A Positive Mass Theorem Based on the Focusing and Retardation of Null Geodesics

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Abstract

A positive mass theorem for General Relativity Theory is proved. The proof is 4-dimensional in nature, and relies completely on arguments pertaining to causal structure, the basic idea being that positive energy-density focuses null geodesics, and correspondingly retards them, whereas a negative total mass would advance them. Because it is not concerned with what lies behind horizons, this new theorem applies in some situations not covered by previous positivity theorems. Also, because geodesic focusing is a global condition, the proof might allow a generalisation to semi-classical gravity, even though quantum violations of local energy conditions can occur there.

I: Introduction

Fifteen years ago, the construction and proof of a suitable positive mass theorem for General Relativity remained one of the major unsolved problems of the theory. The first proof of such a theorem was given by Schoen and Yau,[1] as the culmination of a considerable effort by several workers. The proof they obtained was mathematically sophisticated and not easily accessible to many physicists. Shortly thereafter, Witten[2] found a method of proof which could be given in easily understood terms, though the rigourous mathematical justification of the method had to await subsequent work by other authors.[3,4]

One might pose two questions concerning these efforts, the first pertaining to the generality of the theorems proved. It would seem unreasonable to expect that a theorem could be proved even when the spacetime is nakedly singular (after all, there is the negative mass Schwarzschild solution) or when the non-gravitational energy of the matter present is mostly negative (for one expects gravitational binding energy to be negative as well, so how could the sum be positive?). Yet one still might ask whether the assumptions required for the existing proofs could be weakened or altered, say by relaxing the condition of local positivity of energy or by finding a proof which holds on spacetimes that lack complete spacelike hypersurfaces or do not admit Sen-Witten spinors. It will be seen that, in fact, all these conditions can be relaxed, although other conditions (such as the so-called generic condition, a particularly mild assumption in the present context) have to be added. This brings one to the second and related question, which is more æsthetic in nature. If one is interested in identifying those features of an asymptotically flat spacetime which lead to a positive total mass, one might like to have a theorem whose proof relies as clearly and as deeply as possible on physically intuitive concepts. The second question, then, is whether there exist positive mass theorems which make more explicit use of what one might feel are the physical mechanisms that act to ensure that the mass is positive. The theorem presented herein is hoped to be of this nature.

The idea is to show that negative mass spacetimes are inherently inconsistent. The key to the contradiction is the behaviour of null geodesics in gravitational fields. In 1964, Shapiro[5] noticed that light rays emitted at an infinitely distant point and passing near a positive mass should be delayed with respect to rays passing farther from the mass, and that this effect could be used as the basis for what is today a very accurate test of General Relativity.[6] For present purposes, it is the complementary observation that is of primary interest, specifically that the rays that pass infinitely far from a positive mass "source" are *infinitely advanced* with respect to rays passing nearby the source. In the negative mass case, as one would expect, the distant rays are *infinitely delayed* with respect to the nearby ones. From this effect one may infer that, in the negative mass case, there are points in arbitrarily distant future regions of spacetime which are not causally related

to other points in arbitrarily distant past regions. In any such situation, there will exist null geodesics which traverse the boundary between causally related points and unrelated ones, and in this case it will be possible to show that one such null geodesic does so for an infinite affine distance. (This geodesic will pass nearby or through the source, even though its existence is a consequence of the behaviour of null geodesics passing arbitrarily far from it.)

Yet it is equally possible to argue that this null geodesic cannot exist. On one hand, it cannot have a pair of conjugate points since, if it did, it necessarily could not remain on the boundary between causally related and unrelated points for more than a finite affine distance. (Recall that conjugate points are pairs of points where some Jacobi field — which measures the separation of neighbouring curves of a congruence — vanishes.[7,8] When this happens, *focusing* of the congruence is said to have occurred. See fig. (1).) But on the other hand, the existence of pairs of conjugate points along every null geodesic is a consequence of the *focusing theorems*[9,10,11] which play such a central role in the proofs of the singularity theorems,[7,8] and which are now known to hold under very general conditions. One thus obtains the desired contradiction in any situation in which a focusing theorem can be proved.

The conditions assumed by most focusing theorems express two ideas. One is that each null geodesic should encounter curvature which produces a converging or shearing effect on the cross-section of a congruence of neighbouring null geodesics; * the other is that it should not encounter enough negative energy matter to prevent the ensuing appearance of conjugate points. It is also necessary that the null geodesic of interest should traverse a singularity-free region of the spacetime, which in particular guarantees that it can be extended far enough to develop conjugate points. This is insured by requiring that the subset of spacetime which is outside all event horizons be globally hyperbolic. Incidentally, this form of causality condition means that the positivity theorem will apply in spacetimes with causality violations, provided those violations are hidden behind black hole horizons (or, for that matter, inside white holes). The only other conditions that will be imposed pertain to asymptotic flatness. Asymptotic flatness is first of all used in order that the notion of energy may sensibly be defined, but the correlative existence of geometrical structures at infinity will also prove to be an effective technical tool in the proof.

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* Of course, a generic curvature tensor will produce both shearing and convergence (or divergence if the energy density is negative), but even pure local shearing ("tidal shearing") will produce convergence at second order. Thus the condition means physically that the geodesic passes through a converging gravitational lens.

The theorem thus obtained is quite different in character from the earlier theorems, which have a distinct $3+1$ flavour. ** With those theorems, one relies on the conservation of energy to argue that, since every hypersurface that asymptotes to spatial infinity will carry the same mass, the positivity proof may proceed entirely on a single hypersurface; and, in fact, the arguments used are most often couched in terms of Cauchy data for a spacetime, rather than in terms of the spacetime itself. The present proof, in contrast, is global in nature and is quite reminiscent of the sort of argument used to prove the singularity theorems. It is noteworthy that both primary ingredients of the proof, namely these global techniques and knowledge of the time delay/advance effect, were available long before any of the existing positivity theorems were proved.

The fact that positive energy is derived herein from focusing may be important in connection with the question of the semi-classical stability of Minkowski space. The usual argument for *classical* stability (and also stability against quantum tunnelling) relies on the positivity of mass as proved for stress-energy tensors that obey local energy conditions or energy conditions integrated over spacelike surfaces. Such energy conditions in general fail for quantum matter (or quantum gravitons) coupled to a classical metric, and so the usual type of argument fails semi-classically. However, it is conceivable (and presently very much an open issue) that energy conditions integrated along null geodesics may hold in appropriate circumstances, even in the presence of quantum matter. If such semi-classical focusing theorems could be established, they would lead to an extension of the positive energy theorem presented herein which would be valid in the context of the semi-classical Einstein equation.

In the presentation that follows, Section II.1 states the theorem in a general form, relying on the assumption that null geodesics focus; *i.e.* develop conjugate points. Section II.2 discusses sufficient conditions under which every infinite null geodesic will in fact focus, these being the conditions alluded to above. Section III discusses the behaviour of null geodesics propagating in the asymptotic regions of spacetime. The utility for this purpose of solutions of the Hamilton-Jacobi equation is the subject of Section III.1, while Section III.2 contains the calculations themselves. One can show that an important property of null geodesics from a fixed initial endpoint at past null infinity is governed by the sign of a certain combination of the asymptotic metric coefficients — if this quantity is positive, the geodesics form a sequence receding into the distant future and, if it is negative, they form a sequence converging upon spatial infinity. The quantity that governs this behaviour is a

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** This flavour is less pronounced in the case of the Witten proof which, although still proceeding on a spacelike hypersurface, can be presented in a form where all the basic quantities which appear (including the superpotential for the ADM energy) are 4-dimensional in character[4].

component of the spacetime's 4-momentum, as shown in the appendix. Section IV gives the formal proof of the positivity theorem. Section V points out some purposes for which the present theorem is more general than the others, and contains some speculations on possible further generalisations of the result, particularly concerning the cases of null and zero 4-momentum.

Many of the ideas herein grew out of discussions following a seminar given by one of the authors (R.P.) at Syracuse University in the Spring of 1990. Subsequently, two different but related approaches to proving the result described in this paper were found and have been reported elsewhere.[12,13] This work is intended to give a full proof of the results stated in those previous works and to generalise them. For an early statement of some of the ideas behind this proof see Ref. (14) and the related ideas in Ref. (15); for an alternative development of part of the work, see Ashtekar and Penrose.[16]

The conventions adopted are those of the Landau-Lifshitz Spacelike Convention used in Refs. (7, 17, 18). Specifically, the signature of the metric is taken to be $(-, +, +, +)$, and the Ricci tensor is defined so that it is the positive sign of timelike and null components of R_{ab} which leads to focusing (see Subsection II.2 for a more complete discussion). Small roman letters are used for abstract indices, while small greek indices refer to components in chosen coordinates, usually some quasi-Cartesian system near infinity as in Section III. *Spacetime* means a manifold (Hausdorff, paracompact, 4-dimensional except where higher-dimensional generalisations are discussed) with a Lorentzian metric; it may possess singularities (provided they are suitably attired; again, see subsequent discussion). A *causal curve* in spacetime will generally refer to a curve-without-parametrisation — the image of a continuous Lipshitz map from an interval of $\mathbb{R}$ into spacetime (the Lipshitz condition is automatic for causal curves). The *chronological (resp. causal) future* $I^+(\mathcal{X})$ (resp. $J^+(\mathcal{X})$) consists of all points joined to the point $\mathcal{X}$ or to points in the set $\mathcal{X}$ by future timelike (*resp.* future causal — future timelike or null) curves from $\mathcal{X}$. Single points are considered curves of zero length, and therefore $p \in J^+(p)$ but $p \notin I^+(p)$. Lastly, $\mathcal{A} \backslash \mathcal{B}$ denotes the complement of the set $\mathcal{B}$ in the set $\mathcal{A}$, *i.e.* those points of $\mathcal{A}$ that are not in $\mathcal{B}$.

