Math 201 A1—Maple Lab 8

Objectives: To use Laplace transforms to investigate resonance phenomena in systems subjected to a periodic discontinuous or non-smooth disturbance.

Maple Commands:

`dsolve({deq,y(0)=a,D(y)(0)=b},y(t),method=laplace);` Solves boundary value problem using Laplace transforms.

`plot(expr,x=a..b);` Plots Maple expression `expr`. To plot more than one expression on a single set of axes, enclose the list of expressions in curly braces. You can also specify the range using `y=c..d`.

`diff(y(x),x$2);` Maple notation for the second derivative $y''(x)$.

`Heaviside(t-a);` The Heaviside step function; zero for $t < a$ and one for $t > a$.

This lab is closely based on Problem 16, p. 157 of *Differential Equations with Maple*, by Coombes *et al.* In an earlier lab, we invesitgated resonance. This occurred when the term on the right-hand side of an inhomogeneous differential equation describing an electric circuit was periodic, with period equal to the *natural period* of solutions of the associated homogeneous equation. When this happened, solutions were oscillations whose amplitude grew with time. In this lab, we will investigate whether this occurs when the inhomogeneous term is discontinuous. We begin with an inhomogeneous equation:

$$\frac{d^2y(t)}{dt^2} + y(t) = h(t) \tag{1}$$

The associated homogeneous equation has general solution $y(t) = c_1 \cos t + c_2 \sin t$, which is clearly periodic with period 2π. What if $h(t)$ is periodic with the same period?

Exercise 1: In this exercise, we will construct a discontinuous periodic $h(t)$. We can make a single pulse of height 1 that turns on at $t = \pi$ and off at $t = 2\pi$ by multiplying together two step functions, one of which is zero for $t < \pi$ while the other is zero for $t > 2\pi$:

```
sqPulse:=t->Heaviside(t-Pi)*Heaviside(2*Pi-t);
plot(sqPulse(t),t=0..10);
```

Write a Maple function `h:=t->expr` that consists of five successive such square pulses in a row, recurring with period 2π. In other words, the first pulse turns on at $t = \pi$ and off at $t = 2\pi$, the next pulse turns on at $t = 3\pi$ and off at $t = 4\pi$, and so on. [Hint: Build your function by adding together some copies (translated in t) of `sqPulse()`. This approach will be helpful later.] The last one turns off at $t = 10\pi$. Plot your function, using the domain $t \in [-2, 35]$. You do not need to turn in any answer, but will need your function `h` to answer the next question.

Now let's solve equation `(1)` using the function we have just constructed as the right-hand side. We choose initial data $y(0) = y'(0) = 0$, so our system simply sits still until hit by the first pulse from `h`:

```
soln:=dsolve({diff(y(t),t$2)+y(t)=h(t),y(0)=0,
      D(y)(0)=0},y(t),method=laplace);
```

We insist Maple use Laplace transforms because we know `h` to be discontinuous. Maple can make errors when solving such equations using other techniques. Now plot this solution:

```
plot(rhs(soln),t=0..40);
```

We choose this domain because it is similar to the domain during which `h` is non-zero.

> *Exercise 2:* Pass in a copy of the plot you just made. Do you see resonance (defined as growth in the amplitude of oscillations with time)? If you use a much larger domain for the plot, do the oscillations continue to grow, and can you explain this?
>
> Now plot $h(2t)$. Notice that this has period π. If $h(2t)$ replaces $h(t)$ on the right-hand side of equation `(1)`, do you see resonance?
>
> Finally, plot $h(t/2)$. What is its period? Using this as the right-hand side of equation `(1)`, do you see resonance?
>
> In some sense, it is misleading to say merely that a function has just one period. When we refer to the period, we usually mean the *minimum period*. For example, if a function repeats itself whenever its dependent variable changes by 2π, then it certainly repeats itself when that variable changes by 4π, although it may not repeat itself if the variable changes by only π. Considering this point and your observations above, can you conjecture what properties $h(t)$ should have for resonance to occur in systems governed by equation `(1)`?

Do we see the same effects if h is continuous, but not smooth? In the last lab, you constructed a single tooth of the sawtooth function. If the tooth starts at $t = a$, rises to height 1 at $t = a + T/2$, and falls back to 0 at $t = a + T$, then it can be defined by

```
tooth:=(a,T,t)->
    (2*(t-a)/T)*Heaviside(t-a)*Heaviside(a+T/2-t)
    +((a+T-t)/Pi)*Heaviside(t-a-T/2)*Heaviside(a+T-t);
```

The first line of the right-hand side above makes the rising edge of the tooth, while the second line makes the descending edge. As an example, to plot a tooth that starts at $t = 0$ and has total width 2π, we type:

```
plot(tooth(0,2*Pi,t),t=-1..10);
```

We start the plot domain at `t=-1` because Maple may complain if the domain begins or ends at a place where `tooth(0,2*Pi,t)` isn't smooth, such as at `t=0`. Now we can make a five-toothed saw of period 2π using

```
h:=t->sum(tooth(2*Pi*n,2*Pi,t),n=0..4);
```

`plot(h(t),t=-1..35);` Make sure this looks right before going on.

> *Exercise 3:* Using the five-toothed saw with period 2π as the right-hand side $h(t)$, have Maple solve equation `(1)` and plot the solution. Do you see resonance? Do you see resonance if the period is changed to 4π?