is a key factor regulating the population dynamics of the predator-prey system. It describes the rate at which a predator catches its prey at different prey densities and can thus determine the efficiency of a predator in regulating prey populations (Murdoch and Oaten [30]). The monotone functional response curves may represent an increasing linear relationship (Type I), an increasing at a decelerating rate (Type II), or a sigmoidal increase in consumption rate (Type III), see Holling [19, 20].

From the microbial dynamics or chemical kinetics point of view, the functional response describes the uptake of substrate by micro-organisms. Microbiologists have often faced similar discussions in describing the growth of bacteria or protozoa on some substrate or in the study of competition through resource depletion (see Grover [16]).

In the classical Lotka-Volterra model, a linear functional response \( p_1(x) = mx \), (Holling type I in ecology [19] and Blackman in microbiology [3]), is assumed. Such a response function was almost simultaneously used by Lotka in 1925 in studying a hypothetical chemical reaction and by Volterra in 1926 in modeling a predator-prey interaction. In order to overcome the unboundedness of such a response function, the most famous response function \( p_2(x) = mx/(a + x) \) was proposed by Michaelis and Menten in 1913, now popularly referred to as a Michaelis-Menten or Holling type II (in ecology 1959 [19]) and Monod (in microbiology 1942 [28]) functional response. As all the names suggest, it has several different mechanistic explanations depending on the subfield of biology (enzyme kinetics, resource competition, predator-prey dynamics).

Another type, known as the Holling type-III response function, takes the form \( p_3(x) = mx^2/(a + x^2) \) and in general cases, the functional response \( p_4(x) = mx^n/(a + x^n) \), \( n > 2 \), Real [31] in ecology and Moser [29] in microbiology, is known as the sigmoidal response function. Similar types of such famous response functions can be found in Taylor [35] or Smith [32], e.g., \( p_5(x) = mx^2/((A + x)(B + x)) \). For details of the derivation of this functional response, see Collings [8].

Ivlev [23] in 1961 in ecology and Teissier [36] in 1936 in microbiology, derived the functional response of a consumer by arguing that the rate of consumption should depend on how hungry it is. This functional response takes the form \( p_6(x) = m(1 - \exp(-ax)) \).

For a nice explanation of the above functional responses in ecological models and derivation of these functions for predator-prey systems, see Hassell [17], Holling [19, 20], Ivlev [23], Grover [16], Taylor [35] and Collings [8].

The main characteristic of these types of functional responses (\( p_1, p_2, p_3, p_4, p_5 \) and \( p_6 \)) is that these functions are monotone. Many papers are devoted to these and some other monotone functional responses, in many predator-prey interaction models (see, e.g., Freedman [13], Hsu [21], Kooij and Zegeling [24], Kuang and Freedman [25], May [27], Sugie et al. [34], and the references cited therein).

As discussed in Freedman and Wolkowicz [14] there are experiments such as Andrews [1], Boon and Landelout [7], Edwards [9], Yang and Humphrey [37] and etc., that indicate nonmonotone responses occur at the microbial level: when the nutrient concentration reaches a high level an inhibitory effect on the specific growth
rate may occur. To model such an inhibitory effect, Andrews [1] suggested the function $p_7(x) = mx/(a + bx + x^2)$, called the Monod-Haldane (or Holling type-IV) functional response. In experiments on the uptake of phenol by pure culture of *Pseudomonas putida* growing on phenol in continuous culture, Sokol and Howell [33] proposed a simplified Monod-Haldane function of the form $p_8(x) = mx/(a + x^2)$, and found that it fits their experimental data significantly better and is simpler since it involves only two parameters.

Another type of phenomenon in population dynamics where nonmonotone response occurs, involves group defense. Group defense is a term used to describe the ability of prey to better defend or disguise themselves resulting in a decrease or even prevention of predation when their numbers are large enough. Freedman and Wolkowicz [14] first introduced the model for predator-prey interaction with group defense. For a nice explanation of this phenomenon and analysis of related models, see Freedman and Wolkowicz [14], Wolkowicz [39] and Zhu et al. [40].

The main part of our work in this paper is related to the monotone functional responses. In fact, our results can be applied to the functional responses which are monotone in a certain interval. This interval can be determined by some of the coefficient parameters of the predator-prey system, if the functional response is not a monotone function such as $p_7$ and $p_8$.