II: Focusing and Positivity

II.1: Statement of the Theorem:

The result to be proved refers to spacetimes which are asymptotically flat in a sense to be made precise in Section III (see especially equations (III.2.1) and (III.2.2) and the discussion following equation (A.3) in the appendix). The spacetime manifold $\mathcal{M}$, with Lorentzian metric g_{ab}, will be regarded as embedded in a larger manifold $\widetilde{\mathcal{M}}$ (with conformally related metric $\widetilde{g}_{ab}$) in order that one may refer to points at infinity in the usual manner.[18] Thus, one defines the boundary $\mathcal{I}$ of $\mathcal{M}$ in $\widetilde{\mathcal{M}}$, and its disjoint subsets, $\mathcal{I}^+$

$(\mathcal{I}^-) =$ *future (past) null infinity* and $\{i^0\} =$ *spatial infinity*. Note that $\widetilde{\mathcal{M}}$ is taken to extend slightly "beyond infinity" in order that it can be a manifold — albeit not a C^∞ one — even at i^0.

Notice that $\mathcal{I}$ is the union of the null geodesics* emanating from i^0. Notice also that $\mathcal{I}$ represents only a single asymptotic region of $\mathcal{M}$; if there are others, they will be ignored, and only the mass associated to the one whose "boundary at infinity" is $\mathcal{I}$ will be studied. Thus $\mathcal{I}^+$, for example, consists of the ideal endpoints of out-going null geodesics which reach arbitrarily great radii in the asymptotic region in question. One also defines the *domain of outer communications* $\mathcal{D} = I^+(\mathcal{I}^-) \cap I^-(\mathcal{I}^+)$ (the set of events which "can communicate with the asymptotic region"; an example is depicted in fig. (2)). It will be assumed that $\mathcal{D}$ satisfies the following conditions:

- Every infinite null geodesic in $\mathcal{D}$ possesses a pair of conjugate points.

(In fact, this condition can be weakened to one pertaining only to achronal null geodesics, as will be seen later.)

- $\mathcal{D} \cup \mathcal{I}$ is *globally hyperbolic* as a subset of $\widetilde{\mathcal{M}}$; a globally hyperbolic set being one which is *strongly causal* and which contains $\langle\langle p, q \rangle\rangle := J^+(p) \cap J^-(q)$ as a compact subset, for each p and q in the set.

The conditions for asymptotic flatness will be taken to hold on the intersection $\mathcal{N}$ of some neighbourhood of $\mathcal{I}$ with $\mathcal{D}$. One has then:

Theorem II.1.1: The 4-momentum of $\mathcal{M}$ is future-causal.

Equivalently, the theorem states that the ADM energy is non-negative in every rest-frame. Notice that the theorem as presented above does not explicitly assume Einstein's equations. Rather, those features of the curvature which are essential for the theorem to hold have been extracted and posed as the assumptions. Positivity of mass emerges as a property of general Lorentzian manifolds that focus null geodesics and have an adequate asymptotic flatness property. Upon assumption of Einstein's equations, one can replace the conditions on the curvature by conditions on the matter tensor. In the next subsection, it will be seen that reasonable conditions on the matter tensor act to enforce the conjugate point condition. It may also be appropriate to mention here that such causality conditions as will be needed will apply only to $\mathcal{D}$; there is no need to forbid, for example, closed timelike curves hidden behind event horizons.

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* That is, null geodesics with respect to $\widetilde{g}_{ab}$. Recall that conformally equivalent metrics possess the same null geodesics.

II.2: Focusing Theorems and the Existence of Conjugate Points:

The condition requiring the existence of conjugate points may be imposed by assuming instead other conditions which suffice to imply the necessary focusing, as mentioned in the introduction. The traditional way to do this has been to require that the energy density be pointwise non-negative, and to assume in addition that the curvature tensor satisfies the so-called generic condition defined below. For details, the reader may consult Refs. (7,8,19). However, recent work shows that geodesic focusing occurs under conditions considerably more general than originally known. An example of such conditions would be that along every infinite null geodesic the integral condition sometimes (somewhat inappropriately) called the Averaged Weak Energy Condition hold, with every such geodesic encountering a point of generic curvature. Similar conditions have been used in recent discussions of causality violating spacetimes[20] and are drawn essentially from work of Tipler[9]. However, even Tipler's conditions can be weakened, as was shown by Borde[10], who proves the following theorem.

> Borde's Focusing Theorem 2: Let γ be a complete causal (*i.e.* null or timelike) geodesic with affine parameter t and tangent T^a; and let $T_{[a}R_{b]cd[e}T_{f]}T^cT^d \neq 0$ somewhere on γ. Suppose further that for each $\epsilon > 0$ there exists $B > 0$ such that, for every pair of affine parameter values $t_1 < t_2$, there exists an interval I_1 with length $\geq B$ and endpoints $< t_1$, and an interval I_2 with length $\geq B$ and endpoints $> t_2$, such that
>
> $$\int_{t'}^{t''} R_{ab}T^aT^b \, dt \geq -\epsilon \quad \forall t' \in I_1 \quad , \quad \forall t'' \in I_2 \quad .$$
>
> Then γ contains a pair of conjugate points.

Also, Kánnár[11] recently has produced a variant of this theorem which imposes an integral condition directly on components of the Riemann tensor instead of on components of the Ricci tensor.

A spacetime in which $T_{[a}R_{b]cd[e}T_{f]}T^cT^d \neq 0$ somewhere on every geodesic (with T^a tangent to the geodesic) is said to obey the *generic condition*. For present purposes, it suffices that this hold on every *null* geodesic, and the word *generic* will be used in that sense. Satisfaction of this condition is not at all a strong assumption, being indeed generic in a suitably rigourous sense.[19,21] This last fact means in particular that, even for spacetimes which do not satisfy the condition, there should be nearby generic spacetimes with almost the same total mass. Thus a proof of positivity of mass for generic spacetimes should imply the same result for arbitrary ones.

The motivation for studying general conditions under which focusing might occur came from attempts to analyse the singularity theorems (and more recently the so-called chronology protection conjecture[22] forbidding formation of closed timelike curves) in the context of classical gravity coupled to quantum fields (including the graviton field).[23] The present work raises in addition the possibility of an application to proving a positive energy theorem in this same "semi-classical" context.

If integral energy conditions of the sort used in Borde's theorem were to hold for arbitrary states of arbitrary quantum fields in arbitrarily curved spacetimes then the proof of positivity of mass presented herein would generalise straightforwardly to the case of semi-classical gravity. Although some encouraging results in this direction are available** (in particular the theorems of Refs. (24, 25) and the indications from recent work of Ford and of Roman that the uncertainty principle places strong constraints on the negative energy distributions arising from quantum fields in flat and certain curved spacetimes[26]), it now appears that integral energy conditions can be violated by *test* quantum fields in certain curved spacetimes.[24] However, no such violation is known in the presence of the semi-classical Einstein equation, and even in situations where one would expect that approximate equation to fail (such as for an exploding black hole) the obvious candidates for "non-focusing" null geodesics turn out, on closer consideration, to be likely to possess pairs of conjugate points.

Finally, it is noteworthy that focusing theorems can be generalised to certain metrics that fail to be C^2, for example ones with shock wave discontinuities in the curvature or the connection.[27] This makes it likely that a version of Theorem II.1.1 will still hold in such cases, thereby providing a positivity result appropriate to such generalised solutions of Einstein's equation.

** In this connection, it might prove important that the proof given in Section IV below deals with a null geodesic which is achronal by construction. In deriving the required contradiction it therefore suffices to know that every *achronal* null geodesic has a pair of conjugate points (and therefore is self-contradictory). Hence it would suffice to have available an integral energy condition known to hold only for achronal null geodesics, as in Ref. (24). Notice in particular that although, for compact spacetimes, the Casimir energy may be negative, this need not imply that the integrated energy conditions fail on achronal null geodesics.

III: Null Geodesics in a Neighbourhood of Infinity

III.1: The Phase Delay and the Retarded and Advanced Time Functions

In 1960, Plebański[28] published a study of the behaviour of null geodesics propagating in $\mathcal{O}(1/r)$ weak gravitational fields. He used his results to compute light bending, but did not explicitly note the time delay effect, nor did he comment on the complementary effect, *i.e.* the infinite time advance of null geodesics as one approaches infinity in a positive mass spacetime. His method of solving the Hamilton-Jacobi equation perturbatively about a flat background metric (namely that entailed by asymptotic flatness) will be employed here. The analysis of this section will be valid only on a suitable neighbourhood of infinity, one on which an appropriate background structure may be defined, as detailed in the next subsection.

Fix a generator Λ^- of $\mathcal{I}^-$, let p be some point on Λ^-, and consider the set of all null geodesics originating at p. This set is a subset of the non-rotating congruence of all null geodesics whose initial endpoints lie on Λ^-. Since this is a non-rotating congruence, it can be described by a single scalar function S, which is known by many names, the Hamilton-Jacobi function, the eikonal, the phase (of the waves carried along by the congruence), *etc.* It obeys

$$g^{\mu\nu}\partial_\mu S\partial_\nu S = 0 \quad . \tag{III.1.1}$$

Its significance is that the null geodesics leaving Λ^- generate the level surfaces of S, so knowledge of S defines in a succinct manner the set of null geodesics from each initial point $p \in \Lambda^-$. When S is normalised so that it reduces on Λ^- to a suitably defined affine parameter, it will be called the advanced time function and denoted by S^-.

