

The Cosmic Censor Forbids Naked Topology

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Abstract

For any asymptotically flat spacetime with a suitable causal structure obeying (a weak form of) Penrose's cosmic censorship conjecture and satisfying conditions guaranteeing

focusing of complete null geodesics, we prove that active topological censorship holds. We do not assume global hyperbolicity, and therefore make no use of Cauchy surfaces and their topology. Instead, we replace this with two underlying assumptions concerning the causal structure: that no compact set can signal to arbitrarily small neighbourhoods of spatial infinity (“ i^0 -avoidance”), and that no future incomplete null geodesic is visible from future null infinity. We show that these and the focusing condition together imply that the domain of outer communications is simply connected. Furthermore, we prove lemmas which have as a consequence that if a future incomplete null geodesic were visible from infinity, then given our i^0 -avoidance assumption, it would also be visible from points of spacetime that can communicate with infinity, and so would signify a true naked singularity.

I. Introduction

The active topological censorship theorem of Friedman, Schleich, and Witt [1] (hereinafter FSW) states that, under suitable conditions (including asymptotic flatness and an energy condition), any causal curve beginning and ending at infinity in a globally hyperbolic spacetime must be deformable to a curve that is always near infinity, and so cannot “probe the topology of spacetime.”

Since the original FSW result, several key issues have been resolved. Among them, the FSW result has been extended to prove simple connectedness of the domain of outer communications [2] and a “local” form of the censorship theorem has been established [3]. A counter-example to the so-called passive topological censorship theorem has been given [4]. In addition, the theorem has been applied to answer questions concerning the topology of black hole horizons [5,6].

In this paper, we wish to clarify the role of a chief assumption of the FSW theorem, namely global hyperbolicity. To gain a heuristic understanding of this issue, consider

that the FSW theorem is inspired by the singularity theorems of Gannon [7], who showed that under physically reasonable conditions, there are future-incomplete geodesics lying in the future development of any non-simply connected partial Cauchy surface that is “regular near infinity” (a form of asymptotic flatness). Given the FSW assumptions of global hyperbolicity and asymptotic flatness at null infinity, then we expect these incomplete geodesics to pass behind horizons, so as not to be visible from any point that can signal future null infinity. Then one could interpret the FSW theorem as the statement that the horizons arise early enough to also trap any causal curves which would otherwise “traverse the topology” (and connect up with a causal curve near infinity to form a non-trivial loop).

The role played by horizons in this discussion suggests that the global hyperbolicity assumption is being used to enforce an underlying assumption, namely (weak) cosmic censorship. This has led us to seek a proof of the active topological censorship theorem that does not assume global hyperbolicity, but assumes instead that no naked singu-

larities appear. The purpose of this paper is to present such a theorem, replacing the global hyperbolicity condition by the requirement that there are no future-incomplete null geodesics “visible” (in a sense to be made precise) from a point of the domain of outer communications.

However, global hyperbolicity also plays another role in the FSW theorem. This theorem rests on a lemma which holds only if causal curves beginning in any fixed compact set in spacetime cannot arrive at \mathcal{I}^+ at arbitrarily early times — we term this “ i^0 -avoidance.” This in turn follows from global hyperbolicity, though strictly this inference entails extending the concept of global hyperbolicity to refer to causal curves ending on \mathcal{I} . We will need to use the i^0 -avoidance property herein, but since we make no global hyperbolicity assumption from which it can be derived, we instead impose it *a priori*. It can be rephrased as a type of causality condition imposed only at spatial infinity; specifically, it is the condition of *causal continuity at i^0* .

In the next section, we prove our topological censorship theorem. We first prove

that a failure of topological censorship would result in an incomplete null geodesic visible from a point of \mathcal{I}^+ . We then prove that such a geodesic would necessarily also be visible from points within the domain of outer communications. In the final section, we give some remarks on the importance of the assumption of causal continuity at i^0 and on the geodesic focusing condition that we introduce below.