Recently, the question of existence of periodic solutions of a nonautonomous semi-ratio-dependent predator-prey system with functional responses for the case of discrete time has been studied by Fan et al. in [10]. The result of this work will appear in Section 2, Theorem D. The related continuous system has been investigated by Wang et al. in [38]. The main result of their work will be stated in Theorem B, in Section 2. Also, this continuous system with a special functional response, which is called the Leslie-Gower system, has been investigated by Huo et al. in [22]. Their result can be seen in Theorem C, in Section 2. Then, Bohner et al. in [4] considered this system on time scales and obtained Theorem A, in Section 2. This theorem covers Theorems B and D.

In this paper we improve the above theorems by dropping some crucial assumptions. In fact, we drop all of the assumptions on the parameters of the system and obtain sharp sufficient conditions for the existence of periodic solutions. The interesting point is that for the celebrated functional responses such as Holling type-II (Michaelis-Menten or Monod), Holling type-III, sigmoidal [e.g., Real and $mx^2/((A + x)(B + x))$, Ivlev, $mx$ (Holling type I)], the system has always at least one $\omega$-periodic solution. Moreover, for some widely recognized functional responses which are not monotone such as Monod-Haldane or Holling type-IV, as introduced before, we derive some sufficient conditions for the existence of periodic solutions of this system.

**Notation.** We use the following notations throughout this paper, related to the time scale $\mathbb{T}$ and the $\omega$-periodic function $f \in C_{rd}(\mathbb{T})$,

$$k = \min\{(0, \infty) \cap \mathbb{T}\}, \quad I_\omega = [k, k + \omega] \cap \mathbb{T}, \quad f^t = \min_{t \in I_\omega} f(t), \quad \bar{f} = \frac{1}{\omega} \int_{I_\omega} f(s) \Delta s.$$  

In this paper we consider the following dynamical system on the time scale $\mathbb{T}$:

$$x_1^\Delta(t) = a(t) - b(t) \exp(x_1(t)) - f(t, \exp(x_1(t))) \frac{\exp(x_2(t))}{\exp(x_1(t))},$$

$$x_2^\Delta(t) = c(t) - d(t) \frac{\exp(x_2(t))}{\exp(x_1(t))},$$

(1)
where $f(t, x)$ is a prey-dependent functional response, $a, c \in C_{rd}(T, \mathbb{R})$ with $\bar{a}, \bar{c} > 0$, and also $b, d \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ are $\omega$-periodic functions. The symbol $\Delta$ stands for the delta-derivative.

In Section 2, by taking advantage from a continuation theorem of Gaines and Mawhin, we prove our main results. Moreover, we compare our results with the previous results in both cases of differential and difference equations.

2. Existence of periodic solutions. In this section we explore the existence of periodic solutions of system (1). First, we collect some of the previous results about the existence of periodic solutions of systems (1), (2) and (3), then we state our results.

We start with a theorem of Bohner as follows.

**Theorem A** (Bohner et al. [4], Theorem 3.4]). Assume that in system (1)

(A1) The functional response $f : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}^+$ is rd-continuous and $\omega$-periodic with respect to the first variable and is differentiable with respect to the second variable, and $f(t, 0) = 0$, $\frac{\partial f(t, x) \partial x}{} > 0$, for all $t \in \mathbb{T}$ and $x > 0$, and also $\frac{\partial f(t, x) \partial t}$ is bounded with respect to $t$.

(A2) There exists an $\omega$-periodic function $g \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that $f(t, x) \leq g(t)x$, for all $t \in \mathbb{T}$ and $x > 0$.

(A3) $\exp((\bar{a} + |a| + \bar{c} + |c|)\omega) \leq \frac{\bar{b}d}{\bar{c}}$.

Then this system has at least one $\omega$-periodic solution.

In Theorems 1 and 2 we declare firmly that we can eliminate condition (A3), that is the assumption on the parameters $a, b, c$ and $d$ of system (1). Moreover, by replacing conditions (A1) and (A2) with weaker conditions, we make our theorems applicable to more famous functional responses (monotone and nonmonotone functional responses).

Wang et al. in [38] considered the following nonautonomous semi-ratio-dependent predator-prey system with functional responses:

$$
\dot{u}_1(t) = (a(t) - b(t)u_1)u_1 - f(t, u_1)u_2,
$$

$$
\dot{u}_2(t) = \left( c(t) - d(t)\frac{u_2}{u_1} \right) u_2,
$$

where $a, c \in C(\mathbb{R}, \mathbb{R})$ with $\bar{a}, \bar{c} > 0$, and also $b, d \in C(\mathbb{R}, \mathbb{R}^+)$ are $\omega$-periodic functions. They asserted the following theorem for the existence of periodic solutions of this system.