Consistent with asymptotic flatness, let there exist quasi-Cartesian coordinates $\{x^\mu\}$ in a neighbourhood of infinity, by which is meant coordinates in which the (inverse) metric takes the form

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \quad , \quad h^{\mu\nu} = \mathcal{O}(1/r) \quad , \tag{III.1.2}$$

where $t = x^0$, $r^2 = x^j x^j$ (with j running from 1 to 3) and $\eta^{\mu\nu}$ is the (inverse) Minkowski metric, having components $\mathrm{diag}(-1, 1, 1, 1)$. The precise sense in which $\mathcal{O}(1/r)$ is to be understood will be described shortly. Then one may expand S about its Minkowski space form as

$$S = \eta_{\mu\nu}k^\mu x^\nu + \Delta S \quad , \quad \text{where} \quad \eta_{\mu\nu}k^\mu k^\nu = 0 \quad , \tag{III.1.3}$$

with k^μ having constant components in the x^μ system. The Hamilton-Jacobi equation (III.1.1) becomes

$$k^\mu\partial_\mu\Delta S = -\frac{1}{2}h^{\mu\nu}k_\mu k_\nu + \mathcal{E}(x^\mu) \quad , \quad k_\mu := \eta_{\mu\nu}k^\nu \quad ,$$

$$\mathcal{E}(x^\mu) = -\frac{1}{2}\left\{\eta^{\mu\nu}\partial_\mu\Delta S\partial_\nu\Delta S + h^{\mu\nu}\partial_\mu\Delta S\partial_\nu\Delta S + 2h^{\mu\nu}k_\mu\partial_\nu\Delta S\right\} \quad , \tag{III.1.4}$$

where $\mathcal{E}$ represents an error term of second order in $h^{\mu\nu}$ which will vanish like $1/r^{1+\epsilon}$ or faster as $r \to \infty$. Note the definition of k_μ, which implies that it also has constant components. Neglecting the error term, one may integrate equation (III.1.4) to yield

$$
\Delta S(t, x^j) = -\frac{1}{2k^0} \int_{t_0}^{t} dt' \int d^3x' \; k_\mu k_\nu h^{\mu\nu}(t', x^{j'})
$$
$$
\delta^{(3)}(x^j - x^{j'} - (t - t')\hat{n}^j) + C(t_0) \; ,
$$
(III.1.5)

where $\hat{n}^j = k^j/k^0$ is the unit vector in the k^j-direction and C is a constant of integration. When t_0 is taken to negative infinity with a suitable choice of integration constants,[*] $S = S^-$ describes a plane wave emanating from the generator Λ^- of $\mathcal{I}^-$. Note that the support of the integrand in equation (III.1.5) is along the Minkowskian-null line through (t, x^j) parallel to k^μ. In what follows, k^μ will be taken to have components $(1,0,0,1)$, so that $\eta_{\mu\nu}k^\mu x^\nu = z - t$ describes a plane wave propagating in the z-direction.

Analogous to S^- is another function, the retarded time function, S^+ which will also be needed in what follows. It furnishes a function on spacetime which is constant along each null geodesic whose future endpoint lies on Λ^+, the generator of $\mathcal{I}^+$ that "continues" Λ^-. This function also obeys the Hamilton-Jacobi equation (III.1.1) and the resulting equations (III.1.3) with a ΔS given (approximately) by (III.1.5) with the same k^μ but the integral taken from t to ∞ rather than from $-\infty$ to t.

An important caveat is that S^+ (like S^-) is in general not everywhere well defined. The problem is that null geodesics focus, so two or more null geodesics that have different endpoints on Λ^+ may pass through some given point x in spacetime, assigning to that point two or more different values of S^+. However, these values always have a lower bound, since null geodesics from any fixed x in spacetime cannot arrive at $\mathcal{I}^+$ arbitrarily early. The greatest lower bound of S^+ at x is physically significant in that it gives the earliest (retarded) time of arrival at Λ^+ of any null geodesic (and hence any signal whatsoever) from x. In what follows, however, one is able to limit consideration to null geodesics contained within the asymptotic region, and for them the (obviously single-valued) approximation to S given by (III.1.5) will suffice for present purposes[**] (in fact, all that is needed is that

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[*] In order that ΔS have a limit as $t_0 \to -\infty$, $C(t_0)$ generally will have to diverge logarithmically with t_0, $cf.$ equations (III.2.9) and (III.2.11) below. Note that S^- can also be described as that solution of the Hamilton-Jacobi equation which agrees on Λ^- with a certain Bondi time coordinate, as defined in Ref. (29).

[**] Of course, from a fixed point of $\mathcal{I}^-$, unphysical (but causal) signals may reach any generator of $\mathcal{I}^+$ arbitrarily early by passing through i^0. Less trivially, one may see that causal

the error made by using (III.1.5) be uniformly bounded on neighbourhoods of infinity of the sort discussed in Section (III.2)).

It is worthwhile to interpret the preceding equations before proceeding. The phase of a hypothetical (massless) plane wave that couples only to the background flat metric is $\eta_{\mu\nu}k^{\mu}x^{\nu}$. Equation (III.1.5) may be thought of as giving the *phase delay* experienced by a wave propagating through a refractive medium, following to zero[th] order of approximation the Minkowskian-null direction k^{μ}, but experiencing refraction due to its coupling to $h^{\mu\nu}$. To compute null trajectories to *first* order of approximation, one must note that photons propagate so as to keep the true phase S of equation (III.1.3) constant, not the background phase, so one must consider equations (III.1.3) and (III.1.5) together. With the choice of k^{μ} as having components $(1,0,0,1)$, the equation $S = const.$ determines a surface of constant phase ruled by null geodesics that would be described in 3+1-language as normally outbound from a coordinate 2-plane $z = z_0$ in the asymptotic region at $t = t_0$. Then ΔS describes the amount by which this surface differs from one ruled by lines null in the background metric $\eta_{\mu\nu}$. Thus, one may use ΔS to extract the *time delay*[5,6] (possibly negative) due to $h^{\mu\nu}$, experienced by a plane wave evolving in the asymptotic region from the initial plane surface. To see this, one may ask at what value of coordinate time t does a generator of the $S = const.$ surface reach a point of fixed spatial coordinate x^{i}, and what would be the value of t if $h^{\mu\nu}$ were neglected. The resulting time delay is given by $\Delta t = (1/k^0)\Delta S$, where k^0 is the projection of k^{μ} onto the normal to the surfaces of constant time. If this normal is $u_{\mu} = \partial_{\mu}t$, one may write the formula

$$\Delta t = \frac{\Delta S}{k^{\mu}u_{\mu}} \quad , \tag{III.1.6}$$

describing the Shapiro time-delay effect. This is somewhat implicit, as ΔS depends on Δt but, consistent with the first approximation, the final time required in the evaluation of ΔS may be taken to be the time at which the Minkowskian-null line tangent to k^{μ} has spatial coordinates x^{i}.

Note that there is an interpretation in physical terms associated with the use of the first order approximation described above. It is the approximation in which account is taken of the time delay experienced by null geodesics, but no account is taken of their

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curves in spacetime may reach Λ^{+} arbitrarily early from any point of $\mathcal{I}^{-}\backslash\Lambda^{-}$ by smoothing out the aforementioned unphysical curves so as to avoid i^{0}, and the retarded time thus is unbounded below on approach to such points. See particularly the discussion of equation (III.2.19), the remark after the proof of Theorem II.1.1 in Section IV, and fig. (3). Communication involving physical signals from Λ^{-} to Λ^{+} is key to our argument; for this case, see Lemma III.2.1.

consequent bending. Such an approximation clearly is valid for geodesics which propagate entirely at very large radii; we intend to provide a more rigorous justification for its use in another place.

III.2: Behaviour of the Retarded and Advanced Time Functions Near Infinity

The aim of this subsection is to evaluate the time delay/advance for null geodesics propagating near infinity. One begins with a formulation of the conditions of asymptotic flatness that will be required and which will refer to some neighbourhood of infinity, $\mathcal{N} \subseteq \mathcal{M}$. It is most direct to impose conditions on the inverse metric, namely that there be a quasi-Cartesian coordinate system for $\mathcal{N}$ in which it may be written as

$$g^{\mu\nu} = \eta^{\mu\nu} - 4\frac{m^{\mu\nu}(x^\lambda/r)}{r} + \varphi^{\mu\nu} \ , \tag{III.2.1}$$

where $r^2 = x^2 + y^2 + z^2$. The function $m^{\mu\nu}$ describes what will be called the *DC part* of the asymptotic metric; it is effectively a function on the cylinder $S^2 \times \mathbb{R}$, and will be required to be C^1 and bounded. It is intended that $\varphi^{\mu\nu}$ describe, to leading order, the radiative part of the metric, so it should fall off appropriately and satisfy some manner of transversality condition. It will be required, specifically, that $\varphi^{\mu\nu}$ be $\mathcal{O}(1/r)$, and that $\varphi^{\mu\nu}l_\mu l_\nu$ be $\mathcal{O}(1/r^{1+\epsilon})$ for some $\epsilon > 0$ when l_μ is taken to be either the out-going, coordinate-null vector $l^+_\mu = \partial_\mu(r - t)$ or its in-going counterpart $l^-_\mu = \partial_\mu(r + t)$.