II. Censorship Theorem

We will assume the boundary-at-infinity of an asymptotically flat spacetime to be a null cone with vertex i^0 representing spatial infinity. We denote it by $\mathcal{I} = \mathcal{I}^- \cup \{i^0\} \cup \mathcal{I}^+$ and its domain of outer communications by

$$\mathcal{D} := I^+(\mathcal{I}^-) \cap I^-(\mathcal{I}^+) \quad .$$

It is convenient to extend a little beyond \mathcal{I} ; then \mathcal{D} is embedded in a larger manifold and \mathcal{I} is its boundary. We remark that in the more general case of a boundary which is

a disjoint union of cones, our arguments apply to each cone and its associated domain of outer communications individually. We will also assume that there is a neighbourhood \mathcal{N} of \mathcal{I} such that $\mathcal{N} \cap \mathcal{D}$ has topology $\mathfrak{R} \times \Sigma$ where the \mathfrak{R} fibres are timelike and the surfaces Σ are spacelike and have the topology of the region exterior to a sphere in \mathfrak{R}^3 — note that this yields what are referred to as simply connected neighbourhoods of infinity (one of these surfaces may be completed by appending i^0 to it, the rest are bounded by 2-sphere cuts of \mathcal{I}).

It is helpful to introduce the following definition.

Definition 1: A future-inextendible causal curve ζ will be said to be *visible* from a point x iff it is contained in $\overline{I^-(x)}$ (the overhead bar indicates closure).

Although ζ may lie entirely outside $J^-(x)$, if it is visible from x in the sense of Definition 1 then $I^-(\zeta) \subseteq I^-(x)$; *i.e.* every point of the *indecomposable past set* defined by ζ can signal x . It is in this manner that we interpret the term “visible.” This sense is used when discussing visibility of singularities whose presence is indicated by future-

incomplete null geodesics. This point of view is similar to that previously put forth by Penrose [8], in which the version of cosmic censorship denoted by “CC8” is essentially that which we use as assumption (iii) in our first (and key) result:

Theorem 2: *Let \mathcal{D} be the domain of outer communications of an asymptotically flat spacetime \mathcal{M} and let $\mathcal{H}^+ := I^+(\mathcal{I}^-) \cap \partial I^-(\mathcal{I}^+)$ be the union of all future event horizons that may be present. If*

- (i) *causal continuity holds at i^0 ,*
- (ii) *every complete null geodesic $\gamma \subseteq \mathcal{D} \cup \mathcal{H}^+$ possesses a pair of conjugate points, and*
- (iii) *every future-inextendible null geodesic visible from any point $q \in \mathcal{I}^+$ is future-complete in \mathcal{M} ,*

then \mathcal{D} is simply connected.

The following remarks discuss, in order, assumptions (i–iii).

Remark 2.1: The set-valued function I^+ is causally continuous at a point x iff for

each compact set K that does not meet $\overline{I^+(x)}$, there is a neighbourhood $O \ni x$ such that, for any $y \in O$, K does not meet $\overline{I^+(y)}$; dually for I^- . Causal continuity is said to hold at x if both I^+ and I^- are causally continuous there. At $x = i^0$, since no subset of spacetime meets $\overline{I^-(i^0)} \cup \overline{I^+(i^0)}$, causal continuity reduces to the statement that for any compact set $K \subseteq \mathcal{D}$, then $i^0 \notin \overline{I^-(K)} \cup \overline{I^+(K)}$. It is this phrasing which we employ in the proof.

Remark 2.2: By standard theorems, complete null geodesics always have a pair of conjugate points provided the so-called *null generic condition*¹ and a suitable energy condition hold. While the pointwise null energy condition $R_{ab}n^an^b \geq 0$ for all null n^a suffices, so do the much milder integrated energy conditions, such as the Averaged Null Energy Condition [9] and the Borde condition [10].

¹ This is the condition that for each complete null geodesic, if n^a is tangent to the geodesic, then $n^an^bn_{[c}R_{d]ab[e}n_{f]}$ must be non-zero somewhere along the geodesic.

