Causal Precedence Formalism

by

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Abstract

In this talk, I describe recent work in collaboration with R.D. Sorkin. On lorentzian manifolds, we introduce a new causal relation, called the precedence relation, which is the smallest closed and transitive relation that contains the chronology relation I^+ as a subset. This new relation allows us to extend the techniques of global causal analysis to metrics which may not be differentiable or, in some cases, even invertible. We apply this relation, together with standard order-theoretic arguments, to give a simple proof of the familiar theorem that the space of causal curves joining fixed compact sets of a globally hyperbolic manifold is itself compact.

Causal Precedence Formalism

In riemannian manifolds, metrical properties play an important role not only at the obvious geometric level, but also at the topological level and therefore at the level of global structure as well. Consider for example the Hopf-Rinow theorem [1], that metric completeness is equivalent to geodesic completeness and implies existence of minimal-length curves joining any pair of points. However, in the case of lorentzian manifolds, the topology and the global structure are in certain ways more directly related to the causal order of points than to the full lorentzian metric. With this in mind, consider the following theorem, which is fundamental to the field of global causal analysis and does lead to something of an analogue of Hopf-Rinow (namely, the existence of "fastest causal curves" in globally hyperbolic manifolds).

Theorem 1: Let $\mathcal{A}$ and $\mathcal{B}$ be compact subsets of a globally hyperbolic

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lorentzian manifold $\mathcal{M}$. Then the space $C(\mathcal{A}, \mathcal{B})$ of causal curves joining $\mathcal{A}$ to $\mathcal{B}$ is bi-compact (compact and Hausdorff).

The proof usually given for this theorem is arguably not conceptually simple.[1] Perhaps the reason for this is that the proof involves techniques of *differential topology*, to borrow from the title of Ref. [2], which are often more suited to (and often borrowed from) the riemannian case.

In the formulation of global causal analysis, one may thus feel that it should be possible to replace certain differential topological constructs by notions which are more appropriate to lorentzian manifolds, and so more elementary. Here we have in mind order-theoretic notions which capture the causal order of a lorentzian manifold. R.D. Sorkin and myself, in recent work [3], have reformulated certain basic concepts in causal theory using order-theoretic ideas instead of differential-topological ones. In accordance with the idea that the order-theoretic approach may be more elementary, we have been able to use this formulation to give a simple and direct proof of the compactness theorem quoted above, which is valid even in cases where the lorentz metric may be only C^0 (an extension beyond standard proofs, such as that given in [4]). In what follows, I will attempt to provide an outline of the key ideas and a sketch of the logic entering into the compactness proof, although details cannot be provided for reasons of space limitations. It is intended that the interested reader will refer to [3] for details and for proofs.

Herein, $\mathcal{M}$ is taken to be a C^0-lorentzian manifold — a manifold is C^k-lorentzian iff it is time-orientable, has a C^{k+1} differentiable structure, and possesses a C^k lorentzian metric. We make use of the concept of *relations* on $\mathcal{M}$; these are just subsets of $\mathcal{M} \times \mathcal{M}$. An example is the chronology relation, which contains the ordered pair (p, q) iff $q \in I^+(p)$. Here we say that there is a timelike curve from p to q, and we write $q \in I^+(p)$ or $p < q$, iff there is a C^0, piecewise C^1 curve with everywhere future-timelike tangent that begins at p and ends at q.

