CAUSAL STRUCTURE OF KINK SPACETIMES

E. Woolgar[1]

Dept. of Mathematics, University of Saskatchewan, 106 Wiggins Ave., Saskatoon, SK, Canada S7N 5E6

and

Dept. of Mathematical Sciences, University of Alberta, Edmonton, AB, Canada T6G 2G1

Abstract. We examine the causal structure of Finkelstein-Misner kink spacetimes, and discuss the conjecture that a singularity theorem can be predicated on the existence of non-zero kink number. As evidence in favour of this conjecture, arguments are given which prove a limited special case.

I. PRELIMINARIES

The concept of gravitational kinks was first introduced by Finkelstein and Misner [1], who called them M-geons. Consider an oriented embedded surface Σ in a spacetime $\mathcal{M}$. If Σ "carries a kink," then we may think of it as having light cones that "tumble over" as one traverses Σ, with the kink number counting the number of complete tumbles. Precise definitions of kink number are given elsewhere in this volume [2].

The concept applies to any hypersurface in spacetime, or any boundary hypersurface that spacetime might acquire as the result of a conformally isometric embedding in a larger Lorentzian manifold. It is not even necessary that the spacetime be time-orientable, although we will assume it is in what follows, since we will eventually invoke causality considerations. An easily visualized case is a spacetime consisting of a ball cut from Minkowski spacetime. The ball can be assigned a 3-sphere boundary carrying kink number 1.

[1] E-Mail: ewoolgar@math.ualberta.ca

It had been conjectured that compact manifolds with a single S^3 boundary having kink number other than 1 would contain closed timelike curves [3]. This is but one example of the counter-intuitive nature of kinks, for the conjecture, although it appears quite reasonable, is false. I came to understand this result, due to Chamblin and Penrose [4], through discussions with Ken Dunn during my year at Dalhousie University. It seems appropriate therefore, as this session is dedicated to Ken's memory, to return to the still incompletely understood relationship between kink number and causal structure. In particular, I will concentrate on evidence for another intuitively plausible but as yet undecided conjecture concerning a possible relationship between kinking and geodesic incompleteness.

We must first extend our notion of kink from a surface to an entire spacetime. Even in such a simple case as $(2+1)$-dimensional Minkowski spacetime one can find a surface—even one that extends to spatial infinity dividing the spacetime in two—which carries a kink (*e.g.* attach a handle to a $t = \mathit{const}$ slice; the light cones will "tumble over" as one traverses the handle). The presence, therefore, of a single kink surface is insufficient to characterize the whole spacetime as a kink. The point, however, is that the surface of this example cannot belong to a *foliation* of Minkowski spacetime, so by *kink spacetime* we will mean a topological product $\mathcal{M} = R \times \Sigma$ such that each embedded copy of Σ carries a kink. (I usually will not take care to distinguish between Σ and any sub-manifold that is its image under an embedding $\Sigma \to \mathcal{M}$.)

If a product manifold admits two metrics with different kink numbers, then this means that the space of Lorentzian metrics on it is not path connected. There is no analogue of this for Riemannian geometries, since any two Riemannian metrics on a given manifold are homotopic.[2] It would therefore seem that the topology of the space of Lorentzian metrics is richer than that of Riemannian metrics, and moreover one might expect this richness to result in physical effects, for example in the quantum theory of gravity.

However, it may be the case that metrics with non-zero kink number possess physically undesirable or even pathological properties, such as negative energy densities, violation of certain causality conditions, or unacceptable types of singularities. While the preceding talks, as described in the resulting articles [2, 5], have concentrated on a review of past and recent results, in this talk and article, a more speculative approach will be followed, concentrating on the open question of whether kinking gives rise to singularities. We will examine the causal structure of various kinks on $\mathcal{M} = R \times \Sigma$ with Σ compact, and give evidence suggesting that these kinks cannot be completed to non-singular manifolds unless energy conditions or strong causality are violated. This is consistent with a similar conjecture that asymptotically flat kinks obeying energy and causality conditions cannot be geodesically complete [6].

Example kinks will be discussed in Section II. Section III will contain a discussion of the causal completion of kinks and a singularity theorem valid when that completion takes a certain very specialized form appropriate to our example kinks. Section IV summarizes and includes some brief further comments concerning the role of kinks in cosmology and quantum gravity. Appendices review the terminology of indecomposable past and future sets and record certain properties of spherically symmetric, and in particular de Sitter, kinks.