**Theorem B** (Wang et al. [38], Theorem 3.3].) Assume that in system (2) the following conditions hold.

(B1) The functional response $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\omega$-periodic with respect to the first variable and is differentiable with respect to the second variable, and $f(t, 0) = 0$, $\frac{\partial f(t, x) \partial x}{} > 0$, for all $t \in \mathbb{R}$ and $x > 0$, and also $\frac{\partial f(t, x) \partial t}$ is bounded with respect to $t$.

(B2) There exists a constant $c_0 > 0$ such that $f(t, x) \leq c_0x$, for all $t \in \mathbb{R}$ and $x > 0$.

(B3) $\frac{c_0x}{\omega} \leq \exp(2(\bar{a} + \bar{c})\omega) < 1$ and $a, c$ are positive functions.

Then this system has at least one positive $\omega$-periodic solution.

By applying our results to system (2), we see that condition (B3) can be eliminated from the assumptions of Theorem B. Furthermore, we weaken conditions
(B1) and (B2) in order to study this system with some well-known functional responses. In fact, we demonstrate that this predator-prey system with a large class of prey-dependent functional responses, has always at least one $\omega$-periodic solution.

In [22], Huo and Li considered system (2) with the functional response $f(t, x) = g(t)x$, which is called the Leslie-Gower system, obtained the following theorem for the existence of periodic solutions of this system.

**Theorem C** (Huo and Li [22], Theorem 2.1). Assume that in system (2) with the functional response $f(t, x) = g(t)x$, we have

(C1) $c(t) \geq d(t)$ for all $t \in \mathbb{R}$.

Then this system has at least one positive $\omega$-periodic solution.

We show that condition (C1) is not necessary for this theorem. In fact, from Theorem 1, we conclude that the nonautonomous Leslie-Gower system has at least one positive $\omega$-periodic solution.

Fan and Wang [10] investigated the following discrete predator-prey system:

$$
x_1(k + 1) = x_1(k) \exp \left( \frac{a(k) - b(k)x_1(k) - f(k, x_1(k))}{c(k) - d(k)} \frac{x_2(k)}{x_1(k)} \right),
$$

$$
x_2(k + 1) = x_2(k) \exp \left( \frac{c(k) - d(k)}{c(k) - d(k)} \frac{x_2(k)}{x_1(k)} \right),
$$

where the parameters are $\omega$-periodic sequences of positive real numbers with $k, \omega \in \mathbb{Z}$ and $\omega > 1$. They stated the following theorem related to this discrete system.

**Theorem D** (Fan and Wang [10], Theorem 2.1). Assume that the following assumptions hold on system (3).

(D1) The functional response $f : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{R}^+$ is $\omega$-periodic with respect to the first variable and is differentiable with respect to the second variable, and $f(k, 0) = 0, \frac{\partial f}{\partial x}(k, x) > 0$, for all $k \in \mathbb{Z}$ and $x > 0$, and also $\frac{\partial^2 f}{\partial x^2}(k, x)$ is bounded with respect to $k$.

(D2) There exists a positive $\omega$-periodic sequence $\{g(k)\}_{k \in \mathbb{Z}}$ such that $f(k, x) \leq g(k)x$, for all $k \in \mathbb{Z}$ and $x > 0$.

(D3) $\exp \left( \frac{2(b + c)}{d} \omega \right) < \frac{bd}{ac}$ and $a, c$ are positive functions.

Then this system has at least one positive $\omega$-periodic solution.

In order to get better results for system (3), similar to the previous theorems, we eliminate condition (D3) of this theorem, and weaken conditions (D1) and (D2). On the other hand, our results cover and extend Theorem 1 in [12]. The main tools in our proofs are a theorem from Gaines and Mawhin and a new estimation technique.

Let $X$ and $Y$ be two Banach spaces, $L : \text{dom } L \cap X \to Y$ be a linear mapping and $N : X \to Y$ be a continuous mapping. The mapping $L$ is called a Fredholm mapping of index zero if $\text{dom } L = \text{codim } \text{Im } L < \infty$ and $\text{Im } L$ is closed in $Y$. If $L$ is a Fredholm mapping of index zero, then there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $\text{Im } P = \text{Ker } L$ and $\text{Im } L = \text{Ker } Q = \text{Im } (I - Q)$. It follows that the mapping $L |_{\text{dom } L \cap \text{Ker } P} : (I - P)X \to \text{Im } L$ has an inverse mapping, denoted by $K_p$. For an open bounded subset $\Omega$ of $X$, the mapping $N$ is called $L$-compact on $\Omega$ if $QN(\Omega)$ is bounded and $K_p(I - Q)N : \Omega \to X$ is compact.

Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \to \text{Ker } L$.

The Gaines-Mawhin theorem from [15, p.40], is a remarkable tool which employs the topological degree theory to justify the existence of a solution for some operator equations.
Theorem E (Continuation theorem). Let $L$ be a Fredholm mapping of index zero. Assume that $N : \Omega \to Y$ is $L$-compact on $\Omega$, where $\Omega$ is an open bounded set in $X$. Furthermore, assume:

(a) For each $\lambda \in (0, 1)$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \notin \partial \Omega$.
(b) $QN x \neq 0$ for each $x \in \partial \Omega \cap \text{Ker} L$ and the Brouwer degree,

$$ \text{deg}\{QN x, \Omega \cap \text{Ker} L, 0\} \neq 0.$$

Then the operator equation $Lx = Nx$ has at least one solution in $\text{dom} L \cap \overline{\Omega}$.

Now, we prove a theorem related to system (1). This theorem says that, when we use the monotone prey-dependent functional responses bounded by some polynomials, then system (1) always has at least one $\omega$-periodic solution. For the proof of this theorem we need the following inequalities whose proofs can be found in [4].

Lemma 1. Let $t_1, t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is $\omega$-periodic, then

$$ f(t) \leq f(t_1) + \int_{t_1}^{t} |f^\Delta(s)| \Delta s \quad \text{and} \quad f(t) \geq f(t_2) - \int_{t_2}^{t} |f^\Delta(s)| \Delta s. $$

Theorem 1. Assume that in system (1) the following conditions hold.

(i) The functional response $f : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}^+$ is rd-continuous and $\omega$-periodic with respect to the first variable and is differentiable with respect to the second variable. $f(t, 0) = 0$ and $\frac{\partial}{\partial x} f(t, x) > 0$, for all $t \in \mathbb{T}$ and $x > 0$, and also $\frac{\partial}{\partial x} f(t, x)$ is bounded with respect to $t$.
(ii) There exist $m \in \mathbb{N}$ and $\omega$-periodic rd-continuous functions $a_i : \mathbb{T} \to \mathbb{R}^+ \cup \{0\}$, $i = 0, \ldots, m-1$ such that $f(t, x) \leq a_0(t)x^m + \ldots + a_{m-1}(t)x$, for all $t \in \mathbb{T}$ and $x \in \mathbb{R}^+$.

Then this system has at least one $\omega$-periodic solution.

Proof. In order to apply Theorem E to system (1), for the existence of periodic solutions, let

$$ X = Y = \{x = (x_1, x_2)^T \mid x_i \in C_{rd} \text{ and } x_i(t + \omega) = x_i(t); \quad i = 1, 2\}, $$

$$ |x| = \max_{t \in [a, b]} |x_1(t)| + \max_{t \in [a, b]} |x_2(t)|, \quad \text{for } x \in X(\text{or } Y), $$

$$ \text{dom} L = \{x \in X \mid x_i \in C_{rd} \text{ for } i = 1, 2\}. $$

It is easy to see that $X$ and $Y$ are both Banach spaces, if they are endowed with the above norm $\|\|$. For $x \in X$, we define

\[
N_x = \begin{bmatrix} N_1(t) \\ N_2(t) \end{bmatrix} = \begin{bmatrix} a(t) - \frac{f(t, \exp(x_1(t)))}{\exp(x_1(t))} & \exp(x_2(t)) - b(t) \exp(x_1(t)) \\ c(t) - d(t) \exp(x_1(t)) & \exp(x_2(t)) \end{bmatrix}, \\
L_x = \begin{bmatrix} x_1^\Delta \\ x_2^\Delta \end{bmatrix}, \quad P_x = Q_x = \begin{bmatrix} \frac{1}{\Delta} \int_{t}^{t+\Delta} x_1(t) \Delta t \\ \frac{1}{\Delta} \int_{t}^{t+\Delta} x_2(t) \Delta t \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]

It is not difficult to show that $\text{dim Ker } L = 2 = \text{codim Im } L$. Since $\text{Im } L$ is closed in $Y$, then $L$ is a Fredholm mapping of index zero. It is easy to show that $P$ and $Q$ are continuous projections and $\text{Im } P = \text{Ker } L$, $\text{Im } L = \text{Ker } Q = \text{Im } (I - Q)$. Obviously, $QN$ and $K_{P}(I - Q)N$ are continuous. It can be shown that $N$ is $L$-compact on $\overline{\Omega}$, for every open bounded set, $\Omega \subset X$.