Remark 2.3: Any null geodesic beginning on \mathcal{I}^- that is visible from a point of \mathcal{I}^+ is necessarily contained in $\mathcal{D} \cup \mathcal{H}^+$. If such a geodesic is future-complete in \mathcal{M} , it is also future-complete in any open set containing $\mathcal{D} \cup \mathcal{H}^+$.

Proof: By way of contradiction, we will assume that conditions (i) and (ii) hold and that \mathcal{D} is *not* simply connected, and prove that (iii) cannot hold in these circumstances.

We begin by extending \mathcal{D} somewhat into any black hole regions that may be present. Then $\mathcal{H}^+ = I^+(\mathcal{I}^-) \cap \partial I^-(\mathcal{I}^+)$ is an achronal hypersurface in $I^+(\mathcal{I}^-)$ (viewed as a spacetime in its own right). Let h_{ab} be a Lorentzian metric on \mathcal{M} strictly narrower than the given metric g_{ab} (*i.e.*, any causal vector of h_{ab} is timelike in g_{ab}). Then \mathcal{H}^+ is an edgeless acausal hypersurface — *i.e.*, a partial Cauchy surface — in the spacetime $(I^+(\mathcal{I}^-), h_{ab})$. Let \mathcal{U} be the interior of the domain of dependence of \mathcal{H}^+ in $(I^+(\mathcal{I}^-), h_{ab})$. Finally, let $\mathcal{D}' = \mathcal{D} \cup \mathcal{U}$. Observe that by construction there is a deformation retraction of \mathcal{D}' onto $\mathcal{D} \cup \mathcal{H}^+$, whence $\pi_1(\mathcal{D}') = \pi_1(\mathcal{D} \cup \mathcal{H}^+) = \pi_1(\mathcal{D})$, where the second equality follows, for example, by the Seifert-Van Kampen theorem.

We will employ the universal covering spacetimes and projections $\pi : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$ and $\pi' : \tilde{\mathcal{D}}' \rightarrow \mathcal{D}'$. By our construction, we may embed $\tilde{\mathcal{D}}$ in $\tilde{\mathcal{D}}'$; then the region $\tilde{\mathcal{D}}' \setminus \tilde{\mathcal{D}}$ is not in causal contact with future null infinity (if it were otherwise, the causal curves mediating such contact would project to curves which would causally connect a black hole in spacetime to \mathcal{I}^+).

The boundary-at-infinity of $\tilde{\mathcal{D}}'$ will have at least two disjoint components (we will extend π' to this boundary). Let S be a set indexing these components. By a result of [2], there exist $\alpha, \beta \in S$ (with $\alpha \neq \beta$) such that the distinct components \mathcal{I}_α and \mathcal{I}_β are joined by a causal curve, say beginning on \mathcal{I}_α^- and ending at some point $Q \in \mathcal{I}_\beta^+$.²

² The argument of [2] may be summarized as follows. By definition, every point of \mathcal{D} lies on some causal curve from \mathcal{I}^- to \mathcal{I}^+ . By lifting these curves, we see that every point of $\tilde{\mathcal{D}}$ lies on a causal curve from some \mathcal{I}_α^- to some \mathcal{I}_β^+ (where possibly $\alpha = \beta$), so these causal curves cover $\tilde{\mathcal{D}}$. If every infinite causal curve in $\tilde{\mathcal{D}}$ were to begin and end on the same component of its \mathcal{I} , then $\{I^+(\mathcal{I}_\alpha^-) \cap I^-(\mathcal{I}_\alpha^+) \mid \forall \alpha \in S\}$ would be a disjoint open