We now introduce a precise definition for the symbol $\mathcal{O}(1/r)$. This definition entails in particular a fall-off which is uniform on neighbourhoods of compact portions of $\mathcal{I}$. A function f on $\mathcal{N}$ will be said to be of class $\mathcal{O}(1/r^\beta)$ with respect to an asymptotic coordinate system iff, for each function $h(r)$ of the class $r + o(r)$, there exists a bound $B > 0$ such that $|f(x)| < B/r^\beta$ for all $x \in \mathcal{N}$ such that $|x^0| < h(r)$. (By a function of the class $r + o(r)$ is meant one for which $|h(r) - r|/r \to 0$ as $r \to \infty$, for example the function $h(r) = r + a\log(r/a) + T$.) Thus, in the cases of $\varphi^{\mu\nu}$ and $m^{\mu\nu}$ one has for such x,

$$\left|\varphi^{\mu\nu}\right| < \frac{\kappa_1}{r} \ , \tag{III.2.2a}$$

$$\left|\varphi^{\mu\nu}l^\pm_\mu l^\pm_\nu\right| < \frac{\kappa_2}{r^{(1+\epsilon)}} \ , \tag{III.2.2b}$$

$$\left|m^{\mu\nu}\right| < \kappa_3 \ , \tag{III.2.2c}$$

for any convenient choice of norm $|\cdot|$ (*e.g.* the supremum over the components in the x^μ basis, the root-sum-square of the components in this basis, *etc.*) and some constants $\kappa_i > 0$.

Notice that the conditions for asymptotic flatness are Poincaré invariant: independent of the choice of origin which defines the radial direction and also independent of the choice

of asymptotic rest frame. They are weak enough to encompass, for example, the DC fields and radiation emitted by any astrophysically plausible source we know of. (Our particular "gauge choice" of asymptotic background metric has the disadvantage that $r \mp t$ is *not* a good coordinate on $\mathcal{I}^\pm$, but the change of coordinates one would need to make it so would introduce implicitly defined logarithmic terms into the asymptotic metric, a complication we wish to avoid.)

The proof to be given in Section IV will proceed entirely within a certain asymptotic region $\mathcal{U} \subseteq \mathcal{N} \cup \mathcal{I}$ upon which a single set of bounds κ_i on the fall-off of $h^{\mu\nu}$ holds sway. One may now estimate the retardation function ΔS under the assumption of such fixed bounds. Since equation (III.1.5) is linear in $h^{\mu\nu}$, the contribution to ΔS from each of the two $\mathcal{O}(1/r)$ terms in the metric (III.2.1) may be computed separately. Hence, the retarded time function at x, corresponding to the generator $\Lambda^+ \subseteq \mathcal{I}^+$, is

$$S^+(x) = t - z + \Delta S^+(x) \quad , \tag{III.2.3a}$$

$$\Delta S^+(x) = -\frac{1}{2} \int\limits_t^\infty dt' \int d^3x' \; k_\mu k_\nu h^{\mu\nu}(t', \vec{x}') \, \delta^{(3)}(\vec{x} - \vec{x}' - (t - t')\hat{n}) \quad ,$$

$$= \Delta S_{\mathrm{DC}}^+(x) + \Delta S_\varphi^+(x) \quad , \tag{III.2.3b}$$

$$\Delta S_{\mathrm{DC}}^+(x) = 2 \int\limits_t^\infty dt' \int d^3x' \; \frac{k_\mu k_\nu m^{\mu\nu}}{r'} \, \delta^{(3)}(\vec{x} - \vec{x}' - (t - t')\hat{n}) \quad , \tag{III.2.3c}$$

$$\Delta S_\varphi^+(x) = -\frac{1}{2} \int\limits_t^\infty dt' \int d^3x' \; k_\mu k_\nu \varphi^{\mu\nu}(x') \, \delta^{(3)}(\vec{x} - \vec{x}' - (t - t')\hat{n}) \quad , \tag{III.2.3d}$$

with k^0 normalised to unity, and the (possibly infinite) integration constant coming from $C(t_0)$ in (III.1.5) being left implicit in (III.2.3c).

Anticipating the answer, the DC piece will produce an unbounded contribution, so it will suffice to show that the $\varphi^{\mu\nu}$ contribution is bounded, and therefore negligible in the appropriate limit. It will be useful to characterize null geodesics (in $\mathcal{N}$, as are all null geodesics discussed in this subsection) by an *impact parameter*, defined in the usual manner with respect to the asymptotic coordinates as the minimum value of

$$b^2 = x^2 + y^2 \quad . \tag{III.2.4}$$

along the geodesic.

Let $l_\mu = \pm l_\mu^\pm$, where $\pm = \mathrm{sgn}(z)$, so l_μ corresponds to the radial null direction, inbound for $z < 0$ and outbound for $z > 0$, and with $l_0 = -1$. Define

$$s_\mu = k_\mu - l_\mu \quad ,$$
$$\implies k_\mu k_\nu \varphi^{\mu\nu} = l_\mu l_\nu \varphi^{\mu\nu} + 2 l_\mu s_\nu \varphi^{\mu\nu} + s_\mu s_\nu \varphi^{\mu\nu} \quad . \tag{III.2.5}$$

Equations (III.2.2) may be used to bound the magnitude of each term in this expression, since the entirety of the path of integration lies within an asymptotic region where a single set of bounds κ_i on the fall-off of $h^{\mu\nu}$ holds sway. Hence, using $s_\mu = (0, \vec{s})$, one may write that

$$
\begin{aligned}
|\Delta S_\varphi^+(x)| < & \frac{1}{2} \int\limits_t^\infty dt' \int d^3 x' \left(\frac{c_1(\kappa)}{r'^{(1+\epsilon)}} + \frac{c_2(\kappa)}{r'}|\vec{s}(\vec{x}')| + \frac{c_3(\kappa)}{r'}|\vec{s}(\vec{x}')|^2 \right) \\
& \delta^{(3)}(\vec{x} - \vec{x}' - (t-t')\hat{n}) \\
< & \frac{1}{2} \int\limits_t^\infty dt' \int d^3 x' \left(\frac{c_1(\kappa)}{r'^{(1+\epsilon)}} + \frac{c_4(\kappa)}{r'}|\vec{s}(\vec{x}')| \right) \\
& \delta^{(3)}(\vec{x} - \vec{x}' - (t-t')\hat{n}) \quad .
\end{aligned}
\tag{III.2.6}
$$

The $c_i(\kappa)$ are constants determined by the κ_i and the choice of norm. As well, from the definition of s_μ, one obtains

$$
|\vec{s}|^2 = 2(1 - |z|/r) , \tag{III.2.7}
$$

giving

$$
|\Delta S_\varphi^+(x)| < \frac{c_1(\kappa)}{2b^\epsilon} \int\limits_{\mathrm{Tan}^{-1}(\frac{z}{b})}^{\frac{\pi}{2}} (\cos\alpha)^{\epsilon-1} \, d\alpha \; + \; c_5(\kappa)\left(\frac{\pi}{2} - \mathrm{Tan}^{-1}\left(\frac{z}{b}\right)\right) \quad , \tag{III.2.8}
$$

with $c_5 = c_4/\sqrt{2}$ and with $\tan\alpha = z'/b$.

A more succinct expression may be obtained by extending the geodesic back to $t = -\infty$, which yields

$$
|\Delta S_\varphi^+(x)| < \frac{c_1(\kappa)}{2b^\epsilon} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\alpha)^{\epsilon-1} \, d\alpha \; + \; c_5(\kappa)\pi \quad . \tag{III.2.9}
$$

Thus the "radiative contribution" to the retardation of a null geodesic from $x \in \mathcal{N}$ to Λ^+ is uniformly bounded in x, except possibly in the limit $b \to 0$. Small b will not be relevant in the arguments that follow. For now, the chief lesson is that, in the case of large b, the contribution to the phase delay (or advance) due to $\varphi^{\mu\nu}$ is bounded, as claimed.

Next, the contribution from the DC piece is required. Here the definition

$$
k_\mu k_\nu \, m^{\mu\nu}(x^\lambda/r) = m(\tau, \theta, \mu) , \tag{III.2.10}
$$

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will be useful, with $\tau = t/r$, $\mu = z/r$ and $\tan\theta = y/x$. The integral (III.2.3c) reduces to

$$\Delta S_{\mathrm{DC}}^{+}(x) = 2 \int\limits_{t}^{\infty} dt' \; \frac{m(\tau', \theta, \mu')}{|\vec{x} - (t - t')\hat{n}|} \quad , \tag{III.2.11}$$

with the primes in the argument of m denoting evaluation of τ and μ at the translated (retarded) value $z - t + t'$ of the z-coordinate (θ is independent of z). The problem now is that generally this integral diverges; it is essentially the $\epsilon = c_4 = 0$ case of equations (III.2.6–9). This divergence as such merely expresses the fact that, with our choice of asymptotic coordinates, the Minkowskian null lines along which we have evaluated ΔS arrive at Λ^+ at infinitely late (or infinitely early as the case may be) retarded times — that is to say, they actually miss Λ^+ entirely. In the well known case of Schwarzschild, for example, the true out-going null surfaces take the form $r + a\log(r/a) - t = const.$, which diverges logarithmically from the null surfaces defined by $\eta_{\mu\nu}$. As mentioned earlier, it is because of this effect that (III.1.5) requires a t_0-dependent constant of integration, which reappears as a "renormalization" of (III.2.3c) and (III.2.11).