[2] To prove this, it is a simple exercise to explicitly check that for any two Riemannian metrics g_{ab} and h_{ab} and for all $t \in [0,1]$, then $(1-t)g_{ab} + th_{ab}$ satisfies the axioms of a Riemannian metric.

II. Example Kinks

Consider Minkowski spacetime and let us introduce the 4-dimensional radial coordinate ρ and the polar angle coordinate ψ with respect to the t-axis:

$$\rho^2 := t^2 + r^2 \quad , \tag{1a}$$
$$\tan\psi := r/t \quad . \tag{1b}$$

In these new coordinates the Minkowski metric

$$ds^2 = -dt^2 + dr^2 + r^2\Big(d\theta^2 + \sin^2\theta d\phi^2\Big) \tag{2}$$

takes the form

$$\begin{aligned} ds^2 =& -\cos 2\psi d\rho^2 + 2\rho\sin 2\psi d\rho d\psi + \rho^2\cos 2\psi d\psi^2 \\ &+ \rho^2\sin^2\psi\Big(d\theta^2 + \sin^2\theta d\phi^2\Big) \\ =& e^{2\tau}\Big\{ -\cos 2\psi d\tau^2 + 2\sin 2\psi d\rho d\tau + \cos 2\psi d\psi^2 \\ &+ \sin^2\psi\Big(d\theta^2 + \sin^2\theta d\phi^2\Big)\Big\} \\ =& e^{2\tau} d\overline{s}^2 \quad , \end{aligned} \tag{3a}$$

$$\begin{aligned} d\overline{s}^2 :=& -\cos 2\psi d\tau^2 + 2\sin 2\psi d\tau d\psi + \cos 2\psi d\psi^2 \\ &+ \sin^2\psi\Big(d\theta^2 + \sin^2\theta d\phi^2\Big) \quad , \end{aligned} \tag{3b}$$

where

$$e^{2\tau} = \rho^2 = t^2 + r^2 \quad , \tag{4}$$

so the origin of the t, r system (*i.e.* $\rho = 0$) is not covered by the τ, ψ patch.

With the origin removed, then both ds^2 and $d\overline{s}^2$ are kink metrics on $R \times S^3$. This is intuitively obvious: on any 3-sphere embedded in Minkowski space the future arrow of time is necessarily somewhere parallel and somewhere anti-parallel to the outward normal (defined with respect to any convenient fiducial Riemannian metric), so the light cones "tip over" as required of kinks. Clearly, geodesics of the metric ds^2 that passed through the origin become incomplete once the origin is removed, but they are complete in $d\overline{s}^2$ since the conformal transformation from (3a) to (3b) moves this origin out to infinity. The boundaries of the past and future of the removed origin are horizons. Geodesics of ds^2 that end on the scri of Minkowski spacetime become incomplete in $d\overline{s}^2$, since this boundary moves in to finite distance under the conformal transformation. The manifold with metric $d\overline{s}^2$ can be extended through this boundary.

We next consider general spherically symmetric kinks with metric

$$ds^2 = e^{\varphi(r)}\Big\{ -e^{\chi(r)}\cos 2\alpha(r) dt^2 - 2\sin 2\alpha(r) dt dr + e^{-\chi(r)}\cos 2\alpha(r) dr^2 + r^2 d\Omega^2\Big\} \quad , \tag{5}$$

where r takes values in a closed sub-interval I of $[0, \infty)$ and $\alpha : I \longrightarrow [0, k\pi]$ with k the kink number. Dunn, Harriott, and Williams [7, 5] studied the $\varphi(r) = 0$ case of this metric; in recognition of this and for convenience, we will refer to all metrics of the form (5) as *DHW kinks.* These kinks have properties similar to those of black holes, such as horizons located at $\alpha = (2n+1)\pi/4$, where n is an integer. The kink metrics (3a,b) are of course special cases of equation (5).