Now $\overline{I^-(Q)}$ does not contain i_α^0 . To see this, consider any causal curve Λ from \mathcal{I}_α^- to $Q \in \mathcal{I}_\beta^+$, and project it back to $\mathcal{D}' \cup \mathcal{I}$, obtaining a curve λ that begins on \mathcal{I}^- and ends on \mathcal{I}^+ . By joining a null generator of \mathcal{I} on to this curve, one can form a closed curve, which is necessarily a non-trivial loop. By the assumed asymptotic topology, every loop in the asymptotic region is trivial, so λ must pass through the inner boundary of the asymptotic region. Fix an infinite timelike cylinder C surrounding this boundary — clearly λ must meet this cylinder as well, as must any other curve “threading the topology.” Consider the spacelike surface Σ_t (a leaf of the asymptotic topology $\mathfrak{R} \times \Sigma$) whose boundary includes $q := \pi'(Q) \in \mathcal{I}^+$. The surface $S_t := C \cap \Sigma_t$ is a 2-sphere and,

 cover for $\tilde{\mathcal{D}}$, which violates the fact that $\tilde{\mathcal{D}}$ is connected, unless S has only one element. Therefore if distinct components of infinity exist for $\tilde{\mathcal{D}}$, then each such component must be joined to some other by a causal curve. While this argument was made in \mathcal{D} and its covering spacetime, since $\tilde{\mathcal{D}} \subseteq \tilde{\mathcal{D}}'$ and since their boundaries-at-infinity correspond, the result also applies to $\tilde{\mathcal{D}}'$.

following any curve into the past from q , if the curve meets C it does so in the past of S_t . Since S_t is compact, then by assumption $\overline{I^-(S_t)}$ does not contain i^0 , so no sequence of curves of this type, considered prior to their final crossing of C , can approach i^0 . This implies that no sequence of causal curves in $\tilde{\mathcal{D}}'$ from \mathcal{I}_α^- to Q can approach i_α^0 , which is what we set out to show.

Since there is at least one causal curve from \mathcal{I}_α^- to Q and since $i_\alpha^0 \notin \overline{I^-(Q)}$, then $\partial I^-(Q)$ must meet \mathcal{I}_α^- . Let P lie in this intersection. From P , there will be a future-null geodesic Γ which enters $\tilde{\mathcal{D}}'$ (since Γ cannot contain i_α^0), remains on $\partial I^-(Q)$, and does not have a future endpoint other than Q .

Now Γ cannot have conjugate points, since otherwise standard results would imply that it could not lie on $\partial I^-(Q)$. Then its projection $\gamma =: \pi'(\Gamma)$ cannot have conjugate points either. But γ enters \mathcal{D}' (since Γ enters $\tilde{\mathcal{D}}'$) and $\gamma \subseteq \overline{I^-(q)}$ (since $\pi'(\overline{I^-(Q)}) \subseteq \overline{I^-(q)}$), so $\gamma \subseteq \mathcal{D} \cup \mathcal{H}^+$. By assumption, every complete null geodesic in $\mathcal{D} \cup \mathcal{H}^+$ has conjugate points, so γ must be incomplete, considered as a curve in \mathcal{D}' .

By Remark 2.3 this geodesic is incomplete in the spacetime \mathcal{M} . Since it lies in $\overline{I^-(q)}$, it is visible from $q \in \mathcal{I}^+$. ■

We now seek to show that the incomplete geodesic γ is visible not only from a point of \mathcal{I}^+ but also from a point of spacetime which can signal \mathcal{I}^+ . To show this, we give a useful definition and then two lemmata based upon it. Then our final censorship theorem will follow immediately from these results.

Definition 3: We will say that a future-inextendible causal curve γ is *weakly visible* from a point x iff for every t in the domain of γ there is a point x_t such that³

³ Here, and in what follows, we use *nets* rather than sequences. In the present context, this is merely a convenience, allowing us to use the continuous parameter t as an index rather than having to introduce an integer n . The reader more comfortable with sequences may here define an increasing sequence t_n and replace $\gamma(t)$ and x_t by $\gamma(t_n)$ and x_n in the present definition. Similar modifications may then be made in what

- (i) $\gamma(t) \in \overline{I^-(x_t)}$, and
- (ii) the net of points x_t has x as an accumulation point.

