Math 201 A1—Maple Lab 5

Objectives: We will study an inhomogeneous differential equation and investigate the phenomenon of resonance.

Maple Commands:

`dsolve(deq,y(x));` Finds general solution to the differential equation `deq`.

`dsolve({deq,y(a)=y0,D(y)(a)=y1},y(x));` Solves initial value problem with differential equation `deq` and initial data $y(a)=y_0,\ y'(a)=y_1$.

`plot({expr1,expr2,...},x=a..b);` To plot more than one expression on a single set of axes, enclose the list of expressions in curly braces.

`diff(y(x),x$2);` Maple notation for the second derivative $y''(x)$.

In this lab we consider an electric circuit containing one inductor and one capacitor, connected in series to a source of alternating voltage. In such a circuit, charge accumulates on the capacitor. When the charge is too great, the capacitor discharges, creating an electric current in the circuit. The differential equation governing the charge $Q(t)$ on the capacitor as a function of time t is

$$L\frac{d^2Q}{dt^2}+\frac{1}{C}Q=E\cos(\omega_0 t), \tag{1}$$

where L, C, E and ω_0 are constants. The term on the right-hand side describes the alternating voltage applied to the circuit, and makes the equation inhomogeneous. Note that we can compute the current I in the circuit from Q using the definition of current, which is

$$I=\frac{dQ}{dt}. \tag{2}$$

The general solution for the charge on the capacitor as a function of time is

$$Q(t)=c_1\cos(\omega t)+c_2\sin(\omega t)+\frac{E\cos(\omega_0 t)}{L\left(\omega^2-\omega_0{}^2\right)} \tag{3}$$

where $\omega = {}^1\!/\!\sqrt{LC} \neq \omega_0$.

> *Exercise 1:* Using this solution and any convenient method of hand or Maple calculations, compute d^2Q/dt^2 from (3). Using this, compute the left-hand side of equation (1). What do you get? Does this confirm that (3) really does solve (1)?

Let's fix $c_1=c_2=E=L=1$ in (3) and now plot $Q(t)$ for several different choices of ω and ω_0. A convenient way to do that is to define a function that takes ω and ω_0 as input (and t as well):

```
Q:=(w,w0,t)->sin(w*t)+(cos(w0*t))/(w^2-w0^2);
```

To plot the graph using ω=15/8 and ω_0=2 on domain $t\in[0,100]$, we say:

```
plot(Q(15/8,2,t),t=0..100,numpoints=150);
```

Ordinarily, Maple finds 49 points on the graph and then creates the entire plot by joining up these points. To get a smoother curve, we tell Maple to use more points by specifying the `numpoints` option. However, this means Maple will take longer to create the plot.

You should be able to see two different frequencies manifest in the plot, a rapid one and a slower one that affects the amplitude of the wave. This last frequency is called the *beat frequency*.

> *Exercise 2:* Experiment by plotting graphs of $Q(t)$ vs. t, using several different choices of ω and ω_0. Use a large enough domain so that you can see many periods of the oscillation on each graph. Can you enhance the effects of the beat frequency by appropriate choice of ω and ω_0? Hand in a representative plot, labelled by the frequencies ω and ω_0 used to produce it. What do you think happens as $\omega_0 \to \omega$?

Finally, consider the case where $\omega_0 = \omega = 1/\sqrt{LC}$. Then the general solution of `(1)` is

$$q(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{E}{2L\omega} t \sin(\omega t) \qquad (4)$$

You should expect this, since the last term in `(4)` is a particular solution of `(1)`, and since the right-hand side of `(1)` has the same form as solutions of the corresponding homogeneous equation, then the trial particular solution used when applying the Method of Undetermined Coefficients will be of the form of t times the right-hand side.

> *Exercise 3:* From `(4)`, compute d^2q/dt^2 and confirm that `(4)` solves `(1)` when $\omega_0 = \omega = 1/\sqrt{LC}$. Choosing $c_1 = c_2 = E = L = 1$, plot the graph of `(4)` for several different choices of ω. Pass in a copy of one such graph. You should observe that the oscillations in the charge on the capacitor now grow with time t without upper bound. This is the phenomenon of *resonance*.