We introduce a new relation on $\mathcal{M}$, which we call the *precedence relation*. We

[1] Roger Penrose, in his SIAM monograph [2] of 1972, commences his proof of this theorem (for what we call C^2-lorentzian manifolds) with the words, "I do not have a nice simple argument, but I feel sure that one must exist (exercise: find one!)."

define this relation as the smallest closed and transitive relation which contains the chronology relation as a subset. If (p, q) is in this relation, we write $q \in K^+(p)$ or $p \prec q$ (and we say that p *precedes* q — there is an analogous definition for $K^-(p)$ as well). The closure condition means that if $p_n \prec q_n$ for convergent sequences $p_n \to p$, $q_n \to q$, then $p \prec q$, while transitivity is the statement that $x \prec y$ and $y \prec z \Rightarrow x \prec z$. Just as there is a "relative" chronology relation (which we denote by $<_{\mathcal{O}}$, so $p <_{\mathcal{O}} q \Leftrightarrow q \in I^+(p, \mathcal{O})$), one may also define "precedence relative to the subset $\mathcal{O} \subseteq \mathcal{M}$." Specifically, we define this relation to be the smallest closed and transitive relation on $\mathcal{O}$ that contains the $<_{\mathcal{O}}$ relation, and if (p, q) is in this relation, we write $p \prec_{\mathcal{O}} q$ or $q \in K^+(p, \mathcal{O})$. Based on this, we now define a *causal curve* to be the image of a continuous, locally increasing map (called a *causal path*) $\gamma : [0, 1] \to \mathcal{M}$.[2] By locally increasing, we mean that for every open $\mathcal{O} \ni \gamma(t)$, there is a neighbourhood of t in $[0, 1]$ within which $t' < t'' \Rightarrow \gamma(t') \prec_{\mathcal{O}} \gamma(t'')$.

An open set $\mathcal{O} \subseteq \mathcal{M}$ is said to be *K-causal* if $\prec$ induces a partial order on $\mathcal{O}$; that is, if the relation $\prec$ is *asymmetric* on $\mathcal{O}$ (so that, if both $p \prec q$ and $q \prec p$, then $p = q$). This is a causality condition, quite analogous to the familiar strong causality condition (but note that K-causality is stronger, and can be shown to imply strong causality).[3] A key point is that on K-causal manifolds, the definition of causal curves can be considerably simplified, by using standard results from order theory. Namely, these standard results give

Standard Theorem 2: Let Γ be: (a) a compact, connected, topological space containing a countable dense subset; (b) a linearly ordered set (with some order $\leq$), containing both a minimum and a maximum element with respect to the order; and (c) such that the order-closed intervals $\{y | x \leq y \leq z\}$ are topologically closed for all $x, z \in \Gamma$. Then Γ is simultaneously order-

[2] It is convenient here that the parameter of the curve should take values in $[0, 1]$. For other applications, it may be convenient to consider open or half-open domains instead.

[3] Is K^+ equivalent to Seifert's J_s^+? Is K-causality equivalent to stable causality? These open questions and arguments that might suggest affirmative answers were brought to the author's attention by R. Low in private communications.

and topologically isomorphic to the real interval $[0,1]$.

(A set is linearly ordered by $\leq$ iff for every p, q in it either $p \leq q$ or $q \leq p$.) Using this lemma and the available partial order $\prec$, we may show that our definition of causal curves can be simplified on K-causal manifolds. Although the following is actually a lemma that can be derived by applying Standard Theorem 2 to our general definition of causal curves (see [3]), for purposes of this note we take it as a definition:

"Definition" 3: On a K-causal manifold, a *causal curve* is a compact connected set linearly ordered by $\prec$.

We will say that a set $\mathcal{O}$ is *globally hyperbolic* iff it is K-causal and for every pair $p, q \in \mathcal{O}$, the set $K^+(p) \cap K^-(q)$ is compact and contained in $\mathcal{O}$. In [3] it is shown that in the case of C^2-lorentzian manifolds, this definition is equivalent to the definition used in standard references (such as [4]), but the definition given here represents an extension that includes C^0-lorentzian manifolds.