However, what is important here is the *difference* in arrival time at Λ^+ of null geodesics with different impact parameters, and (to leading order) such differences can be found directly from the integral in (III.1.5) or (III.2.11) because the value of the integration constant drops out. This is the procedure that will be followed.[†] Accordingly, consider a family of null geodesics with the same initial $z = z_0$ and $t = t_0$, and the same θ, but differing initial b. One may compute the phase difference between neighbouring finite-length geodesics specified by b and $b + db$, then let the final endpoints go to infinity, and lastly integrate the resulting value of $\partial S^+/\partial b$ to obtain the b-dependence of S^+. (Along all these null geodesics, the same bounds κ_i on the fall-off rates hold.) Thus, taking two points along a null line, one with coordinates (t_0, b, z_0), the other with coordinates (t, b, z) ($t > t_0$ say), and comparing the phase shift to that along the null line joining $(t_0, b+db, z_0)$ to $(t, b+db, z)$, one obtains the result

$$\frac{\partial \Delta S_{\mathrm{DC}}^{+}}{\partial b} = -\frac{2zm^+}{br} + \mathcal{O}(1/b^2) \; . \tag{III.2.12}$$

[†] Another way to describe this procedure is as follows. One wants the difference $S^+(x_0'') - S^+(x_0')$. To find it, one propagetes null geodesics γ' and γ'' from x_0' and x_0'' respectively, to Λ^+ (*i.e.* exact null geodesics with respect to the true metric g_{ab}). When γ' and γ'' reach the far asymptotic region, $r \to \infty$, they are running parallel to k^μ, and the difference in their k^μ-normalised time of arrival at Λ^+ is given by the phase difference $k_\mu(x' - x'')^\mu$, where x' and x'' are points on γ' and γ'' in the far asymptotic region. To leading order this phase difference is just that due to the difference in starting phase, $k_\mu(x_0' - x_0'')^\mu$, plus the difference in the phase delay integrals (III.1.5) computed from x_0' and x_0'' to endpoints of equal z (or t) in the far asymptotic region.

Here m^+ denotes the value of m at the point having coordinates (t, b, z). In passing, it may be remarked that the $\mathcal{O}(1/b^2)$ terms come from two sources, one being the initial endpoint contribution $2z_0 m^-/br_0 = \mathcal{O}(1/b^2)$, the other being b-dependence of the arguments of $m(\tau, \theta, \mu)$.

In the limit as the final endpoint goes to infinity, equation (III.2.12) becomes

$$\frac{\partial S_{\mathrm{DC}}^+}{\partial b} = -\frac{2M^+}{b} + \mathcal{O}(1/b^2) \quad , \tag{III.2.13}$$

where M^+, the limiting value of m^+, is a constant (one might have expected it to retain the θ-dependence, but θ parametrises a circle whose angular size at fixed b becomes negligible as its centre approaches infinity). In terms of k^μ one has from (III.2.10) that

$$M^+ = \lim m^+ = m^{\mu\nu}(k^\lambda/k^0) \, k_\mu k_\nu \quad . \tag{III.2.14}$$

Now take $z_0 = 0$, in which case (since M^+ is independent of θ) one may treat S_{DC}^+ as a function of b alone. Then, integrating (III.2.13) and adding in the contribution ΔS_φ^+ (which has already been seen to be bounded, by (III.2.9)) yields

$$S^+(b) = -2M^+ \log(b/b_0) + \mathcal{O}(1) \quad , \tag{III.2.15}$$

where b_0 is a constant of integration and $\mathcal{O}(1)$ indicates terms bounded in the large b limit. A similar analysis gives the advanced time function at $z = 0$ (and sufficiently large b) as

$$S^-(b) = +2M^- \log(b/b_0) + \mathcal{O}(1) \, , \tag{III.2.16}$$

with $M^- = m^{\mu\nu}(-k^\lambda/k^0) \, k_\mu k_\nu$.

Finally, for curves that have their initial endpoints on Λ^- and final endpoints on Λ^+, one may define a *time-of-flight* as the difference between the (retarded) time of arrival and the (advanced) time of departure. In terms of this concept one may state the following lemma, which will be used in the proof in Section IV.

Lemma III.2.1: Let γ consist of the union of two null geodesics, one that joins Λ^- to a point $w \in \mathcal{N}$ with $z = 0$ and radial coordinate b, and one that joins w to Λ^+. For a fixed set of bounds κ_i in (III.2.2), there exist positive constants R_0 and C_0 such that, if γ remains always at radii greater than R_0 then, to within a correction of magnitude less than C_0, its time-of-flight from Λ^- to Λ^+ is given by

$$-2(M^+ + M^-) \log \left(b/R_0 \right) . \tag{III.2.17}$$

Proof: Consider a neighbourhood $\mathcal{N}(\kappa)$ of a compact portion of infinity on which the inequalities (III.2.2) hold with some triple of *fixed* bounds $\kappa = (\kappa_i)$, and let $w \in \mathcal{N}$ have $z = 0$. If the radial coordinate at w is b, then the functions $S^{\pm}(w)$ may be computed as in this section, provided the Minkowskian-null trajectory $z - t = const.$ through w remains within $\mathcal{N}(\kappa)$, and it is easy to choose $\mathcal{N}(\kappa)$ so that this is so. Then, from (III.2.15) and (III.2.16), one has

$$S^+(w) - S^-(w) = -2(M^+ + M^-) \log(b/R_0) + \mathcal{O}(1) \quad . \tag{III.2.18}$$

The lemma follows. $\square$

In interpreting the time-of-flight, recall that the normalisations of $S^{\pm}$ are both fixed by the null vector k^a which determines the generators $\Lambda^{\pm}$ of $\mathcal{I}^{\pm}$. It is possible to define time-of-flight for curves joining arbitrary generators. Consider for example the generator $\Gamma^+ \subseteq \mathcal{I}^+$ associated to the null vector $\frac{\partial}{\partial t} + \cos\theta \frac{\partial}{\partial z} + \sin\theta \frac{\partial}{\partial x}$. The time-of-flight of the curve that leaves Λ^-, follows a null geodesic to a point w with coordinates (b, z), then follows another null geodesic to Γ^+ is given by

$$S^+(w) - S^-(w) = -2(M^+ + M^-) \log(b/R_0) - 2M^+ \log\left(1 + \frac{z}{b}\tan\theta\right) + \mathcal{O}(1) \quad . \tag{III.2.19}$$

For non-zero θ, the point w may be adjusted to make this quantity arbitrarily negative, whence such curves may join arbitrarily late points on Λ^- to arbitrarily early ones on Γ^+, as remarked in an earlier footnote. See also the illustration in fig. (3).

IV: The Positivity Proof

Now it is possible to prove the theorem stated in Section II.

Proof of Theorem II.1.1: Choose a future-pointing asymptotic null vector k^a, as before. This vector determines a unique null geodesic generator Λ^+ of $\mathcal{I}^+$ and (the past-pointing null vector $-k^a$ determines) a unique null geodesic generator Λ^- of $\mathcal{I}^-$. Recall that, by assumption, there exists a neighbourhood of infinity $\mathcal{N}$ and a quasi-Cartesian coordinate system x^{μ} for $\mathcal{N}$ in terms of which the spacetime metric $g_{\mu\nu}$ fulfils the asymptotic flatness conditions introduced in Section III. Recall also that $\mathcal{D} \cup \mathcal{I} = [I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)] \cup \mathcal{I}$ is assumed to be globally hyperbolic as a subset of $\widetilde{\mathcal{M}}$ (fig. (2)). We seek a "fastest causal curve" γ from Λ^-, through $\mathcal{D}$, to Λ^+. The curve γ will be constructed as the limit of a sequence of causal curves γ_n.

In order to obtain γ_0, the initial curve of our sequence, let us begin by choosing $x_0 \in \mathcal{D}$ and a causal curve from x_0 to $q_0 \in \Lambda^+$. (Such a curve is

easily constructed; one can, for example, pick x_0 in the asymptotic region $\mathcal{N}$ and proceed from there in a null direction coinciding asymptotically with k^a.) Similarly, we can find some causal curve joining x_0 to a point $p_0 \in \Lambda^-$. All our considerations henceforth will be confined to the *interval* $\langle\langle p_0, q_0 \rangle\rangle := J^+(p_0) \cap J^-(q_0)$, by definition a compact subset of the globally hyperbolic set $\mathcal{D} \cup \mathcal{I}$. For future reference, let us introduce also a radius R_0 large enough so that all points $x \in \langle\langle p_0, q_0 \rangle\rangle$ with (finite) radial coordinate $r(x) > R_0$ belong to $\mathcal{N}$, and let $\mathcal{U} = \{x \in \langle\langle p_0, q_0 \rangle\rangle \,|\, r(x) > R_0\}$. (Here we also admit $r(x) = \infty$, so that $\langle\langle p_0, q_0 \rangle\rangle \cap \mathcal{I} \subseteq \Lambda^- \cup \{i^0\} \cup \Lambda^+$ is included in $\mathcal{U}$.) Clearly, $\mathcal{U}$ is open in $\langle\langle p_0, q_0 \rangle\rangle$, whence its complement, $\mathcal{K} := \langle\langle p_0, q_0 \rangle\rangle \backslash \mathcal{U}$, is a compact subset of $\mathcal{D}$.

