Math 201 A1—Maple Lab 10

Objectives: We will study the musical tone that arises from a plucked guitar string.

If two different musical instruments are used to play the same set of notes, it is usually easy to distinguish the sounds made by the two different instruments. The reason for this is that the instruments do not succeed in generating pure tones. Instead, the actual sounds they create are coloured by tones that are present because of the design and structure of the instrument. This leads to a distinctly recognizable sound unique to that type of instrument.

For example, stringed instruments sound quite distinct from horns. In this lab, we will examine the shape of a string after it is plucked, to see what tones are actually created.

Exercise 1: The one-dimensional wave equation describing the amplitude of a standing wave on a string of length L and tension per unit mass density α^2 is

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

Show that this equation, subject to the boundary conditions $u(0,t) = u(L,t) = 0$ for all t, has a solution of the form $u = T(t)X(x)$, where $X(x)$ satisfies

$$X''(x) + \omega^2 X(x) = 0 \tag{2}$$

and $\omega = n\pi / L$. What ordinary differential equation does $T(t)$ obey?

The general solution of these equations can be found. It is

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi\alpha t}{L} + b_n \sin \frac{n\pi\alpha t}{L} \right] \sin \frac{n\pi x}{L} \tag{3}$$

If this wave exists on the string of a guitar, then L should represent the distance from the "bridge" where the string is fastened at one end to the "fret" against which the other end of the string is held by the player's finger while the player's other hand plucks the string.

To make a pure tone, say middle C, which vibrates with period 512 cycles per second, the idea is to try to have all the a_n and b_n be zero except when $n = 1$. Then we get

$$u(x,t) = \left(a_1 \cos \frac{\pi\alpha t}{L} + b_1 \sin \frac{\pi\alpha t}{L} \right) \sin \frac{\pi x}{L}. \tag{4}$$

We then try to arrange that $\pi\alpha / L$ be $2\pi \times 512/\text{s}$.

Exercise 2: What value must α have if the length of the string from the point where it is fastened against the bridge to the fret for middle C (512 Hz, or a period of 1/512 seconds) is 1 m? Guitars are tuned by turning a screw which adjusts the tension in the string, and so changes α.

The distinct sound of a guitar arises in part because the process of plucking a string results in "exciting" modes that are not $n = 1$; in other words, some of the a_n and b_n for $n > 1$ are not zero. The shape of a guitar string, when being plucked, resembles a sawtooth. Let's assume this sawtooth has amplitude 1 mm and the 1 m string is plucked a distance 0.1m along its length. A function that correctly gives the shape of the string on the domain [0,1] is

```
g:=x->0.01*x-0.0111*(x-0.1)*Heaviside(x-0.1);
plot(g,0..1);
```

We will need to write *g* as a Fourier series. For now, let's use an approximation by a finite sum that is reasonably close on the domain [0,1].

$$f(x) = 0.0111 \sum_{n=1}^{5} \frac{\sin n\pi x}{20n} \tag{5}$$

```
f:=x->0.0111*sum(sin(Pi*n*x)/(20*n),n=1..5);
```

`plot({f,g},0..1);` Do these two graphs resemble each other?

The finite upper limit on the sum means that Maple will be able to compute *f* faster than if the sum had an infinite number of terms. By the way, (5) is not what you'd get if you wrote the full Fourier series for *g* and just kept the first 5 terms, but it does approximate the correct shape rather well.

<u>*Exercise 3:*</u> If equation (5) were a correct model for the shape of the guitar string as it is being plucked, and if plucking results in $\frac{\partial u}{\partial t}(x,0) = 0$, what would be the solution for the shape of the string at all times? What are the ratios of the amplitudes of middle C (512 Hz) to treble C (1024 Hz) to the $n = 3$ mode (1536 Hz, not a C but I don't know what note it corresponds to) to high C (2048 Hz) in the tone that is produced?

We can now see another reason to truncate the sum in (5) after only a few terms. A non-zero term with $n = 6$ in (5) would produce a non-zero $n = 6$ term in the solution (4), whose frequency we can see would be about 3000 Hz. While this is still in the range of human hearing, we wouldn't hear it very well, so discarding these terms has little effect on the sound we hear from this guitar string.

As an aside, you can have Maple animate the motion of this guitar string. For example, using equation (3) with $\alpha = 200$, $L = 1$, and say $a_1 = b_1 = 0.05$, $a_2 = 0.02$, and all other coefficients zero, can be animated as follows:

`with(plots):` First load the `plots` package.

```
u:=(0.05*cos(200*Pi*t)+0.05*sin(200*Pi*t))*sin(Pi*x) + 0.02*cos(400*Pi*t)*sin(2*Pi*x);
animate(u,x=0..1,t=0..5,frames=100);
```

To start the animation, push the `play` button (it is marked by a triangle shape) after the graph appears. Try this with the solution you found in Exercise 3.

With a real guitar, the plucked string merely acts as the initial condition setting up a sound wave in the air inside the guitar box. The shape of the box determines the boundary conditions for that sound wave, and is therefore very important in determining the overall mix of harmonics in each note. We have ignored this point above.