Some further discussion of DHW kinks, including the example of the de Sitter kink, is provided in Appendix II. We note here certain easily verifed properties which

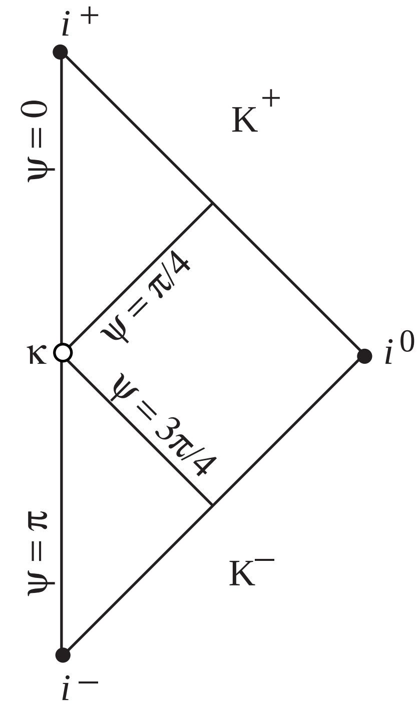

Figure 1. Penrose conformal diagram for the punctured Minkowski spacetime 1-kink. The puncture is labelled κ, while the surfaces labelled $K^{\pm}$ are scri for the metric of (3a) and are a boundary at finite distance for the metric (3b).

will be pertinent to our considerations in the next section. These properties are evident in Figure 2.

Remark 1: Let $\mathcal{M} := R \times \Sigma$ be a DHW kink. Let t parametrize the R factor and let Σ_α be the embedded kink surface corresponding to t_α. Let $\subset$ denote *proper* subset. Then:

(a) $I^+[\Sigma_2] \subset I^+[\Sigma_1]$ and (simultaneously) $I^-[\Sigma_2] \subset I^-[\Sigma_1]$ whenever $t_1 < t_2$.[3]

(b) $\mathcal{F} := \bigcap_t I^+[\Sigma_t]$ is a non-empty future set.

(c) $\mathcal{P} := \bigcap_t I^-[\Sigma_t]$ is a non-empty past set.

(d) $\mathcal{F} \cap \mathcal{P}$ is empty.

For example, if the surfaces Σ_t are the $t = const.$ surfaces of DHW 1-kinks obeying equation (5), then $\mathcal{F}$ is the set of all points for which $\alpha \leq \pi/4$, while $\mathcal{P}$ is the set of all points for which $\alpha \geq 3\pi/4$.

Before leaving this section, it is worthy of mention that Klösch and Strobl [8] describe some particularly interesting kinks in two dimensions. They construct incomplete but inextendible kinks of arbitrary kink number $k > 1$ as k-fold coverings of $(R^2 - \{pt\}, g_{ab})$ with g_{ab} an arbitrary non-kinked metric. (Note that kink number k in

[3] If the sign of the off-diagonal term in (5) is reversed or, equivalently, if the kink number is negative, then one must reverse the direction of the inequality between t_1 and t_2 here (or, equivalently, the direction of the set containment signs).

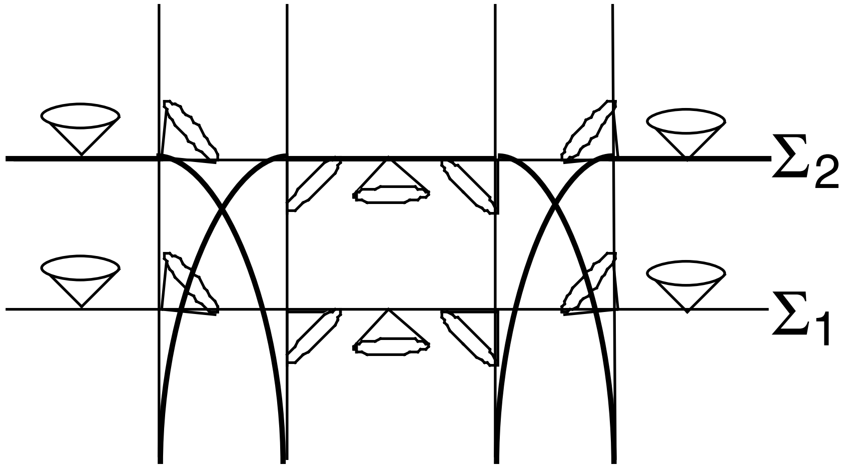

Figure 2. A DHW kink. The heavy lines depict the boundaries of the past and of the future of Σ_2. Notice that they intersect off Σ_2.

the convention of this article becomes $2k$ in the convention of Klösch and Strobl.) Moreover, given a kink with a Killing field, they are able to construct from it locally isometric but globally inequivalent kinks by cutting out parts of the manifold and matching up points of the resulting boundaries along the Killing trajectories. It would be interesting to know if there are suitable generalizations of these constructions in higher dimensions.