Now what will be called a "faster causal curve" than γ_0 from Λ^- to Λ^+ will be one which departs (weakly) later *and* leaves (weakly) earlier than γ_0. Such a curve will by definition have its initial and final endpoints in $\langle\langle p_0, i^0 \rangle\rangle$ and $\langle\langle i^0, q_0 \rangle\rangle$ respectively. In order to home in on a fastest curve, let us introduce a partial ordering $\leq$ on $\langle\langle p_0, i^0 \rangle\rangle \times \langle\langle i^0, q_0 \rangle\rangle$ by taking $(p', q') \leq (p'', q'')$ iff $\langle\langle p', q' \rangle\rangle \subseteq \langle\langle p'', q'' \rangle\rangle$ (*i.e.* iff $p' \in J^+(p'')$ and $q' \in J^-(q'')$). Then, defining F as the set of pairs of points in $\langle\langle p_0, i^0 \rangle\rangle \times \langle\langle i^0, q_0 \rangle\rangle$ which are joined by a causal curve through $\mathcal{D}$, let $(p, q) \in \overline{F}$ be any element of the closure of F which is *minimal* with respect to $\leq$.

By the definition of p and q, there exists a sequence of points $p_0, p_1, p_2, \ldots$ ascending Λ^- to p, and a sequence $q_0, q_1, q_2, \ldots$ descending Λ^+ to q, such that each pair (p_n, q_n) is joined by a causal curve γ_n through $\mathcal{D}$. We claim (passing to a subsequence if necessary) that the curves γ_n approach a limit γ, as depicted in fig. (4). This follows from the known fact that, for arbitrary compact subsets A and B of any globally hyperbolic set, the space $\mathcal{C}(A, B)$ of causal curves from A to B is compact. (Here the topology of $\mathcal{C}(A, B)$ can be taken to be the so-called Vietoris topology, as will be shown in a forthcoming paper[30] wherein the existence of limit curves like γ will be proved by a new method.) Thus, since $\langle\langle p_0, p \rangle\rangle$ and $\langle\langle q, q_0 \rangle\rangle$ are compact subsets of the globally hyperbolic set $\mathcal{D} \cup \mathcal{I}$, the sequence $\{\gamma_n\}$ must possess within $\mathcal{C}(\langle\langle p_0, p \rangle\rangle, \langle\langle q, q_0 \rangle\rangle)$ an accumulation point, γ, as claimed. By construction, γ is a fastest causal curve from p to Λ^+, in the sense that no causal curve from p (and which enters $\mathcal{M}$) can arrive earlier on Λ^+. In consequence, γ must remain always on the boundary of $J^+(p)$, whence it must be a *null geodesic without conjugate points* (if it enters $\mathcal{M}$ at all).[7,8,31]

The curve γ will be our self-contradictory null geodesic *if* we can rule out the possibility that the γ_n "escape to infinity" (in which case we would have $\gamma = \langle\langle p, i^0 \rangle\rangle \cup \langle\langle i^0, q \rangle\rangle \subseteq \Lambda^- \cup \{i^0\} \cup \Lambda^+$, which never even enters $\mathcal{M}$). In excluding this possibility, one may distinguish two cases: either an infinite number of the

γ_n meet the compact set $\mathcal{K}$ (defined above) or they do not. In the former case $\gamma = \lim \gamma_n$ must also meet $\mathcal{K}$ since, were it to lie in its complement $\mathcal{K}' := \widetilde{\mathcal{M}} \backslash \mathcal{K}$ (an open set), the γ_n would eventually also have to lie in $\mathcal{K}'$ according to the definition of the topology of $\mathcal{C}(A, B)$. Thus γ must enter $\mathcal{M}$, as desired.

Passing to the latter case, then, we may assume (by possibly omitting a finite number of curves from the sequence) that none of the γ_n meet $\mathcal{K}$ or, in other words, that (except for their endpoints on $\Lambda^\pm$) they lie entirely within the subset $\Omega \subseteq \mathcal{D}$, $\Omega = \{x \in \langle\langle p_0, q_0 \rangle\rangle \,|\, R_0 < r(x) < \infty\}$. But our conditions of asymptotic flatness imply that, within Ω (which is contained in the compact set $\langle\langle p_0, q_0 \rangle\rangle$), the inequalities (III.2.2) hold with *fixed* bounds κ_i, and we can assume that the R_0 defining $\mathcal{K}$ has been chosen at least as large as that occurring in Lemma III.2.1. The conditions for invoking Lemma III.2.1 are then met, except for the fact that the γ_n are not necessarily null geodesics. To remedy this, let us find* on each γ_n a point s_n at which the z-coordinate vanishes, and having found it, modify γ_n so that it becomes a union of "fastest curves" $\gamma_n^\pm$ from s_n to $\Lambda^\pm$. Such fastest curves $\gamma_n^\pm$ can be constructed just as before for γ itself, and must be null geodesics for the same reason.** Since they are fastest, the replacement $\gamma_n \to \gamma_n^- \cup \gamma_n^+$ makes γ_n faster than it was, and accordingly does not change the fact that the modified endpoints p_n and q_n also converge to the same limits p and q. Thus we can assume without loss of generality that each γ_n has the form $\gamma_n^- \cup \gamma_n^+$, where $\gamma_n^\pm$ is a null geodesic from s_n to $\Lambda^\pm$. Notice finally that, again as before, we can also reduce ourselves to the case where all of the γ_n remain within Ω.

Invoking Lemma III.2.1, we thus can conclude that, for $M^+ + M^- < 0$, the time-of-flight of γ_n from Λ^- to Λ^+ is bounded below by

$$-2(M^+ + M^-) \log \frac{b_n}{R_0} - C_0, \tag{IV.1}$$

where b_n is the value of $\sqrt{x^2 + y^2}$ at s_n and C_0 is a constant. In other words, the difference between the retarded time coordinate of q_n and the advanced time coordinate of p_n is at least that given by expression (IV.1).

Assume then that $M^+ + M^- < 0$. If the b_n diverged to $+\infty$, then expression (IV.1) would eventually become greater than the time-of-flight between p_0 and

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* Another approach would be to broaden Lemma III.2.1 to apply to arbitrary causal curves.

** Alternatively, their existence follows from the fact that $\partial J^\pm(s_n)$ must contain a ruling null geodesic through q_n (respectively p_n).

q_0, contradicting the definition of p_n and q_n; hence there is some upper bound R_1 which the b_n cannot exceed. But this means precisely that the γ_n all meet the compact set $\mathcal{K}_1$ defined like $\mathcal{K}$, but with R_1 replacing R_0 or, to put it slightly differently, that we can choose R_0 large enough so that all of the γ_n, and therefore γ itself, meet the compact set $\mathcal{K} \subseteq \mathcal{D}$.

We thus conclude that, unless $M^+ + M^- \geq 0$, γ must enter $\mathcal{D}$, in which case it (or more precisely its intersection with $\mathcal{D}$) will be an infinite null geodesic without conjugate points, providing the desired contradiction. Appealing to the appendix in order to replace $M^+ + M^-$ by $-2P \cdot k$, we can conclude, finally, that the total 4-momentum P^a must obey $P_a k^a \leq 0$, or, since any future-null vector could have been chosen as k^a, that P^a must be future-causal (*i.e.* future-timelike or -null). □

Remark: One may wonder what happens when $k \cdot P < 0$, such as for a positive mass Schwarzschild metric. In that case, one cannot confine the γ_n to meet any compact subset of spacetime; in fact, the sequences $\{p_n\}$ and $\{q_n\}$ will converge to i^0, each curve γ_n having successively larger impact parameter, and the limit curve γ will be the "null geodesic" joining i^0 to itself! (In particular, for any $p \in \mathcal{I}^-$, $I^+(p)$ contains all of $\mathcal{I}^+$ when P^a is future-timelike.)

To make a possibly greater appeal to intuition, it may be useful to make the point another way.[13] The formulæ of Section III may be used to evaluate implicitly the z-coordinate at which a given null geodesic with endpoint at $p \in \mathcal{I}^-$ meets a particular Cauchy surface parametrised by t, simply by setting $S = constant$. Then the methods of Section III.2 yield

$$
z(t) - z_0 = t - 2 \int\limits_{z_0/r_0}^{z/r} d\mu \; \frac{m(\tau(b,\mu),\theta,\mu)}{1 - \mu^2} + \mathcal{O}(1) \quad , \tag{IV.2}
$$

where $\mathcal{O}(1)$ signifies order one (bounded) in b. One now may ask what would be the shape of the wavefront defined by the meeting of a surface ruled by null geodesics of constant advanced time with a Cauchy surface of fixed t. To answer this, compute $(\partial z/\partial b)_{S^-,t}$. The analysis follows as before. One takes the limit $t_0 \to -\infty$, $z_0 \to -\infty$, $\mu_0 \to -1$, which places the vertex of the light cone at $p \in \mathcal{I}^-$. To $\mathcal{O}(1/b)$, one obtains

$$
\frac{\partial z}{\partial b} = 2 \frac{M^-}{b} \quad , \tag{IV.3}
$$

which in turn implies that the wavefront formed by the null geodesics from $p \in \mathcal{I}^-$ meets the chosen Cauchy surface (say $t = 0$) in a two-surface that is asymptotically of the form

$$
z = 2M^- \log b + \mathcal{O}(1) \quad . \tag{IV.4}
$$

Similarly, a wavefront of past-directed null geodesics from $q \in \mathcal{I}^+$ would also form a logarithmic surface, this time described by

$$z = -2M^+ \log b + \mathcal{O}(1) \quad , \tag{IV.5}$$

obtained by holding the retarded time function constant. The condition that q be in the causal future of p is the condition that these two wavefronts meet, for which it suffices that $M^+ + M^- > 0$ (*i.e.* that $k \cdot P < 0$). When this inequality holds, the wavefronts are "swept forward near infinity" and necessarily overlap there, whence q is actually in the chronological future of p. When the inequality holds in the opposite direction (so that $k \cdot P > 0$), then the wavefronts are "swept back near infinity" and, if they meet at all, it is not within some neighbourhood of infinity, which is the key point. Choosing p and q so that the wavefronts "just touch" gives a "fastest null geodesic" γ, as before, and thence the contradiction. These wavefronts are depicted in fig. (5), with the Minkowski space case given there as well. Fig. (6) depicts the resulting set $\mathcal{I}^+ \backslash I^+(p)$ for $M = 0$ (and for $M < 0$).

