Lab 14—The Gram-Schmidt Procedure

Recall:

`norm(v)`	Returns the length of the vector v.
`dot(a,b)`	Takes dot product of two vectors a and b.
`inv(A)`	Inverse of matrix A.
`eig(A)`	Computes the eigenvalues of matrix A.
`[P,D]=eig(A)`	Finds the diagonal form D of A and the matrix P that diagonalizes A (so $D=P^{-1}AP$).
`A'`	(A-apostrophe) Computes the transpose of A.

The Gram-Schmidt Procedure for a 3-dimensional space:

Let $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ be a basis for a 3-dimensional (sub-)space. From it, we can produce an orthogonal basis as follows:

$$\mathbf{w}_1 = \mathbf{u}_1$$
$$\mathbf{w}_2 = \mathbf{u}_2 - \mathrm{proj}_{\mathbf{w}_1}\mathbf{u}_2$$
$$\mathbf{w}_3 = \mathbf{u}_3 - \mathrm{proj}_{\mathbf{w}_1}\mathbf{u}_3 - \mathrm{proj}_{\mathbf{w}_2}\mathbf{u}_3$$

where of course $\mathrm{proj}_{\mathbf{w}}\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$.

We then get an orthonormal basis $B = \{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ by normalizing, so

$$\mathbf{v}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|},\quad \mathbf{v}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|},\quad \mathbf{v}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_{31}\|}.$$

1. The span of the set of vectors $\{(1,4,1,3),(-2,-2,1,2),(2,-1,2,3)\}$ is a subspace of R^4. Starting with this set, use the Gram-Schmidt procedure (see the hint below) to generate an orthonormal basis for this subspace.

   Remark: In general, you should not expect to get the same basis in both parts of this question. However, since the two bases are both orthonormal, they are related to each other by a change-of-basis matrix which is itself an orthogonal matrix.

   Hint: You will need to compute quantities like $\mathrm{proj}_{\mathbf{w}}\mathbf{u}$. To help you do this, we have installed a routine. Simply type

   `project(u,w)`

   to obtain $\mathrm{proj}_{\mathbf{w}}\mathbf{u}$. If the routine does not work, you can instead have MATLAB explicitly execute:

`(dot(u,w)/dot(w,w))*w` since this equals $\text{proj}_{\mathbf{w}}\mathbf{u}$.

2. Rotation about an arbitrary axis in R^3:

   Find the standard matrix of rotation counter-clockwise by angle $\pi/3$ about the axis that passes through the origin and points in the $(1,1,2)$ direction in R^3. Use the following method:

   (i) First let $\mathbf{w}_1 = (1,1,2)$. Then find two other vectors $\mathbf{w}_2$ and $\mathbf{w}_3$ such that $\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$ is an orthogonal basis. Normalize these vectors to get an orthonormal basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ where $\mathbf{v}_i = \mathbf{w}_i/||\mathbf{w}_i||$. There are several different ways to go about this:

   Method 1: Find any vector $\mathbf{w}_2$ that solves $\mathbf{w}_1 \cdot \mathbf{w}_2 = 0$. Next, find a vector $\mathbf{w}_3$ that solves the system $\mathbf{w}_1 \cdot \mathbf{w}_3 = 0$, $\mathbf{w}_2 \cdot \mathbf{w}_3 = 0$—you can do this by explicitly solving these last two equations or by computing the cross-product $\mathbf{w}_1 \text{x} \mathbf{w}_2$. These vectors are an orthogonal basis. Normalize the vectors $\mathbf{w}_i$ to get an orthonormal basis; call the normalized vectors $\mathbf{v}_i$, so $\mathbf{v}_i = \mathbf{w}_i/||\mathbf{w}_i||$.

   Method 2: Pick two new vectors $\mathbf{x}$ and $\mathbf{y}$ such that $\{\mathbf{w}_1,\mathbf{x},\mathbf{y}\}$ are linearly independent; for example, you might choose them to be $\mathbf{i}$ and $\mathbf{j}$. If so, then $\{\mathbf{w}_1,\mathbf{i},\mathbf{j}\}$ is a basis for R^3, but it's not orthonormal. Apply the Gram-Schmidt procedure, exactly as in the last problem, to get an orthonormal basis.

   (ii) To make sure the basis is right-handed, enter the vectors $\mathbf{v}_i$ as the columns of a matrix and take the determinant. If it is +1, you have a right-handed basis; if it's −1, you have a left-handed basis, so take one of your basis vectors (any one will do) and multiply it by −1 to get a right-handed basis.

   (iii) Write the *B*-matrix for rotation about $\mathbf{v}_1$ "counter-clockwise" by angle $\pi/3$. [Hint: This matrix has precisely the same form as the standard matrix for rotation about the x_1-axis by the same angle.]

   (iv) Write down the change-of-basis matrix P that will transform this *B*-matrix into the standard matrix for this rotation. Confirm that $P^{-1}=P^T$ (this means that P is an orthogonal matrix). Now use the change-of-basis formula $[R]_S = P[R]_B P^{-1}$ to obtain the standard matrix for this rotation.

   (v) Compute $[R]_S[\mathbf{w}_1]$, and use the result to confirm that $\mathbf{w}_1$ is an eigenvector of $[R]_S$ with eigenvalue 1.

   (vi) What vector results from the application of this rotation to the vector $\mathbf{x} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$?