Now, we are in the position to build up the suitable open bounded subset $\Omega$ of $X$, and apply Theorem E to system (1) for the existence of periodic solutions. Suppose that $x = (x_1, x_2)^T \in X$ is a solution of the operator equation $Lx = \lambda Nx$,.
for a certain $\lambda \in (0, 1)$. By integrating from both sides of this operator equation over the set $I_w$, we obtain

$$\tilde{a}\omega = \int_{I_w} \left[ \frac{f(t, \exp(x_1(t)))}{\exp(x_1(t))} \exp(x_2(t)) + b(t) \exp(x_1(t)) \right] \Delta t, \quad (4)$$

$$\tilde{c}\omega = \int_{I_w} \left[ d(t) \frac{\exp(x_2(t))}{\exp(x_1(t))} \right] \Delta t. \quad (5)$$

From the operator equation $Lx = \lambda Nx$, (4) and (5) we conclude that

$$\int_{I_w} |x_1^\Delta(t)| \Delta t < \int_{I_w} a(t) \Delta t \quad (6)$$

$$+ \int_{I_w} \left[ \frac{f(t, \exp(x_1(t)))}{\exp(x_1(t))} \exp(x_2(t)) + b(t) \exp(x_1(t)) \right] \Delta t = (\tilde{a} + |\tilde{a}|)\omega,$$

$$\int_{I_w} |x_2^\Delta(t)| \Delta t < \int_{I_w} c(t) \Delta t + \int_{I_w} \left[ d(t) \frac{\exp(x_2(t))}{\exp(x_1(t))} \right] \Delta t = (\tilde{c} + |\tilde{c}|)\omega. \quad (7)$$

We observe in light of $(x_1, x_2)^T \in X$ that there exist $\xi_i$ and $\eta_i \in I_w$, $i = 1, 2$, such that

$$x_i(\xi_i) = \min_{t \in I_w} x_i(t) \quad \text{and} \quad x_i(\eta_i) = \max_{t \in I_w} x_i(t). \quad (8)$$

Therefore, from (4) and (8) we obtain

$$\tilde{a}\omega \geq \int_{I_w} b(t) \exp(x_1(\xi_1(1))) \Delta t = \tilde{b}\omega \exp(x_1(\xi_1(1))),$$

from this inequality, (6) and Lemma 1, it follows

$$x_1(t) \leq x_1(\xi_1(1)) + \int_{I_w} |x_1^\Delta(t)| \Delta t \leq \ln(\tilde{a}/\tilde{b}) + (\tilde{a} + |\tilde{a}|)\omega := M_1. \quad (9)$$

We deduce from (5) and (8) that

$$\tilde{c}\omega = \int_{I_w} \left[ d(t) \frac{\exp(x_2(t))}{\exp(x_1(t))} \right] \Delta t \geq \tilde{d}\omega \frac{\exp(x_2(\xi_2))}{\exp(x_1(\eta_1))}. \quad (10)$$

In view of this and (9), we get

$$x_2(\xi_2) \leq \ln(\tilde{c}/\tilde{d}) + x_1(\eta_1) \leq \ln(\tilde{c}/\tilde{d}) + M_1,$$

and from (7) and Lemma 1, we have

$$x_2(t) \leq x_2(\xi_2) + \int_{I_w} |x_2^\Delta(t)| \Delta t \leq \ln(\tilde{c}/\tilde{d}) + M_1 + (\tilde{c} + |\tilde{c}|)\omega := M_2. \quad (11)$$

It follows from (4) and (8) that

$$\tilde{a}\omega \leq \tilde{b}\omega \exp(x_1(\eta_1)) + \exp(x_2(\eta_2)) \int_{I_w} f(t, \exp(x_1(t))) \frac{\exp(x_2(t))}{\exp(x_1(t))} \Delta t.$$

From this and assumption (ii) we have

$$\tilde{a} \leq \tilde{b} \exp(x_1(\eta_1)) + (\tilde{a}_0 \exp(x_1(\eta_1)))^{m-1} + \ldots + \tilde{a}_{m-2} \exp(x_1(\eta_1)) + \tilde{a}_{m-1} \exp(x_2(\eta_2)). \quad (12)$$

To proceed further, we require to obtain $M_3$ and $M_4$ such that $x_1(t) \geq M_3$ and $x_2(t) \geq M_4$, for all $t \in T$. To this end, we consider the following two cases.