Finally, there is a third version of the argument, which is somewhat closer to that of Ref. (12) and which avoids the introduction of causal curves which are not geodesic. In this version, one begins with $p \in \Lambda^-$ and chooses a sequence of *finite* points $p_n \in \mathcal{D}$ descending a causal curve to p. Because p_n is not at infinity, the boundary of its future will meet Λ^+ at q_n (say) and will contain a null geodesic γ_n from p_n to q_n with no finite pair of conjugate points. (Proof: One knows that some past null geodesic ruling $\partial J^+(p_n)$ originates at q_n and can leave $\partial J^+(p_n)$ only at p_n.) One then deduces, along the lines of the analysis of Section III, that, for $P \cdot k > 0$, the initial tangent vectors to the γ_n must avoid a neighbourhood of the future direction along Λ^- in order that the q_n not recede into the infinite future along Λ^+ (which they cannot do, since $J^+(p_n) \subseteq J^+(p_{n+1})$). Obtaining a limiting geodesic γ as before, one concludes that its tangent is not along Λ^- (whence it enters $\mathcal{M}$) and that it is also free of conjugate points (since a limit of conjugate-point-free null geodesics is also such). This is the desired contradiction.

V: Final Observations

In the introduction it was mentioned that one potential application of this theorem would be to semi-classical gravity. Another potential application would be to spacetimes possessing closed timelike curves shrouded within event horizons. Most singularity theorems break down in the presence of acausalities, and Finkelstein has suggested that causality violations might be found instead of singularities, were one to investigate the interior of certain black holes. In such a situation there might be no edgeless spacelike hypersurfaces,

but Theorem II.1.1 would still apply as long as the region outside the hole were globally hyperbolic.

Returning to the semi-classical situation for a moment, there is a potential *extension* of Theorem II.1.1 which would also be of interest in that context. Classically, the fact that the energy is bounded below is often taken to imply the stability of flat space, and an extension of this fact to semi-classical gravity* would be a step toward proving stability in the fully quantum case. However the result presented here does not deal with the case of zero mass. For completeness, one would like such an extension even in the classical case, of course, but it has a greater potential interest semi-classically. There, unless one can rule out the existence of non-flat solutions of zero energy, one leaves open the interesting possibility of inequivalent ground states between which tunnelling would be expected to occur.

It seems that any $P^a = 0$ result would be obtained most naturally in conjunction with a further generalisation, a proof that the 4-momentum cannot be null. The reason is that, in both situations, one would be dealing with the case $P \cdot k = 0$ for an appropriate k^a; in the $P^a = 0$ case any choice of k^a will do, while in the other case one chooses k^a parallel to P^a. For such a k^a, $k^a k^b h_{ab}$ falls off faster than $\mathcal{O}(1/r)$, say perhaps as $\mathcal{O}(1/r^2)$. If one considers null geodesics originating at p on the generator of $\mathcal{I}^-$ corresponding to this k^a, the result of this fall-off rate is to modify the relative delay of neighbouring null geodesics at large b. In analogy with equation (IV.3), one obtains

$$\frac{\partial z}{\partial b} = \mathcal{O}(1/b^2) \quad . \tag{V.1}$$

When one integrates this, one sees that the relative time delay of different null geodesics (at least, ones that propagate so as to remain always near $\mathcal{I}$) vanishes as $\mathcal{O}(1/b)$, so the null geodesics from p now will form a sequence whose large b limit arrives at $\mathcal{I}^+$ at finite retarded time U (equal to the retarded time of arrival one would compute by ignoring h_{ab} and using the flat metric η_{ab}).** If any of the null geodesics from p with finite b arrive at earlier retarded times than U, then the earliest of these arrivals is a null geodesic ruling the boundary of $I^+(p)$ and so cannot have focused, whence the theorem applies. However, it is conceivable that no geodesic arrives earlier than the limiting time U, in which case there is no apparent contradiction.

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* Once again, the reference here is to stability in the context of the semi-classical Einstein equations. It is not only to the more limited question of the presence or absence of instanton solutions tunneling to classical states of lower energy, which of course are already ruled out by the classical positivity of energy.

** A special case of this behaviour is Minkowski space, where all null geodesics from any fixed $p \in \mathcal{I}^-$ arrive at $\mathcal{I}^+$ at exactly the same retarded time.

This would seem unfortunate, and one might hope that there may be more subtle contradictions that emerge in this case because it seems reasonable to hope that the methods applied herein should yield results known to be true from the earlier techniques. However, there is also a contrary indication that perhaps any such extension of the present methods would prove too much! Specifically, postulate for a moment the existence of N additional *non-compact* spatial dimensions. Then, roughly speaking, finiteness of the ADM mass requires that the metric behave as

$$g_{ab} = \eta_{ab} + \mathcal{O}(1/r^{1+N}) \quad , \tag{V.2}$$

which, again in analogy with equation (IV.3), yields

$$\frac{\partial z}{\partial b} = \mathcal{O}(1/b^{1+N}) , \tag{V.3}$$

even for future-timelike P^a. Again, the integral of this shows that the endpoints of null geodesics of successively larger b approach a finite limiting value of retarded time on $\mathcal{I}^+$, precisely mimicking the behaviour in the usual three spatial dimensions when $P^a = 0$. There is thus a danger that any proof excluding $P^a = 0$ in 3+1-dimensions would also erroneously exclude known asymptotically flat solutions with $P^a \neq 0$ in higher dimensions.

Another possible line of generalisation which might be instructive to pursue concerns alternative gravity theories, or theories with compactified extra dimensions. In some of those theories, positive energy theorems have been proved, and an attempt to reproduce them with the methods of this paper would help disclose the extent to which Theorem II.1.1 depends on special features of Einstein gravity in four dimensions. For example, in 5-dimensional Kaluza-Klein theory, the total energy is positive for some choices of spatial topology but of indefinite sign in others, whereas it is not apparent how the general ingredients entering into the present proof, such as the focusing of null geodesics, would be affected by the dimensionality, or how they could be sensitive to the topology in this way.[†]

[†] The zero-energy solution discussed by Witten[32] provides one interesting example. Although one might think that a theorem analogous to ours ought to rule out such metrics (or really the negative energy solutions with which it is associated[33]), it turns out that spacetime of reference (32) contains a "bubble of nothingness which grows at the speed of light", and our conditions of asymptotic flatness would not be satisfied in the presence of such a disturbance. Kaluza-Klein examples wherein the topological character of $\mathcal{I}$ can be affected in other interesting ways also exist, such as the monopole solution, whose twisted character as a circle bundle would seem to lead to an extended spacetime $\widetilde{\mathcal{M}}$, which is not even a manifold[34].

Similarly, in Kaluza-Klein and other theories, the inertial mass is typically less closely tied to the "active gravitational mass" than it is in General Relativity, and the purely transverse character of radiation can also be lost — both of these being elements on which the present method of proof directly depends.

Finally, it seems likely this proof may be generalised to 4-dimensional metrics with slower fall-offs rates. Although the fall-off conditions assumed herein seem fairly weak, it is hoped that the method of proof will apply as well to metrics with fall-off rates as slow as $1/r^{\frac{1}{2}+\epsilon}$, $0 < \epsilon < 1/2$, on approach to i^0, for such a rate is sufficient to allow for the definition of a convergent mass integral. With such fall-offs, one might expect the $b \to \infty$ logarithmic divergences encountered in Section IV to be replaced by even stronger power law divergences, so it seems reasonable to hope that the method of proof used would apply in those cases as well.

Acknowledgements

Appendix: Superposed Asymptotically Schwarzschild Metrics

With respect to a flat background metric η_{ab}, the Schwarzschild metric may be written in isotropic form as

$$g_{ab} = \left[\left(1 + \frac{M}{2\rho}\right)^4 - \left(\frac{1 - \frac{M}{2\rho}}{1 + \frac{M}{2\rho}}\right)^2\right] U^c \eta_{ac} U^d \eta_{bd} + \left(1 + \frac{M}{2\rho}\right)^4 \eta_{ab} \quad , \qquad \text{(A.1)}$$

where $\eta_{ab} U^a U^b = -1$, and $\rho^2 = \eta_{ab} X^a X^b + (\eta_{ab} X^a U^b)^2$. The 4-momentum of the solution is $P^a = M U^a$. By convention, U^a will be future-directed, so a negative value of P^0 means a negative value of M. Asymptotically, this metric is

$$g_{ab} = \left(1 + \frac{2M}{\rho}\right) \eta_{ab} + \frac{4M}{\rho} U^c \eta_{ac} U^d \eta_{bd} \quad . \qquad \text{(A.2)}$$

From equation (A.2) it is easy to write down the superposition of an arbitrary number of asymptotic Schwarzschild metrics indexed by A. One obtains

$$g_{ab} = \eta_{ab} + 2 \sum_A \frac{M_A \eta_{ab} + 2\eta_{ac} U_A^c \eta_{bd} U_A^d}{\rho_A} \quad . \tag{A.3}$$

For definiteness, the superposition has been written as a discrete sum, but more generally one could include a continuous integral as well (and possibly still more general sorts of superposition). Because the M_A are not restricted to be positive, a superposition of this sort may possess any value of P^a, including spacelike and negative timelike ones.