III. Causal Properties of Kinks

Consider a kink spacetime $\mathcal{M} = R \times \Sigma$ with Σ compact. Then $\mathcal{M}$ has two putative asymptotic ends. One's first expectation might be for conformal completion to result in a manifold-with-boundary $\overline{\mathcal{M}} = [0,1] \times \Sigma$, the boundary being comprised of two disjoint copies of Σ, both being at infinity (if the kink is already geodesically complete).

However, if the kink is a DHW kink, then this is not the case. To see this, notice that if the conformal completion were as just described, then the metric would extend continuously to both boundary components, which would therefore be kink surfaces and so would be somewhere timelike. Let Σ_∞ denote the component corresponding to $t \to \infty$. By Remark 1(a),[4] we note that $Q := I^+[\Sigma_\infty] \cap I^-[\Sigma_\infty]$ will be a subset of $I^+[\Sigma_t] \cap I^-[\Sigma_t]$ for any t, and so will be a subset of $\mathcal{F} \cap \mathcal{P}$, which by Remark 1(d) is empty. This contradicts the assumption that the completion at $t \to \infty$ is achieved by appending a surface Σ_∞, since Σ_∞ would have to be somewhere timelike, whence Q could not be empty.

An example is provided by the metric (3b) in the limit $\tau \to -\infty$ (which corresponds to the limit $t \to +\infty$ in the coordinates used in the metric (5) and in this section). Completion of this spacetime in this limit is achieved by filling in the puncture with a single point κ at infinity, not a boundary surface.

[4]If signs are reversed, as per the footnote to Remark 1(a), then the argument given here applies instead to the boundary component $\Sigma_{-\infty}$ associated to $t \to -\infty$.

A notable property of this point at infinity is that it is both a future endpoint of causal curves and also a past endpoint. All well-behaved kinks have points at infinity with this property (this essentially explains Remarks 1(b) and 1(c)). To see this, note that kinks cannot be globally hyperbolic and apply the following result of Penrose [9]. Penrose's lemma is conveniently expressed in the language of Indecomposable Past (IP) and Future (IF) sets. We review this terminology in Appendix I.

Lemma 2 (Penrose): *A strongly causal spacetime is not globally hyperbolic*

(i) iff there is a Proper IP (PIP) containing a Terminal IP (TIP)

(ii) iff there is a Proper IF (PIF) containing a Terminal IF (TIF).

We will refer to a TIP contained within a PIP as being "visible," since every point in the TIP is visible from a (fixed) point of spacetime (some may prefer the term "naked" instead of visible, but "naked" sometimes means "visible from infinity"). As discussed in Appendix I, TIPs are divided into singular TIPs or ∞-TIPs; if visible, we refer to them as "visible singularities" or "visible infinities" respectively. The lemma ensures that "non-singular" (actually: strongly causal, not visibly singular) kinks have visible infinities. Given our earlier deduction concerning the non-existence of the putative boundary surface Σ_∞ for DHW kinks, it is hard to see how such kinks can have any points at infinity that are not visible.

Conjecture 3: Let $\mathcal{S} := \overline{\mathcal{M}}\backslash\mathcal{M}$ denote the set of points in the conformal completion of a DHW kink that are not points of spacetime. Then $\mathcal{S}$ has a connected component consisting of a compact set, every point of which is visible.

For the punctured Minkowski spacetime, the visible TIP and TIF (there is only one of each) both correspond to the puncture. If the metric is given by (3a), this is a singular TIP/TIF, whereas if the metric is given by (3b) then the conformal transformation places the origin at infinite distance and so we have an ∞-TIP/TIF. The case wherein the compact set of visible points at infinity consists of a single point yields a particularly simple incompleteness theorem.

Definition 4: A point p in the conformal completion $\overline{\mathcal{M}}$ of a spacetime $\mathcal{M}$ is an *isolated point at infinity* if p is both an ∞-TIP and an ∞-TIF possessing a neighbourhood $\mathcal{U} \subseteq \overline{\mathcal{M}}$ with compact boundary in $\mathcal{M}$ that divides $\mathcal{M}$ in two.