If a future-inextendible causal curve γ is visible from x , then it is weakly visible from x , but the converse to this is not always true. However, we have the following lemma:

Lemma 4: *If a future-inextendible causal curve γ is weakly visible from z , then it is visible from all points $w \in I^+(z)$.*

Proof: First we choose some T in the domain of γ and show that $\gamma(T) \in \overline{I^-(w)}$. To do this, note that $I^-(w)$ is a neighbourhood of z . By definition, there is a net of points z_t with accumulation point z such that $\gamma(t) \in \overline{I^-(z_t)}$. It follows that for any choice of T in the domain of γ , and hence for our choice, there is a $t > T$ such that $z_t \in I^-(w)$ and $\gamma(t) \in \overline{I^-(z_t)}$, whence $\gamma(t) \in \overline{I^-(w)}$. But then it is easy to show that $\gamma(T) \in \overline{I^-(w)}$,

follows.

since $\gamma(t) \in \overline{I^-(w)}$ and since $\gamma(T) \in J^-(\gamma(t))$.

But we can repeat the argument (with the same w) for all choices of T in the domain of γ , so $\gamma \subseteq \overline{I^-(w)}$. ■

In virtue of this lemma, we need only show that our incomplete geodesic is weakly visible from some point of spacetime, whence it will also be *visible* from some point. But by the next lemma, weak visibility from a point of spacetime follows, given weak visibility from a point of \mathcal{I}^+ and “ i^0 -avoidance” (our causal continuity assumption).

Lemma 5: *If spacetime contains a future-inextendible causal curve γ weakly visible from a point of \mathcal{I}^+ and if causal continuity holds at i^0 , then γ is weakly visible from some point w in the domain of outer communications.*

Proof: Take γ to be weakly visible from some $q \in \mathcal{I}^+$. Along the null generator of \mathcal{I}^+ from i^0 to q , let Q be the infimum of all points from which γ is weakly visible.

Note first that $Q \neq i^0$. To see this, let K be a compact neighbourhood about some

point $\gamma(T)$ in the incomplete curve. Then $I^+(K)$ will be a neighbourhood of $\gamma(t)$ for each $t > T$. Then by the definition of weak visibility, for each $t > T$ there is a sequence of timelike curves ending on x_t (*cf.* Definition 3) whose past endpoints converge on $\gamma(t)$, so these curves eventually all lie in $I^+(K)$. Then for $t > T$, the x_t themselves obviously lie in $I^+(K)$. It follows that the accumulation points x of the net of points x_t are contained in $\overline{I^+(K)}$. In turn, since the points x that lie along the given generator of \mathcal{I}^+ themselves constitute a net in $\overline{I^+(K)}$, then $\overline{I^+(K)}$ also contains the infimum Q of the points x . But by the assumption of causal continuity at i^0 , $\overline{I^+(K)}$ does not contain i^0 , since K is compact, whence $Q \neq i^0$.

Now let τ be the supremum in \mathfrak{R} of the domain of γ , and parametrize the null generator of \mathcal{I}^+ through Q so that the parameter has value τ at Q and decreases to the future. Denote the points of the generator by $q(s)$. By assumption, γ is weakly visible from $q(s)$ for $s < \tau$, so for each $s < \tau$ there is a net of points $x_t(s)$ accumulating at $q(s)$ and satisfying Definition 3. From these nets, construct the new net of points $x_s(s)$,

and note that it has Q as an accumulation point (incidentally, this construction proves that γ is weakly visible from Q), whence it contains subnets which converge to Q ; let z_t belong to such a subnet.

Choose a neighbourhood $U \ni Q$, small enough so that the contact between $\overline{I^-(Q, U)}$ and \mathcal{I}^+ is along one null generator.⁴ Surround Q by a closed surface $S \subseteq U$ with interior O . Then the z_t are “eventually all” —*i.e.* for $t > T$ — inside O . Since $\gamma(t) \subseteq \overline{I^-(z_t)}$, for each t there is a sequence of causal curves, each curve having future endpoint z_t , whose past endpoints converge to $\gamma(t)$. For $t > T$, these curves all meet and penetrate S , and some may even do so more than once, but there is always a final penetration point along each curve as it enters O for the last time and proceeds to its endpoint at z_t . For each $t > T$, one can construct such a “sequence of (final) penetration points” and it will have an accumulation point on S since S is compact; label this point $a(t)$

⁴ While this is usually the case for all such U , it could fail if causality violations occurred at Q and if U were big enough to wholly contain the curves responsible.

