Calculus Lab 9—Newton's Method

Objective: To use the computer to implement and study Newton's iterative method for finding roots of a function.

Maple Commands:

`plot(expr,x=a..b,y=c..d);` Plots expression `expr`; restricts both domain (x-values) and range (y-values).

`plot({expr1,expr2},x=a..b);` Plots both `expr1` and `expr2` on one set of axes.

`diff(expr,x);` Differentiates expression `expr` with respect to x.

`f:x->(x^2+1)/x;` An example of a Maple function. This one takes x as input and computes $\frac{x^2+1}{x}$. For example:

`f(2);` Evaluates the Maple function `f` defined above at $x = 2$. Should return `5/2` in this example. To accomplish the same thing with Maple expressions instead of functions, use:

`expr:= (x^2+1)/x;` Maple expression representing $\frac{x^2+1}{x}$.

`subs(x=2,expr);` Substitutes $x = 2$ into `expr`. Should return `5/2`.

Maple can easily solve equations that are difficult or impossible to solve by hand. To do this, it sometimes abandons exact analytical techniques and instead uses numerical approximation methods. However, this can result in unexpected or unhelpful answers. Therefore, we should understand something about how computers attempt such solutions, so that we can better judge whether we should rely on the answers.

The simplest useful numerical procedure for finding solutions of equations is *Newton's method*. We will study it below.

Recall that a root of a function $f(x)$ is a value of x such that solves the equation $f(x) = 0$. Roots are very easy to find using graphs—a root occurs whenever the graph of $f(x)$ crosses the x-axis. Here are three methods for finding roots:

1. Plot $f(x)$ and read off the x-values where the graph crosses the x-axis. This method is not always accurate, and you could miss a root that lies outside the domain of your plot.
2. Use Maple: `solve(expr=0,x);` or, to force Maple to return a real floating point number, `evalf(solve(expr=0,x));` where `expr` is the function $f(x)$.

Sometimes the `solve()` command will not return a useful answer unless used in conjunction with `evalf()`, but if you use `evalf()` then Maple returns at most one root, even when there are several.

3. Newton's Method. See formula `(1)` immediately below.

Given a function $f(x)$, the idea of Newton's Method is to first make a guess at a root. We call this guess x_0. Usually this guess will fail, so we improve it. To do this, draw the line that is tangent to the graph of $f(x)$ at $x = x_0$ and ask at what x-value does this tangent line cross the x-axis. This is easy to answer. This x-value is our improved guess, and we call it x_1. It is not hard to show that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad (1)$$

<u>Exercises:</u>

`Q1.` Let $f(x) = x^4 - 2x - 2$. Use a graph to estimate all roots of $f(x)$.

`Q2.` Let's make the not-very-good guess that $x_0 = 2$ is a root of $f(x)$. Use equation `(1)` to find an improved guess x_1.

`Q3.` Find the equation of the tangent line to $f(x)$ at $x = 2$. Graph $f(x)$ and this tangent line on a single set of axes. At what x-value does the tangent line cross the x-axis? Does this bring us closer to the root than our initial guess?

The real power of Newton's method is that it allows you to continually improve on your guess by applying the method over and over again. The value x_1 that emerges from equation `(1)` can be treated as a new guess and substituted back into the right-hand side. An improved guess x_2 then emerges. We can repeat this over and over again, a procedure called *iteration*. In general, the formula for the n^{th} improved guess x_n is

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \qquad (2)$$

`Q4.` Apply Newton's method to estimate a root of the function $f(x) = x^4 - 2x - 2$, starting with the guess $x_0 = 2$. Iterate 5 times to find the approximate root x_5. [Hint: One way of doing this is to make a Maple function which accepts x_{n-1} as input and produces x_n as output.] Check by substituting your answer back into $f(x)$ to see if $f(x_5)$ really is (nearly) zero.

Sometimes the method gives unexpected results, or even fails completely. For example. consider the function

$$f(x) = \frac{(x+1)(x+2)}{x^2+2x+2} \qquad (3)$$

It is easy to see that this function has exactly two roots, $x = -1$ and $x = -2$. But when we try to find them with Newton's method, we may get a surprise.

Q5. Plot this function on the domain [-5,5]. If you start with the guess $x_0 = -0.1$, which of these two roots would you expect Newton's method to eventually find? Try it. Which one does it actually find? Explain what happened (a plot of the function and its tangent line at $x = -0.1$ might help).

Q6. Now try again, this time starting with the guess $x_0 = 2$. What happens? Explain.

Optional

Q7. Derive equation (1) by writing the point-slope form of the equation of the line tangent to $f(x)$ at the first guess x_0 and then solving for the x-value at which this line crosses the x-axis.