**Case 1:** If $x_1(\eta_1) \leq x_2(\eta_2)$, then it follows from (9) and (12) that

$$\tilde{a} \leq (\tilde{b} + \tilde{a}_{m-1} + \tilde{a}_{m-2} \exp(M_1) + \ldots + \tilde{a}_0 \exp(M_1)^{m-1}) \exp(x_2(\eta_2)).$$
Thus, from this inequality, (7) and Lemma 1, for all $t \in \mathbb{T}$, we have

$$x_2(t) \geq \ln \left( \frac{\bar{a}}{\bar{b} + \bar{a}_{m-1} + \bar{a}_{m-2} \exp(M_1) + \ldots + \bar{a}_0 \exp(M_1)^{m-1}} \right) - (\bar{c} + |\bar{c}|) \omega := M_4^1. \quad (13)$$

From (10) we see that $x_2(\xi_2) \leq \ln(\bar{c}/\bar{d}) + x_1(\eta_1)$ which by considering (13) we have $x_1(\eta_1) \geq M_4 - \ln(\bar{c}/\bar{d})$. Again, utilizing Lemma 1, we find

$$x_1(t) \geq x_1(\eta_1) - \int_{L_\omega} |x_1^2(t)| \Delta t \geq M_4^1 - \ln(\bar{c}/\bar{d}) - (\bar{a} + |\bar{a}|) \omega := M_4^1. \quad (14)$$

**Case 2:** If $x_1(\eta_1) > x_2(\eta_2)$, then it follows from (12) that

$$\bar{a} \leq (\bar{b} + \bar{a}_{m-1}) \exp(x_1(\eta_1)) + \bar{a}_{m-2} \exp(x_1(\eta_1))^2 + ... + \bar{a}_0 \exp(x_1(\eta_1))^m. \quad (15)$$

We develop here some machinery which ensures that there exists a lower bound for $x_1(\eta_1)$. To accomplish this consider the function $g : \mathbb{R}^+ \to \mathbb{R}$ with $g(x) = b_0 x^m + b_1 x^{m-1} + ... + b_{m-1} x - b_m$ such that $b_i, i = 0, ..., m - 2$ are nonnegative, $b_{m-1}$ and $b_m$ are positive real numbers and $m \in \mathbb{N}$. Obviously, the function $g$ is increasing and $g(0) = -b_m < 0$, so $g$ has a unique root $x^*$ in $\mathbb{R}^+$. Then for $x \in \mathbb{R}^+$ and $g(x) \geq 0$, we have $x \geq x^*$. Hence it follows from (15) that $\exp(x_1(\eta_1)) \geq x^*$, where $x^*$ only depends on $\bar{a}, \bar{b}, m$ and $\bar{a}_i, i = 0, ..., m - 1$.

According to Lemma 1, for all $t \in \mathbb{T}$, we have

$$x_1(t) \geq x_1(\eta_1) - \int_{L_\omega} |x_1^2(t)| \Delta t \geq \ln(x^*) - (\bar{a} + |\bar{a}|) \omega := M_3^2. \quad (16)$$

Combining (5) and (8), we find

$$\bar{c} \omega = \int_{L_\omega} \left[ d(t) \frac{\exp(x_2(t))}{\exp(x_1(t))} \right] \Delta t \leq \frac{\exp(x_2(\eta_2))}{\exp(x_1(\xi_1))} \omega d \omega.$$ 

This inequality implies $x_2(\eta_2) \geq \ln(\bar{c}/\bar{d}) + x_1(\xi_1)$, substituting (16) into this expression gives, $x_2(\eta_2) \geq \ln(\bar{c}/\bar{d}) + M_3^2$. Hence, Lemma 1 tells us that, for all $t \in \mathbb{T}$,

$$x_2(t) \geq x_2(\eta_2) - \int_{L_\omega} |x_2^2(t)| \Delta t \geq \ln(\bar{c}/\bar{d}) + M_3^2 - (\bar{a} + |\bar{a}|) \omega := M_4^2. \quad (17)$$

Now we take

$$M_3 := \min \left\{ M_1^3, M_2^3 \right\} \quad \text{and} \quad M_4 := \min \left\{ M_1^4, M_2^4 \right\}. \quad (18)$$

Then, in light of (13), (14), (16) and (17) we observe that

$$x_1(t) \geq M_3 \quad \text{and} \quad x_2(t) \geq M_4, \quad \text{for all} \ t \in \mathbb{T},$$

as required. Consequently, from (9) and (11) we have

$$\max_{t \in L_\omega} |x_1(t)| \leq \max \{|M_1|, |M_3|\} := R_1, \quad \max_{t \in L_\omega} |x_2(t)| \leq \max \{|M_2|, |M_4|\} := R_2.$$ 

These inequalities illustrate that $x_1(t)$ and $x_2(t)$ are uniformly bounded in $L_\omega$. Obviously that $R_1$ and $R_2$ are independent of $\lambda$.

