

Lab 15—Power Method and Dominant Eigenvalues

Objective: To see how mathematical techniques are adapted to take advantage of the computer.

MATLAB Commands:

eig(M) Finds eigenvalues of matrix M, including complex eigenvalues when appropriate.

[P,D]=eig(M) Finds matrix P whose columns are the eigenvectors of M, and matrix D whose non-zero entries are the eigenvalues.

Here we will use the theory of eigenvalues and eigenvectors to develop an algorithm for finding certain eigenvectors on computer. The advantage of our algorithm is that it is less subject to round-off error than our row reduction techniques, an important consideration for some computational problems.

Let A be an $n \times n$ matrix with n real distinct eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$; we will assume these are numbered (renumbered, if necessary) so that λ_1 has a larger magnitude than the others. We therefore refer to it as the *dominant* eigenvalue. Then the eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of A form a basis of \mathbf{R}^n , so any vector \mathbf{x} in \mathbf{R}^n can be written as a linear combination of them:

$$\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n \quad (1)$$

1. Derive the following two equations

$$A\mathbf{x} = c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_2 \mathbf{v}_2 + \dots + c_n \lambda_n \mathbf{v}_n \quad (2)$$

$$A^2 \mathbf{x} = c_1 \lambda_1^2 \mathbf{v}_1 + c_2 \lambda_2^2 \mathbf{v}_2 + \dots + c_n \lambda_n^2 \mathbf{v}_n \quad (3)$$

After proving the above two results, it becomes easy to see that

$$A^m \mathbf{x} = c_1 \lambda_1^m \mathbf{v}_1 + c_2 \lambda_2^m \mathbf{v}_2 + \dots + c_n \lambda_n^m \mathbf{v}_n \quad (4)$$

If λ_1 really is larger in magnitude than all the other eigenvalues and if m is a sufficiently large power, then we get from this that

$$A^m \mathbf{x} = c_1 \lambda_1^m \mathbf{v}_1 + \text{small corrections} \quad (5)$$

One last remark: As you know, if \mathbf{x} is an eigenvector with a given eigenvalue, then so is $k\mathbf{x}$ for any k . Thus, only the direction of an eigenvector matters; we can freely choose its length by changing k . Therefore, we will seek eigenvectors of length 1, called *normalized* eigenvectors. If we didn't do this, then terms on the right-hand side of (5) could become very large or very small, leading to problems.

Step 1: Guess an eigenvector \mathbf{x}_0 and normalize it. Call the result \mathbf{y}_0 :

$$\mathbf{y}_0 = \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|}$$

Step 2: Multiply \mathbf{y}_0 once by \mathbf{A} to get a new vector. Normalize the result and call it \mathbf{y}_1 :

$$\mathbf{A}\mathbf{y}_0 = \mathbf{x}_1, \quad \mathbf{y}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$$

Step 3: Multiply \mathbf{y}_1 once by \mathbf{A} to get a new vector. Normalize the result and call it \mathbf{y}_2 :

$$\mathbf{A}\mathbf{y}_1 = \mathbf{x}_2, \quad \mathbf{y}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}$$

Repeat m Times in Total: If m is large enough, \mathbf{y}_{m-1} should equal \mathbf{y}_m to a good approximation. That's when we know to stop. Then \mathbf{y}_m is approximately a normalized eigenvector of \mathbf{A} . Moreover, if it's an eigenvector, we can then write:

$$\mathbf{A}\mathbf{y}_{m-1} \approx \mathbf{A}\mathbf{y}_m = \lambda\mathbf{y}_m$$

which we can use to read off the eigenvalue.

2. For the matrix \mathbf{A} , apply the above method to determine the dominant eigenvalue and its corresponding (normalized) eigenvector, where

$$\mathbf{A} = \begin{bmatrix} 3.72 & 3.19 & 0.93 \\ 0.97 & -3.04 & 2.11 \\ 3.45 & 0.17 & 2.32 \end{bmatrix}$$

Use an initial guess $\mathbf{x}_0 = (1,1,1)$. How many iterations of the method are required in order that the dominant eigenvalue is determined to five digits of accuracy? What happens if you try an initial guess of $\mathbf{x}_0 = (0,-1,-1)$? Does the result make sense? Experiment with a few other initial guesses to see if the choice makes a difference.

3. Find the dominant eigenvalue and corresponding eigenvector for the matrix

$$\mathbf{A} = \begin{bmatrix} 2.9316 & -2.11456 & 2.19873 \\ -0.65052 & 0.81455 & -0.76045 \\ 1.55212 & -0.77982 & 2.4132 \end{bmatrix}$$

4. a) In this procedure, we were careful to normalize the approximate eigenvector at every step. What happens if you do not do this (try it)?

b) Once the dominant eigenvalue and its eigenvector are found, can you suggest how the computer might go about finding the next eigenvalue?