Theorem 5: *If spacetime is strongly causal, obeys the strong energy and generic curvature conditions, and has an isolated point κ at infinity, then it is non-spacelike geodesically incomplete.*

Proof: Choose a timelike geodesic which extends back to the isolated point κ in the past, and assume this geodesic is also future-complete (for if it is not then we are done). Choose a sequence of points $p_{n+1} < p_n$ along this geodesic, converging to κ (here $x < y \Leftrightarrow x \in I^-(y)$). This sequence diverges to infinity in the sense of p. 207 of [10]. Since the geodesic is assumed complete, we can choose another sequence $q_n < q_{n+1}$ along it such that the affine parameter at successive q_ns diverges to infinity as $n \to \infty$. By strong causality, the geodesic cannot be (totally or partially) future-imprisoned within any compact set, and so this sequence also diverges to infinity.

Since κ is an isolated point, we can surround it by a neighbourhood U in spacetime with compact boundary K as above and require $p_n \in U$. As well, if U is small enough then $q_n \notin U$ $\forall n$ (since otherwise the q_n would accumulate to κ, in which case it would be easy to construct a closed causal curve joining p_n to q_n to p_n for some n). Every curve, and hence every causal curve, from p_n to q_n must meet K. Then the spacetime is said to be *causally disconnected* by the set K (*cf.* p. 207 of [10]). Application of Theorem 11.41 on p. 390 of [10] completes the proof. ∎

This result is obviously evidence in favour of the view that kink spacetimes are incomplete. However, it was obtained by making a great many assertions which may not always be true. Moreover, it relies on the strong energy and generic curvature conditions, both of which are sometimes thought to be unnecessarily restrictive.

It seems possible that Remark 1(a) may hold for a much wider class of kinks than DHW kinks, and certainly Remarks 1(b,c) are general. However, it is easy to envisage kinks in which Remark 1(d) fails. If one were to dispense with Remark 1(d), one would have to deal with fully general compact boundaries, including surfaces. Nonetheless, it seems possible that a causal disconnectivity argument like the above, but relying on null lines rather than general non-spacelike lines could result in a theorem for the general case. Such a method would automatically employ a weaker form of the energy and generic conditions as well.

Finally, what if Σ is not compact? For these kinks, one always imposes boundary conditions which amount to existence of a smooth extension of the metric under a one-point compactification of Σ. The metric may not extend differentiably under this procedure, but one might expect reasonable behaviour when the metric falls off sufficiently rapidly near the asymptotic end of Σ. Figure 3 gives a conformal diagram for a DHW 1-kink with an asymptotically flat region. In this particular case, the asymptotically flat region extends back into the past from a point κ at infinity having many of the properties used above, and in fact the above argument still goes through. If the asymptotically flat region had extended into the future instead, a time-reversed version of the above argument would work.

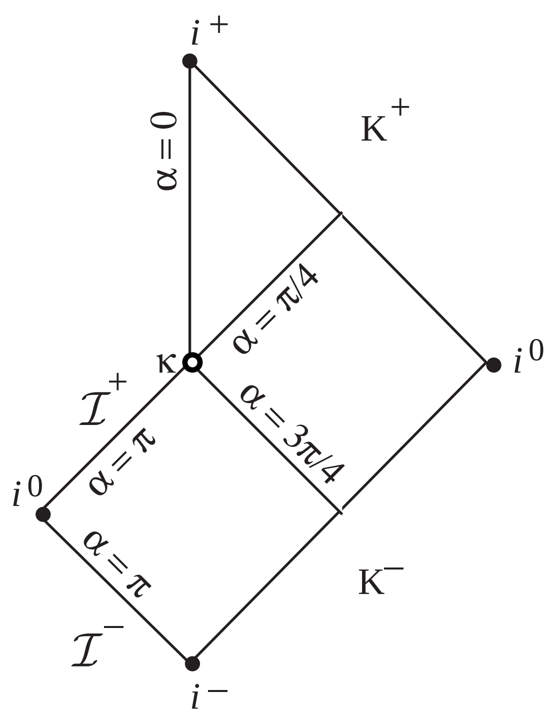

Figure 3. Conformal diagram of a 1-kink with an asymptotically flat region.

IV. CONCLUDING REMARKS

Although the above discussion is far from complete or rigourous, one can deduce from it that many known kink solutions, and sufficiently mild perturbations thereof, will not provide physically acceptable models of black holes without singularities, contrary to the optimism evident in [11].