Let $\mathcal{A}$ and $\mathcal{B}$ be compact subsets of a globally hyperbolic C^0-lorentzian manifold $\mathcal{M}$. Let $C(\mathcal{A}, \mathcal{B})$ be the space of all causal curves ("Definition" 3 suffices here since the space is automatically K-causal) from $\mathcal{A}$ to $\mathcal{B}$. We topologise this space by first considering the Vietoris topology on the spaces $2^{\mathcal{K}}$ of *all* closed non-empty subsets of the compact sets $\mathcal{K} := K^+(x) \cap K^-(y)$, for $x, y \in \mathcal{M}$. The open sets of the Vietoris topology are those generated by taking arbitrary unions of the sets $\mathcal{V}(\mathcal{O}; \mathcal{O}_1, \ldots, \mathcal{O}_n)$, where $\mathcal{O}, \mathcal{O}_1, \ldots, \mathcal{O}_n$ are open subsets of $\mathcal{M}$ and where $\gamma \in \mathcal{V}(\mathcal{O}; \mathcal{O}_1, \ldots, \mathcal{O}_n)$ iff γ is a closed subset of $\mathcal{M}$ that is contained in $\mathcal{O}$ and meets each of $\mathcal{O}_1, \ldots, \mathcal{O}_n$. We refer to limits of nets taken with respect to this topology as *Vietoris limits*. The advantage of this approach is that we may use the following standard theorem.

Standard Theorem 4: The space $2^{\mathcal{X}}$ of closed subsets of the bi-compact (*i.e.* compact and Hausdorff) space $\mathcal{X}$ is bi-compact.

In particular, let Γ_α be a net of curves joining compact sets $\mathcal{A}$ to $\mathcal{B}$. Let p (*resp.* q) be an accumulation point of the initial (*resp.* final) endpoints, and choose any $x \in I^-(p)$, $y \in I^+(q)$. Then $\mathcal{K} := K^+(x) \cap K^-(y)$ is compact (by global hyperbolity) and eventually contains each curve in the net (or at least in a subnet). By Theorem 4 applied to the space $2^{\mathcal{K}}$, the net of curves has a convergent subnet with Vietoris limit in $2^{\mathcal{K}}$. It is an easily proved result that this limit, call it Γ, is a compact connected set, since it is a Vietoris limit of a net whose elements are themselves compact connected sets (in particular, causal curves).

To establish that Γ is linearly ordered by $\prec$, let $r, s \in \Gamma$ and let $\mathcal{O}_1 \ni r$ and $\mathcal{O}_2 \ni s$ be open in $\mathcal{M}$. Since Γ is a Vietoris limit, we have $\Gamma_\alpha \in \mathcal{V}(\mathcal{M}; \mathcal{O}_1, \mathcal{O}_2)$ for α large enough. As the neighbourhoods $\mathcal{O}_1$ and $\mathcal{O}_2$ shrink down, we may extract convergent sequences $r_n, s_n \in \Gamma_n$ such that $r_n \to r$ and $s_n \to s$. If the Γ_α are curves in the K-causal manifold $\mathcal{M}$, then "Definition" 3 implies that either $r_n \prec s_n$ or vice versa (since each Γ_α is linearly ordered), whence by closure $r \prec s$ (or vice versa), whence Γ is linearly ordered by $\prec$. Furthermore, it can be shown that $p \in \mathcal{A}$ and $q \in \mathcal{B}$ are the infimum and supremum with respect to this order on Γ. Combining all these results, we have that Γ is a compact, connected, linearly ordered set — thus, by "Definition" 3, a causal curve — joining $\mathcal{A}$ to $\mathcal{B}$, whence $\Gamma \in C(\mathcal{A}, \mathcal{B})$. Thus, we have shown that every net in $C(\mathcal{A}, \mathcal{B})$ has a convergent subnet, whence $C(\mathcal{A}, \mathcal{B})$ is compact. Lastly, it is Hausdorff, and therefore bi-compact, since it is immediate that if distinct curves can be separated by open sets in $2^{\mathcal{K}}$ (which is Hausdorff by Theorem 4), they will also be so separated in $C(\mathcal{A}, \mathcal{B})$.

References:

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[4] Hawking, S.W., and Ellis, G.F.R., *The Large Scale Structure of Spacetime* (Cambridge, 1973).