For the sake of deriving equation (A.8) below, we assume that a superposition like (A.3) can serve, on approach to spatial infinity, to describe the most general behaviour of the metric at $\mathcal{O}(1/r)$ or, in other words, that such a form captures the asymptotic metric which remains after radiation terms have been excluded. Correspondingly, we reason in the main text as if the most general DC metric can be taken to be of this form, and we limit the metric perturbation $m^{\mu\nu}/r$ of expression (III.2.1) accordingly. While this could be adopted as part of our definition of asymptotic flatness, we suspect that such a limitation would be merely a convenient short-cut which could be avoided at the cost of some extra work needed to derive (A.8) directly from fall-off conditions together with the asymptotic field equations.

Now compute $M^{\pm}$, as defined by equations (III.2.1), (III.2.10), and (III.2.14), for the metric (A.3) and the (future-pointing) null vector k^a. By definition, this is just the limit as $\lambda \to \pm\infty$ of $\frac{-1}{4} r h^{ab}(x) k_a k_b$, where $(x - x_0)^a = \lambda k^a$ and r is the radial coordinate of x. Since M^+ is linear in h^{ab}, it may be computed for (A.2) alone. Denoting the vector $\partial/\partial t$ by e^a and recalling that $k \cdot k \equiv \eta_{ab} k^a k^b = 0$, one has

$$\begin{aligned}
\rho^2 &= \eta_{ab} x^a x^b + (U \cdot x)^2 \\
&= (k \cdot U)^2 \lambda^2 + \mathcal{O}(\lambda) \\
\Longrightarrow \rho &= -\lambda k \cdot U + \mathcal{O}(\lambda^0) \quad ,
\end{aligned} \tag{A.4}$$

and analogously

$$r = -\lambda k \cdot e + \mathcal{O}(\lambda^0) \quad . \tag{A.5}$$

(Here $\mathcal{O}(\lambda^0)$ means boundedness in λ.) Hence, to leading order in $1/r$ and $1/\lambda$, we have

$$\begin{aligned}
h^{ab} k_a k_b &= (\eta^{ab} + h^{ab}) k_a k_b = g^{ab} k_a k_b \\
&= g_{ab} k^a k^b = \left(1 + \frac{2M}{\rho}\right)(k \cdot k) + \frac{4M}{\rho}(U \cdot k)^2 \\
&= \frac{4M(U \cdot k)^2}{-\lambda(U \cdot k)} = -4 \frac{MU \cdot k}{r} k \cdot e \quad , \tag{A.6} \\
\Longrightarrow M^+ = M^- &= M(U \cdot k)(k \cdot e) = (P \cdot k)(k \cdot e) = -k^0 P \cdot k \quad . \tag{A.7}
\end{aligned}$$

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Therefore, if $k^0 = 1$ (as in the main text), then

$$M^+ + M^- = -2P \cdot k \quad , \tag{A.8}$$

for (A.2), and hence for (A.3). This may be inserted into the results of Section III and used in the proof of Theorem II.1.1 presented in Section IV.

In connection with Section III.2, it may be illuminating to consider an example consisting of the addition of two asymptotic metrics of the above form. For sake of the example, the result of this addition will be expressed in a frame where one of the two constituent metrics is time-independent. In this frame, say that the other metric is boosted so that it has 4-velocity proportional to $(1, 0, 0, V)$ with $|V| < 1$. Let $R^2 = X^k X^k$ with $k \in \{1, 2, 3\}$. Then the result is

$$
\begin{aligned}
ds^2 =& \left\{ 1 + \sum_{A=1,2} \frac{2M_A}{R} \left[1 - \frac{T^2}{R^2} + \frac{(T - V_A Z)^2}{R^2(1 - V_A^2)} \right]^{-1/2} \right\} \eta_{ab} dX^a dX^b \\
&+ \frac{4}{R} \left\{ M_1 dT^2 + \frac{M_2}{\sqrt{1 - V^2}} \frac{(dT^2 - 2V\,dZ\,dT + V^2 dZ^2)}{\sqrt{(1 - T^2/R^2)(1 - V^2) + (T/R - VZ/R)^2}} \right\} \quad ,
\end{aligned} \tag{A.9}
$$

where $V_1 = 0$ and $V_2 = V$.

Consider two cases, one where the null geodesics of Section III propagate perpendicular to the boost velocity and one where they propagate parallel to the boost velocity. This corresponds to two different choices of generator of $\mathcal{I}^-$ on which to place the initial point from which the null geodesics emanate. One may read off the function $M(T/R, \theta, \mu)$. The two cases give

$$M_\perp(T/R, \mu) = M_1 + \frac{M_2}{\sqrt{1 - V^2}} \left\{ (1 - V^2)\left(1 - \frac{T^2}{R^2}\right) + \left(\frac{T}{R} - V\frac{Z}{R}\right)^2 \right\}^{-1/2}, \tag{A.10a}$$

$$
\begin{aligned}
M_\parallel(T/R, \mu) = M_1 &+ \frac{M_2}{\sqrt{1 - V^2}} (1 - V)^2 \left\{ (1 - V^2)\left(1 - \frac{T^2}{R^2}\right) \right. \\
&\left. + \left(\frac{T}{R} - V\frac{Z}{R}\right)^2 \right\}^{-1/2} .
\end{aligned} \tag{A.10b}
$$

Taking the limits to $\mathcal{I}^\pm$, one obtains

$$M_\perp^+ + M_\perp^- = 2\left(M_1 + \frac{M_2}{\sqrt{1 - V^2}} \right) \quad , \tag{A.11a}$$

$$M_\parallel^+ + M_\parallel^- = 2\left(M_1 + \frac{M_2(1 - V)}{\sqrt{1 - V^2}} \right) \quad . \tag{A.11b}$$

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Lastly, since the ADM 4-momentum of this configuration is given by adding the 4-momenta of the two constituent solutions, one boosted and one stationary, then

$$P^a = \left(M_1 + \frac{M_2}{\sqrt{1-V^2}}, 0, 0, \frac{M_2 V}{\sqrt{1-V^2}} \right) \quad , \tag{A.12}$$

In agreement with (A.8) this results in

$$M_\perp^+ + M_\perp^- = -2k_\perp \cdot P \quad , \tag{A.13a}$$
$$M_\parallel^+ + M_\parallel^- = -2k_\parallel \cdot P \quad , \tag{A.13b}$$

where $k_\perp^a$ and $k_\parallel^a$ are future-pointing Minkowskian-null vectors normalised so that $k^0 = +1$ in both cases. The spatial components of $k_\perp^a$ are orthogonal to those of the boost vector U^a while those of $k_\parallel^a$ are parallel.

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Figure Captions

Fig. (1a): Focusing of null geodesics. The geodesics enter the crossing region and leave the boundary of the future of the initial point, *en route* to the conjugate points they will eventually encounter when they reach the caustics.

Fig. (1b): Focusing of null geodesics. A single pair of geodesics is shown. At q, they cross and leave the boundary of the future of p, but they must obey local causality, so they remain (for awhile) on the boundary of the future of q. They will eventually reach the caustics.

Fig. (2): An example of the Domain of Outer Communications of a particular asymptotic region of some solution.

Fig. (3): A conformal compactification with i^0 properly displayed as a single point. Spacetime is exterior to the cones which represent infinity. From $p \in \Lambda^-$, one may reach any point of any generator of $\mathcal{I}^+$, *except* Λ^+, by a causal curve in spacetime produced by smoothing out the curve that is the union of Λ^- with $\{i^0\}$ and with the appropriate generator of $\mathcal{I}^+$. Such a smooth causal curve in spacetime is displayed as a dotted line here joining p to q'.

Fig. (4): The sequence of curves $\{\gamma_n\}$ used in the proof of Theorem II.1.1, for the case $M \leq 0$.

Fig. (5a): The $t = 0$ surface in Minkowski space. In flat space, parallel lines, here defined by null geodesics incident on plane wavefronts, meet at infinity and so define the points of $\mathcal{I}^{\pm}$. If the two 2-plane wavefronts shown lying in this Cauchy surface are truly parallel, it is clear they define points of $\mathcal{I}^{\pm}$ that are not in causal contact with each other. If they are not parallel, they must meet somewhere, and causal contact is established. This gives rise to the "missing-half-generator" of fig. (6).

Fig. (5b): A spacelike surface when $M > 0$. Now curvature bends geodesics en route to/from $\mathcal{I}$, producing "logarithmic wavefronts" that always meet. The augmentations that appear on the wavefronts are the result of the focusing depicted in fig. (1).

Fig. (5c): A spacelike surface when $M < 0$. The logarithms of fig. (5b) are reversed. Again, wavefronts may fail to meet. Because the fall-off near infinity is only logarithmic in the impact parameter, only one half-generator of $\mathcal{I}^+$ is excluded from the future of any chosen point in $\mathcal{I}^-$, as depicted in fig. (6).

Fig. (6): Compactified Minkowski space displayed conventionally, so that i^0 is not correctly represented as a single point. This diagram is also appropriate to $M < 0$ space. The accented half-generator of $\mathcal{I}^+$ is the set of all points in $\mathcal{I}^+ \backslash I^+(p)$. Note that i^0 is incorrectly represented — it should be a single point.