Finally, the oral presentation of this material generated questions concerning the possible role of kinks in cosmology, in view of the fact that kinks are analogous to "topological defects" of other field theories, which play an essential role in modern cosmology. Assuming Einstein evolution, then one cannot address the question of whether kinks may "form" from given initial data since the lack of global hyperbolicity implies that there is no Cauchy surface on which to give the initial data. However, the question probably should be treated in a quantum context. But the kink number of the Universe is not a quantum observable. Kink number labels the homotopy class of the spacetime metric; in other words, it labels connected components of the phase space, not the configuration space, of the theory. The question of to which connected component the metric belongs is analogous to the question of through which slit an electron has passed in a double-slit experiment, and is therefore unanswerable.

ACKNOWLEGEMENTS

I'd like to thank Andrew Chamblin, Tina Harriott, Rafael Sorkin, and Jeff Williams for discussions concerning the first three sections of this paper. In particular, the method of proof for the Theorem draws directly from ideas of Chamblin, while the example of the hypersurface with handles in Minkowski space was brought to my attention during a discussion with Sorkin. I would like to thank Bernard Carr, G.F.R. Ellis, and Jack Gegenberg for the questions that led to some of the comments in the final section, and Don Page for a very helpful explanation of quantum cosmology, both generally and as applied to kinks.

REFERENCES

[1] Finkelstein, D., and Misner, C.W., 1959, *Ann. Phys., NY* **12**, 230–243.

[2] Williams, J.G., 1997, this volume.

[3] Hawking, S.W., and Gibbons, G.W., 1992, *Phys. Rev. Lett.* **69**, 1719–21.

[4] Chamblin, H.A., and Penrose, R., 1992, *Twistor Newsletter* **34**, 13–8.

[5] Harriott, T.A., 1997, this volume.

[6] Chamblin, H.A., 1995, DAMTP preprint R95/44.

[7] Dunn, K.A., Harriott, T.A., and Williams, J.G., 1996, *J. Math. Phys.* **37**, 5637–5651.

[8] Klösch, T., and Strobl, T., 1997, preprint gr-qc/9707053.

[9] Penrose, R., 1978, in *Theoretical Principles in Astrophysics and Relativity*, ed Lebovitz, N.R., Reid, W.H., and Vandervoort, P.O., (Chicago: University of Chicago), p 217–42.

[10] Beem, J.K., and Ehrlich, P., 1981, *Global Lorentzian Geometry*, first edition (New York: Dekker).

[11] Finkelstein, D., and McCollum, G., 1975, *J. Math. Phys.* **16**, 2250–9.

[12] Geroch, R.P., Kronheimer, E.H., and Penrose, R., 1972, *Proc. Roy. Soc. Lond.* A**327**, 545–67.

[13] Dunn, K.A., Harriott, T.A., and Williams, J.G., 1994, *J. Math. Phys.* **35**, 4145–4156.

[14] Dunn, K.A., Harriott, T.A., and Williams, J.G., 1997, *Proc. Sixth Can. Conf. Gen. Rel. Rel. Astrophys.*, ed Braham, S.P., Gegenberg, J., and McKellar, R.J. (*Fields Inst. Comm.* **15**, AMS, Providence).

Appendix I: Indecomposable Past and Future Sets

Section III makes use of the Geroch, Kronheimer, and Penrose completion [12] of strongly causal spacetimes. The technique is to construct a new space, the space of IPs, or *Indecomposable Past Sets.* An IP is a past set X (a past set is one that equals its chronological past, so X obeys $I^-[X] = X$; a suitable alternative definition sometimes used is an *open* set that contains its own chronological past) which cannot be decomposed into two proper subsets that are themselves past sets. We also may construct the space of IFs, or *Indecomposable Future Sets*; these sets are dually defined.

All IPs may be represented by sets of the form $I^-[\gamma]$ where γ is a timelike curve. If γ can be assigned a future endpoint p, then the IP is a PIP (a *Proper* IP), and can be represented as $I^-(p)$. If instead γ is inextendible, then the corresponding IP is *Terminal*, and is called a TIP. In a past-distinguishing spacetime,[5] the PIPs are in one-to-one correspondence with the points of spacetime. Dual definitions and results exist for IFs. In particular, in future- and past-distinguishing spacetimes, the PIFs, PIPs, and points of spacetime are all in one-to-one correspondence and so all three can be identified.