Now, consider the following equations in $\mathbb{R}^2$ simultaneously:

$$\bar{a} - \frac{\bar{f}(\exp(u))}{\exp(u)} \exp(v) - \bar{b} \exp(u) = 0,$$

$$\bar{c} - \frac{\bar{d} \exp(v)}{\exp(u)} = 0.$$ 

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where \( \tilde{f}(\exp(u)) := \tilde{f}(\cdot, \exp(u)) \). From assumption (i), it follows that (18) has a unique solution \((u^*, v^*)^T \) in \( \mathbb{R}^2 \). We choose \( R > R_1^* + R_2 \), so large that the unique solution (18) satisfies \( \|(u^*, v^*)^T\| = |u^*| + |v^*| < R \).

Now, we take \( \Omega = \{ x = (x_1, x_2)^T \in X \mid \|x\| < R \} \). This open bounded subset of \( X \) satisfies condition (a) of Theorem E. Let \( x \in \partial \Omega \cap \text{Ker } L = \partial \Omega \cap \mathbb{R}^2 \), then \( x \) is a constant vector in \( \mathbb{R}^2 \) with \( \sum_1^2 |x_i| = R \). From the definition of \( R \) we see that \( QN(x) \neq 0 \).

Finally, by some straightforward calculations we get

\[
\text{deg}(JQN, \Omega \cap \text{Ker } L, 0) = sgn \left\{ \tilde{c}(\tilde{b} \exp(u^*) + \frac{\partial}{\partial x} \tilde{f}(\exp(u^*)) \exp(v^*)) \right\} > 0,
\]

where \( \text{deg}(\ldots) \) is the Brouwer degree and the isomorphism \( J \) of \( \text{Im } Q \) onto \( \text{Ker } L \) can be chosen to be the identity mapping, since \( \text{Im } Q = \text{Ker } L \). Hence condition (b) of Theorem E holds, too. Thus, we have at last checked all of the assumptions of Theorem E. Therefore, there must exist at least one \( \omega \)-periodic solution for system (1).

\[\square\]

**Remark 1.** Theorem 1 tells us that assumption (A3) in Theorem A is not necessary. So, we do not have any assumptions on the parameters of this system. Also, according to assumptions (i) and (iii), for monotone functional responses bounded by some polynomials we can apply Theorem 1. We see that some well-known functional responses such as \( p_1(x, t) = m(t)x, p_2(x, t) = m(t)x/(A(t) + x) \), \( p_3(x, t) = m(t)x^2/(A(t) + x^2), p_4(x, t) = m(t)x^n/(A(t) + x^n) \), \( n > 2 \), \( p_5(x, t) = m(t)x^2/(A(t) + x)(B(t) + x) \) and \( p_6(x, t) = m(t)(1 - \exp(-A(t)x)) \), explained in the introduction, satisfy these conditions, where \( m(t), A(t) \) and \( B(t) \) are positive \( \omega \)-periodic rd-continuous functions. In fact, we should mention that, for a large class of monotone functional responses, system (1) has always at least one \( \omega \)-periodic solution.

**Corollary 1.** Assume that in semi-ratio-dependent predator-prey system (2) we have

(i) The functional response \( f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous and \( \omega \)-periodic with respect to the first variable and is differentiable with respect to the second variable, and \( \frac{\partial}{\partial x} f(t, x) > 0 \), for all \( t \in \mathbb{R} \) and \( x > 0 \), and also \( \frac{\partial}{\partial x} f(t, x) \) is bounded with respect to \( t \).

(ii) There exist \( m \in \mathbb{N} \) and \( \omega \)-periodic functions \( a_i : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}, i = 0, \ldots, m - 1 \) such that \( f(t, x) \leq a_0(t)x^m + \ldots + a_{m-1}(t)x \), for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^+ \).

Then this system has at least one positive \( \omega \)-periodic solution.

**Proof.** Consider system (1) with \( T = \mathbb{R} \), and use this change of variables \( u_1(t) := \exp(x_1(t)) \) and \( u_2(t) := \exp(x_2(t)) \). Then, apply Theorem 1. \[\square\]

**Corollary 2.** Assume that in discrete predator-prey system (3) we have

(i) The functional response \( f : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{R}^+ \) is a \( \omega \)-periodic sequence with respect to the first variable and is differentiable with respect to the second variable, and \( \frac{\partial}{\partial k} f(k, x) > 0 \), for all \( k \in \mathbb{Z} \) and \( x > 0 \), and also \( \frac{\partial}{\partial k} f(k, x) \) is bounded with respect to \( k \).

