Calculus Lab 7—Rational Functions

Objective: To analyse the behaviour of rational functions, paying attention to the determination of extreme points and asymptotes.

New Maple Commands: Long division of polynomials using Maple:

`quo(3*x^2-1,x+4,x);` This command accepts two polynomials as arguments and divides the second polynomial into the first. It reports the result but does not report the remainder. For this example, Maple returns the answer:

```
3x-12
```

`rem(3*x^2-1,x+4,x);` Gives the remainder term. For this example, Maple returns the answer:

```
47
```

These Maple responses are meant to tell us that:

$$\frac{3x^2-1}{x+4} = 3x-12+\frac{47}{x+4}$$

Recall:

`diff(expr,x);` Differentiates expression `expr` with respect to x.

`D(f);` Differentiates function f with respect to its argument.

`limit(expr,x=a);` Takes limit as $x \to a$ of expression `expr`. For infinite limits, use `x=infinity` or `x=-infinity` in second argument of this command.

`solve(expr,x);` Solves `expr=0` for `x`. May not return *all* solutions.

`fsolve(expr,x);` As above, but it returns a decimal (floating point) number. Usually returns only one solution, even when there are many.

When considering a complicated function $f(x)$, often what one needs to know are certain general properties. Does the function have any maxima or minima and if so then where are they? Does the function behave simply if x is very large? very negative? very close to zero?

There are two methods for extracting this information. One can analyse the function using calculus techniques, or one can ask a graphing calculator to graph the function and then one can examine the graph. Sometimes it is helpful to apply both these methods.

Asymptotes provide important information because they tell us about the behaviour of the function for large values of x and y. Recall that the graph of a function $g(x)$ is said to have a horizontal asymptote $y = a$ if either

$$g(x) \to a \quad \text{as} \quad x \to \infty \quad \text{or if} \quad g(x) \to a \quad \text{as} \quad x \to -\infty,$$

where a is a constant. If instead we have

$$g(x) - (mx + b) \to 0 \quad \text{as} \quad x \to \infty \quad \text{or if} \quad g(x) - (mx + b) \to 0 \quad \text{as} \quad x \to -\infty,$$

then we say that $g(x)$ has an oblique asymptote with equation $y = mx + b$. [Notice that if $m = 0$, then the oblique asymptote becomes a horizontal asymptote with equation $y = b$.]

Consider the function:

$$f(x) = \frac{x^3 + 4}{x^2 + 2x - 3}$$

Exercise 1: Does the graph of this function have any vertical asymptotes and, if so, where? Now experiment by plotting the graph of this function using several different domains. Try to determine graphically where there might be maxima or minima, whether the graph has any oblique or horizontal asymptotes, and what the equations of those asymptotes might be. Your results will be treated as guesses which we'll improve upon in the following exercises. Keep in mind that vertical asymptotes may affect the plots in certain ways; therefore choose the domains and ranges of your plots appropriately.

Now let's try to find horizontal and oblique asymptotes and max/min precisely:

Exercise 2: Does $f(x)$, as defined above, have any horizontal asymptotes? If so, give their equations.

Use the result of the command

```
q:=quo(37*x^2-84*x+50,6*x-7,x);
```

to determine whether $f(x)$ has oblique asymptotes as $x \to \infty$ and as $x \to -\infty$. If so, what are the equations of these asymptotes? To verify your answers, find the limit as $x \to \infty$ and as $x \to -\infty$ of $(f(x) - q)$.

Exercise 3: Now compute the derivative of $f(x)$. Find the (x,y) coordinates of all local maxima and minima. (Hint: Maple probably won't do this exactly. If so, you may have to rely on information obtained graphically by plotting the derivative function.) Are there any absolute maxima? absolute minima?

Exercise 4: Consider the function

$$f(x) = \frac{x}{1-(1/x)}$$

Are there any x-values which are not allowed to be in the domain of this function? What is the derivative of this function? Using the derivative, find the (x,y) coordinates of all local maxima and minima. [Do your answers to these questions conflict with each other? If so, comment.]

Find all asymptotes. Plot the function $f(x)$, together with any oblique or horizontal asymptotes, on one graph. Pass in a copy of this graph, labelling maxima, minima, and asymptotes.