The TIPs are further divided into two classes; those which can be written as the past of a timelike curve of infinite proper length into the future are called ∞-TIPs, while those that cannot be written this way are called *singular* TIPs. The idea is to think of the TIPs and TIFs as representing idealized endpoints of timelike curves—since points of spacetime are PIPs/PIFs, these endpoints are boundary points for spacetime. This interpretation is appropriate only if spacetime is strongly causal,[6] since in strongly causal spacetimes timelike curves of infinite proper length cannot remain within any compact subset of spacetime, nor return infinitely many times to any compact set.

Appendix II: DHW kinks and the de Sitter kink

Here we give a few more details concerning the $\varphi = 0$ case of the metric (5) of the text, and discuss a particular example.

Setting $\varphi = 0$ in (5), we easily obtain first integrals of the radial geodesic equations, expressed as functions $t = x^0(\tau)$, $r = x^1(\tau)$ of an affine parameter τ:

$$u^0 := \frac{dx^0(\tau)}{d\tau} = \frac{1 \mp \sin 2\alpha\sqrt{1-\epsilon e^\chi \cos 2\alpha}}{e^\chi \cos 2\alpha} \quad , \tag{A.II.1a}$$

$$u^1 := \frac{dx^1(\tau)}{d\tau} = \pm\sqrt{1-\epsilon e^\chi \cos 2\alpha} \quad , \tag{A.II.1b}$$

[5] A past-distinguishing spacetime is one in which $I^-(p) = I^-(q) \Rightarrow p = q$.

[6] Every strongly causal spacetime is future- and past-distinguishing.

where

$$\epsilon = -g_{ab}u^a u^b \quad , \tag{A.II.2}$$

so ϵ is a constant of the motion and distinguishes whether the geodesic is timelike ($\epsilon > 0$ in our signature), spacelike, or null. These equations are easily solved by quadratures.

By setting $\epsilon = 0$ in (A.II.1b) we see that the radial coordinate induces an affine parameter along radial null geodesics. Radial null geodesics that do not approach infinite r are incomplete.

An interesting 2-dimensional kink arises by restricting to a (t, r) section and setting $\alpha(r) = r$, $\chi = 0$. This results in an isometric universal cover of the Misner torus.

A 4-dimensional example of a $\varphi = 0$ DHW kink is provided by the so-called de Sitter kink [13]. To define this kink, choose $\chi(r) = 0$ and $\alpha(r) = \arcsin(r/\sqrt{2})$. Since $r \in [0, \sqrt{2}]$, then $\alpha(r) \in [0, \pi/2]$, so the r-coordinate covers only half a kink.

On this half-kink, then

$$ds^2 = -(1-r^2)dt^2 - 2r\sqrt{2-r^2}dtdr + (1-r^2)dr^2 + r^2\Big(d\theta^2 + \sin^2\theta d\phi^2\Big) \quad . \tag{A.II.3}$$

The coordinate transformation

$$\hat{t} = \frac{e^{-t+\sqrt{2-r^2}}}{\sqrt{2-r^2}+1} \quad , \tag{A.II.4a}$$

$$\hat{r} = r\hat{t} \quad , \tag{A.II.4b}$$

brings the metric to the form

$$ds^2 = \frac{1}{\hat{t}^2}\Big[-d\hat{t}^2 + d\hat{r}^2 + \hat{r}^2\Big(d\theta^2 + \sin^2\theta d\phi^2\Big)\Big] \quad . \tag{A.II.5}$$

This form of the de Sitter metric is familiar and shows it to be conformal to a region of Minkowski spacetime. The region covered by the "half-kink" is the wedge-shaped region of Minkowski spacetime lying above the cone $r = \sqrt{2}\hat{t}$. This region is globally hyperbolic. The surfaces corresponding to the $t = const.$ surfaces are the level sets of

$$f(\hat{t}, \hat{r}) = \frac{e^{\sqrt{2-\hat{r}^2}}}{\hat{t}(1+\sqrt{2-\hat{r}^2})} \quad . \tag{A.II.6}$$

The full kink is constructed by taking a second wedge of Minkowski spacetime, this one lying between the cone $r = -\sqrt{2}\hat{t}$ and the negative t-axis. The two wedges are then joined along the cones. This cannot be done smoothly [14].