(ii) There exist \( m \in \mathbb{N} \) and nonnegative real value \( \omega \)-periodic sequences \( \{a_i(k)\}_{k \in \mathbb{Z}}, i = 0, \ldots, m - 1 \) such that \( f(k, x) \leq a_0(k)x^m + \ldots + a_{m-1}(k)x \), for all \( k \in \mathbb{Z} \) and \( x \in \mathbb{R}^+ \).

Then this system has at least one positive \( \omega \)-periodic solution.
Corollary 3. Consider system (1) with $T = \mathbb{Z}$, and use this change of variables $u_1(t) := \exp(x_1(t))$ and $u_2(t) := \exp(x_2(t))$. Then, apply Theorem 1.

**Remark 2.** In view of these corollaries we see that assumptions (B3) and (D3) in Theorems B and D are not necessary. Also, by using the weaker conditions (ii) and (ii) instead of (B2) and (D2), we make these theorems much more applicable. On the other hand, from Corollary 1 we conclude that assumption (C1) of Theorem C is not necessary, therefore the nonautonomous Leslie-Gower system has always at least one $\omega$-periodic solution.

Notice that Theorems 1, A, B, C and D are not applicable to functional responses which are not globally monotone in $\mathbb{R}^+$, such as Monod-Haldane or Holling type-IV.

Here, we have another theorem which is applicable for such functional responses.

**Theorem 2.** Assume that in system (1) the following conditions hold.

(i) The functional response $f : T \times \mathbb{R}^+ \to \mathbb{R}^+$ is rd-continuous and $\omega$-periodic with respect to the first variable and is differentiable with respect to the second variable, also $f(., x)$ is monotone function for $0 < x < \bar{a}/\bar{b}$, and $f(t, 0) = 0$
for all $t \in T$.

(ii) There exist $m \in \mathbb{N}$ and $\omega$-periodic rd-continuous functions $a_i : T \to \mathbb{R}^+ \cup \{0\}, i = 0, \ldots, m - 1$ such that $f(t, x) \leq a_0(t)x^m + \ldots + a_{m-1}(t)x$, for all $t \in T$ and $0 < x < \frac{a}{b} \exp((|a| + |\bar{a}|)\omega)$.

Then this system has at least one $\omega$-periodic solution.

**Proof.** First of all notice that, assumption (ii) in Theorem 1 is used only in (12). By considering (9) we see that (12) holds if assumption (ii) is satisfied for $0 < x < \bar{a}/\bar{b} \exp((|a| + |\bar{a}|)\omega)$. Thus, the above assumption (ii) can be used instead of assumption (ii) in the proof of Theorem 1.

Secondly, for every solution $(u^*, v^*)^T$ of (18) we have $\exp(u^*) < \bar{a}/\bar{b}$. From the above assumption (i) we see that $\bar{f}(., x)$ is a monotone function in the interval $[0, \bar{a}/\bar{b}]$. Thus, (18) has a unique solution in $\mathbb{R}^2$. Moreover, it follows from (i) that the Brower degree of $JQN$ is positive. Hence, (i) can be replaced by (i).

Therefore, we have the following result for the Monod-Haldane and Holling type-IV functional response, i.e., $p_\tau(t, x) = m(t)x/(A(t) + B(t)x + x^2)$ and $p_b(t, x) = m(t)x/(A(t) + x^2)$, respectively, where $m(t), A(t)$ and $B(t)$ are positive $\omega$-periodic rd-continuous functions.

**Corollary 3.** Consider system (1) with functional responses $p_\tau$ or $p_b$. If $A^i > (\bar{a}/\bar{b})^2$, then this system has at least one $\omega$-periodic solution.

See the following example, for an application of this theorem.

**Example 1.** Consider system (1) with the Monod-Haldane or Holling type-IV functional response on the time scale $\mathbb{T}$, $a(t) = 1 + 0.5 \sin(\pi t), b(t) = 2 + \cos(\pi t), c(t) + (n/2) \sin(\pi t), d(t) = m + (m/(1 + m)) \cos(\pi t), A(t) = 2 + \sin(\pi t), B(t) = h + (h/(1 + n)) \sin(\pi t)$ and $m(t) = k + (k/3) \cos(\pi t)$ and $n, m, h, k \in \mathbb{R}^+.$

For $\mathbb{T} = \mathbb{R}$ and $\mathbb{Z}$, for all $n, m, h, k \in \mathbb{R}^+$, by Theorem 2 (or Corollary 3) this system always has at least one 2-periodic solution.

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