Math 201 A1—Maple Lab 9

Objectives: We will solve a second-order differential equation with non-constant coefficients. The solution will be an unfamiliar function which we can then describe in terms of its graph and its Maclaurin series expansion.

Maple Commands:

`soln:=dsolve({deq,y(0)=y0},y(t));` Solves first-order boundary value problem and stores result in variable called `soln`.

`plot(expr,x=a..b,y=c..d);` Plots Maple expression `expr`. To plot more than one expression on a single set of axes, enclose the list of expressions in curly braces.

`plot(rhs(soln),x=a..b);` Plots solution to boundary value problem above.

`diff(y(x),x$2);` Maple notation for the second derivative $y''(x)$.

`taylor(expr,x=a,n+1);` Computes n^{th} order Taylor polynomial for Maple expression `expr` expanded about centre `x=a`.

`convert(%,polynom);` Converts the previous expression (`%`) to a polynomial. If the previous expression is the output of `taylor()`, then `convert()` removes the `O(x^n)` symbol that appears in that output. This is necessary if the output is to be further manipulated or graphed.

If $F(s)$ is the Laplace transform of $f(t)$, then recall we have the formula

$$\mathcal{L}[t^n f(t)](s) = (-1)^n \frac{d^n F}{ds^n}$$

for n a positive integer.

As well, we have the results for the Laplace transform of the first and second derivatives of a function $y(t)$ whose Laplace transform is $Y(s)$:

$$\mathcal{L}[y'(t)](s) = sY(s) - y(0)$$

$$\mathcal{L}[y''(t)](s) = s^2 Y(s) - sy(0) - y'(0)$$

Exercise: From the above equations, deduce the Laplace transforms of

(i) $ty(t)$ in terms of $Y(s)$ and its derivatives, and

(ii) $ty''(t)$.

(iii) Use this information to show that the Laplace transform of the Bessel equation

$$ty''(t) + y'(t) + ty(t) = 0 \qquad (1)$$

is

$$\left(1+s^2\right)Y'(s) + sY(s) = 0 \qquad (2)$$

(iv) Solve this separable first-order equation (by hand or using `dsolve()`), using the boundary condition $Y(0) = 1$.

`(v)` Have Maple compute the inverse Laplace transform of the solution $y(t) = \mathcal{L}^{-1}[Y(s)]$. Maple's response will probably be unhelpful but you can plot the result. Pass in a copy of this plot (choose the domain so that the plot gives a reasonable illustration of the function—you might start by trying `t=-20..20`). This function is called the Bessel function of order zero. Does the graph resemble any of the solutions to equations studied previously? (*e.g.* overdamped harmonic oscillator? forced harmonic oscillator with resonance? damped harmonic oscillator?)

`(vi)` Have Maple compute the fifth-order Taylor polynomial $T_5(t)$, expanded about centre $t = 0$, of the solution. Notice it has only three terms. Plot this polynomial and the actual solution together on the domain `t=-5..5`. (See the descriptions of the `taylor()`, `convert()`, and `plot()` commands at the beginning of the lab sheet.) Do they agree well? When does the agreement appear (visually) to break down? Where does the agreement appear to fail if we increase the Taylor polynomial to include ten non-zero terms (you may have to expand the domain of the graph somewhat)?

`(vii)` Compute the first and second derivatives of the fifth-order Taylor polynomial $T_5(t)$ and plug them into the left-hand side of equation `(1)`. What should you get? Zero? What do you actually get? Explain.

`(viii)` From Maple's Taylor polynomial calculations, can you guess what the general n^{th} term of the Taylor series about $t = 0$ (*i.e.* Maclaurin series) for your solution is? Using this guess, write the series using sigma notation.

Equation `(1)` arises in a large number of practical situations. For example, it describes the behaviour of a vibrating membrane, such as a musical drum. The solution $y(t)$ we found above is not a simple combination of known functions, but does have a not-too-complicated Maclaurin series, so it is often defined in terms of that series.

By direct substitution, it should be possible to check that the series you write down in part `(viii)` indeed solves `(1)`. This suggests a new idea, that we could try to write any solution of any differential equation as a series $y(x) = \sum a_n x^n$. If we substitute this form into a differential equation that we wish to solve, we will get algebraic equations which we can then solve to find the coefficients a_n. In this way, we obtain the solution as a Maclaurin series. This is a useful technique for equations with non-constant coefficients, such as `(1)` above.