Math 201 A1—Maple Lab 6

Objective: To study qualitatively systems of two coupled first-order differential equations.

We will begin with a system of the form

$$\frac{dx}{dt} = Ax + By$$
$$\frac{dy}{dt} = Cx + Dy$$

where A, B, C, and D are constants.

You have by now encountered two methods for solving this system. In Math 102, such systems were written as a single matrix equation and "decoupled" by diagonalizing the matrix that appears on the right-hand side. In Math 201, we replace the system by a single second-order linear equation with constant coefficients, which we can then solve.

Let's consider the system, as above, for which $A = D = -2$, $B = C = -1$.

```
deq1:=diff(x(t),t)=-2*x(t)-y(t);
deq2:=diff(y(t),t)=-x(t)-2*y(t);
soln:=dsolve({deq1,deq2},{x(t),y(t)});
```

We can also solve initial value problems for this system. Say $x(0) = 5$, $y(0) = 0$.

```
soln:=dsolve({deq1,deq2,x(0)=5,y(0)=0},{x(t),y(t)});
```

`soln[1];` This accesses the solution for $x(t)$.

`soln[2];` This accesses the solution for $y(t)$. Let's graph these:

```
plot({rhs(soln[1]),rhs(soln[2])},t=0..5);
```

Now imagine a vertical line drawn on this plot at $t = t_0$. This line determines the values $x_0 = x(t_0)$ and $y_0 = y(t_0)$ where it cuts the two curves. But the parameter t_0 is actually unimportant; what matters is that x_0 and y_0 are uniquely related. This suggests we can express the solution using a single curve of the form $y = f(x)$. Such a curve will be tangent to the vector

$$\mathbf{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (-2x - y, -x - 2y).$$

This means we can draw a field of vectors tangent (or *tangent field*, a direction field but with arrows instead of line segments) to solution curves $y = f(x)$ by plotting at each point this vector field. Maple provides us with several ways of plotting this field. Here's one:

`with(DEtools):` We need to load the `DEtools` package for this.

```
DEplot({deq1,deq2},[x(t),y(t)],t=0..20,x=-1..5,y=-1
..5);
```

We can also plot solution curves on top of this diagram, if we are careful to use `DEplot()` correctly. `DEplot()` solves differential equations numerically at many values of the parameter *t* and plots the resulting curve. We will tell `DEplot()` how often it is to do this by specifying the `stepsize` option. Too big a value will result in inaccurate solution curves. Too small a value can also produce inaccurate curves, and can take too long.

```
DEplot({deq1,deq2},[x(t),y(t)],t=0..20,[[x(0)=0,y(0)=
5],[x(0)=3,y(0)=3],[x(0)=5,y(0)=0]],x=-1..5,y=-1    ..5,
stepsize=0.1);
```

To extend these curves back past their initial points, use negative values of the parameter t.

Exercise 1: For each of the initial data sets

(i) $t = 0 \Rightarrow (x, y) = (0,5)$,

(ii) $t = 0 \Rightarrow (x, y) = (3,3)$, and

(iii) $t = 0 \Rightarrow (x, y) = (5,0)$,

sketch $x(t)$ vs. t and $y(t)$ vs. t on the same set of axes. Are there any minima or maxima of these functions? How do they behave as $t \to \infty$? Using instead the graphs of y vs. x for these initial data, as just obtained with `DEplot()`, can you recognize the same features?

From the output of `DEplot()`,can you say anything about the behaviour of *any* solution curve as $t \to \infty$? Can you see how the form of the differential equations themselves leads to this behaviour?

Powerful qualitative methods like tangent fields allow us to examine even *non-linear* systems, including so-called *predator-prey* models. Let $x(t)$ represent the population of a prey species and $y(t)$ the population of a predator species that eats species x. The change in these populations with time t can be modelled using familiar equations of population growth, such as the logistic growth model, but with an added contribution. The equation for dx/dt should have a negative contribution from $y(t)$, since more predators will imply that more of the prey species are eaten. Similarly, the more food, the faster the population of species y grows, so the dy/dt equation gets a positive contribution from $x(t)$. One such model is

$$\begin{aligned} \frac{dx}{dt} &= x(4 - x - y) \\ \frac{dy}{dt} &= (x - 2)y \end{aligned} \qquad (1)$$

Exercise 2: Use `DEplot()` to have Maple draw the field of tangent vectors for this system.

If x and y represent populations, then they should never be negative. Use the plot of the tangent field to explain why solution curves of this system can never cross an axis.

Eventually, no matter what the initial condition, all solution curves of this system approach the same point, a so-called *stable fixed point*. What are the coordinates of this point? Using the equations (1), compute dx/dt and dy/dt at this point.

For three randomly chosen sets of initial data (all positive values), have Maple plot solution curves of the form y vs. x. For one of these initial data sets, have Maple plot solution curves of the form $y(t)$ vs. t and $x(t)$ vs. t. Does Maple's solution $(x(t), y(t))$ make sense; in particular, can the value of one of the populations become negative? [Hint: Pay careful attention to the domain of the parameter t.] Is it obvious from both forms of plots that solutions converge to the fixed point?