Math 201 A1—Maple Lab 11

Objectives: To practise Fourier series and observe the Gibbs phenomenon.

Maple Commands:

`Heaviside(expr);` Heaviside step function, takes value zero when `expr` is negative and one when `expr` is positive.

`plot({expr1,expr2},x=a..b,y=c..d);` Plots two expressions on one set of axes. In this example we've specified the domain $[a,b]$ and the range $[c,d]$, but it's not required.

Recall that the Fourier series of a piecewise-continuous function on an interval $[-L,L]$ is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right] \qquad (1)$$

where the coefficients are given by

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}\,dx \quad , \quad n=0,1,2,\ldots \qquad (2)$$

$$b_n=\frac{1}{L}\int_{-l}^{L}f(x)\sin\frac{n\pi x}{L}\,dx \quad , \quad n=1,2,3,\ldots \qquad (3)$$

Given a Maple expression for $f(x)$, which here we'll denote by `expr`, and a numerical value for L, the following Maple functions can be defined and used to compute the coefficients a_n, b_n:

```
a:=n->(1/L)*int(expr*cos(n*Pi*x/L),x=-L..L);
b:=n->(1/L)*int(expr*sin(n*Pi*x/L),x=-L..L);
```

where of course instead of `expr` and `L` you insert the appropriate expression and numerical value. To have Maple return, for example, a_5, you would simply type

```
a(5);
```

To have Maple take a partial sum of all terms in the Fourier series (1) for $f(x)$ up to and including the N^{th} term, use the function

```
S:=N->(a(0)/2)+sum(a(n)*cos(n*Pi*x/L)
+b(n)*sin(n*Pi*x/L),n=1..N);
```

Exercise 1: Let $f(x)=\begin{cases}-1 & , \quad -\pi\le x<0\\ +1 & , \quad 0<x\le\pi\end{cases}$ be defined on $[-\pi,\pi]$ (except at $x=0$). Find the $N=5$ partial sum of the Fourier series for $f(x)$. Plot the result, together with the original function. Notice the "overshoot" where soon after passing through $x=0$ the partial sum exceeds $f(x)$.

Increase the number of terms in the partial sum and, simultaneously, decrease the domain of the plot. By including enough terms in the sum, does it appear that this overshoot problem will go away? Try it. In percentage terms, by how much does the height of the plotted partial sum initially exceed the height of the function it is supposed to represent? Submit one copy of a plot showing this overshoot. While the width of this overshoot vanishes as more terms are added to the partial sum, the height does not. The effect occurs whenever a function with a jump discontinuity is represented by a Fourier series, and is called the Gibbs phenomenon.

A function defined on just $[0,L]$ can have two different Fourier series. For such a function, it's *Fourier sine series* is given by

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad , \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad , \quad n = 1,2,\ldots \tag{4}$$

while its Fourier cosine series is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad , \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad , \quad n = 0,1,2,\ldots \tag{5}$$

Exercise 2: By any convenient method, compute both the Fourier sine series and the Fourier cosine series of the function $f(x) = \cos x$ whose domain is $[0,\pi]$. In the case of the sine series, plot a few partial sums and compare to the original function. How do they compare if we extend the domain (of the plot only; don't compute new Fourier series!) to $[-\pi,\pi]$? Submit a copy of one such plot.