(if there are multiple accumulation points, choose any one). Increasing t within the domain of γ , we obtain a net of accumulation points $a(t)$, and this net will have its own accumulation point $y \in S$ (again, if there are multiple accumulation points, choose any one). Note that γ is weakly visible from y .

The point y must lie in $\overline{I^-(Q, U)}$ but cannot lie along \mathcal{I}^+ (since Q is an infimum for the set of points on the given generator of \mathcal{I}^+ from which γ is weakly visible). Therefore, $y \in \mathcal{D}$. ■

Finally, we state our ultimate result, whose proof now follows trivially from the preceding results.

Theorem 6: *If an asymptotically flat spacetime with domain of outer communications \mathcal{D} obeys conditions (i) and (ii) of Theorem 2 and if no future-incomplete null geodesic is visible (cf. Definition 1) from any point $x \in \mathcal{D}$, then \mathcal{D} is simply connected.*

Proof: By Theorem 2, a non-simply connected domain \mathcal{D} could be compatible with

assumptions (i) and (ii) of that theorem only if there were to exist an incomplete null geodesic ζ visible — and therefore *weakly* visible — from a point of \mathcal{I}^+ . But this would violate the new assumption of Theorem 6, for by Lemma 5, ζ would be weakly visible from some point $y \in \mathcal{D}$ and hence, by Lemma 4, visible from any point $x \in I^+(y)$. Therefore, if the assumptions of Theorem 6 hold, then \mathcal{D} must be simply connected. ■

Final Remarks

We have given an active topological censorship theorem in which the usual assumption of “extended” global hyperbolicity is discarded in favour of two conditions which would ordinarily follow from it under the circumstances, namely cosmic censorship and “ i^0 -avoidance,” the latter expressed as a causal continuity condition at i^0 .

A question that arises is whether one might eventually weaken or altogether do away with this last condition. In the introductory section, we argued that the causal continuity condition at i^0 , sometimes in the guise of a generalized global hyperbolicity condition, plays an important role in previous proofs of topological censorship. More-

over, there is some evidence that one cannot completely dispense with conditions of this general sort. In particular, we refer to recent work of Schein and Aichelburg [11], taken together with an example quoted in [12] and attributed there to Bardeen. Ref. [11] gives a traversable wormhole solution satisfying energy conditions. This solution has incomplete geodesics and violates the i^0 -avoidance assumption; indeed, through every point of the asymptotic region there is a closed timelike curve. However, the incomplete geodesics arise from singularities of the sort familiar from the extended Reissner-Nordström solution, and according to the Bardeen example it should be possible to remove them. If so, the resulting spacetime (or, rather, a perturbation thereof, which appears necessary to enforce the null generic condition and concomitant focusing lemma) would provide an example of a spacetime that obeys all our conditions except the causal continuity condition at i^0 , and in which topological censorship violations arise as a result.

We close with a remark concerning our requirement that complete geodesics should have conjugate points. Remark 2.2 notes that this follows from integrated energy con-

ditions and the null generic condition. While the FSW theorem also follows from these assumptions, it follows as well from an integrated energy condition on null geodesics that are complete in only the future direction [2]. In this form of the FSW theorem, the generic condition is not needed. Then the question arises as to whether our present theorem can be recast in a form that is so explicitly independent of the null generic condition. Preliminary investigations show that it can, but at the cost of introducing new assumptions concerning the global structure which serve to obscure somewhat the role of cosmic censorship. Whether a more suitably direct theorem along these lines can be proved remains an open issue.

Acknowledgements

We would like to thank the Banach Center, Polish Academy of Sciences, and the organizers of its Minisemester on Mathematical Aspects in Theories of Gravitation for hospitality during the initiation of this work.

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