Math 201 A1—Maple Lab 4

Objectives: To investigate underdamping, overdamping, and critical damping.

Maple Commands:

`dsolve(deq,y(x));` Finds general solution to the differential equation `deq`.

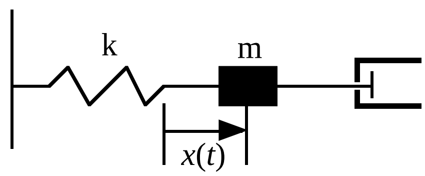
`dsolve({deq,y(a)=y0,D(y)(a)=y1},y(x));` Solves initial value problem with differential equation `deq` and initial conditions $y(a)=y_0$, $y'(a)=y_1$. To get a numerical solution, use:

`dsolve({deq,y(a)=y0,D(y)(a)=y1},y(x),numeric);`

`plot(expr,x=a..b,y=c..d);` The `plot()` command with domain $x \in [a,b]$ and range $y \in [c,d]$.

`plot({expr1,expr2,...},x=a..b,y=c..d);` To plot more than one expression on a single set of axes, enclose the list of expressions in curly braces.

`diff(y(x),x$2);` Maple notation for the second derivative $y''(x)$.



Consider the following model for a shock absorbing system. We begin with a spring with spring constant k attached to a mass m. We also attach the mass to a piston whose other end is immersed in a viscous medium. This provides a damping force proportional to the velocity and opposing the motion of the mass, say $-bv$ where c is a positive constant. The mass then obeys the differential equation:

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx = 0 \qquad (1)$$

Here x measures the displacement of the mass from its equilibrium or rest position. The auxiliary equation for (1) is

$$mr^2 + br + k = 0 \qquad (2)$$

When $b^2 - 4mk > 0$, the general solution is easily found to be

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad (3)$$

where $r_1 = \dfrac{-b+\sqrt{b^2-4mk}}{2m}$ and $r_2 = \dfrac{-b-\sqrt{b^2-4mk}}{2m}$. Since m, b, and k are positive constants, then r_1 and r_2 are negative constants.

Exercise 1: Consider the case $r_1 = -1$, $r_2 = -3$. Use Maple's `plot()` command to verify that when the constants c_1 and c_2 have the same sign, the spring never returns to its equilibrium position $x = 0$. Now choose these constants have different sign; in particular, take $c_1 = -1$ and $c_2 = 2$. How many times does $x(t)$ pass through $x = 0$ now? Try a few other values of these constants. Does the number of zero-crossings change? Is it ever more than one? How do solutions behave as $t \to \infty$? Pass in one example plot.

The above case is called the *overdamped* spring. Next, we study the *underdamped* case. Then $b^2 - 4mk < 0$ and the general solution takes the form

$$x(t) = e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right). \qquad (4)$$

Exercise 2: Consider the case $b = 6$, $m = 5$, $k = 5$, and find the particular solution corresponding to $x(0) = 2$, $x'(0) = 0$.

Now on the same set of axes, plot this solution together with the two functions $y_1(t) = Ae^{\alpha t}$ and $y2(t) = -Ae^{\alpha t}$, where $A = \sqrt{c_1{}^2 + c_2{}^2}$. Submit a copy of this plot.

These functions sandwich the graph of the solution between them, touching the solution curve tangentially so that the latter never passes through the enveloping curves. The form of this plot suggests something that can be proved by using trig identities, that we can rewrite (4) as

$$x(t) = Ae^{\alpha t}\sin(\beta t + \phi)$$

where ϕ is a constant determined from c_1 and c_2. but we won't pursue that issue here.

Finally, we consider the *critically damped* case $b^2 - 4mk = 0$. The only root of the auxiliary equation is then $r = {-b}/{2m}$. Then the general solution is

$$x(t) = \left(c_1 + c_2 t\right)e^{-rt} \qquad (4)$$

Let's consider the case $b = m$, so $k = {m}/{4}$. Then $r = {-1}/{2}$. We may check that (4), with the appropriate subsitution for *r*, really does solve the differential equation in this case:

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deq:=m*diff(x(t),t$2)+m*diff(x(t),t)+(m/4)*x(t)=0;
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`test:=subs(x(t)=(c1+c2*t)*exp(-t/2),lhs(deq));` Substitute the proposed solution into the left-hand side of the equation.

`simplify(test);` It should be zero, and therefore will equal the rhs (right-hand side), which is of course also zero.

Exercise 3: Show that in this case the spring does not oscillate. How many times does it pass through the equilibrium position, assuming it begins moving at $t = 0$? assuming it has been moving for all time? By choosing appropriate values of c_1 and c_2, give a plot of a case where it does pass through zero.