Calculus Lab 6—Exponential Growth and Decay

Objective: To study exponential growth and decay, and in particular to confirm empirically that the exponential function is proportional to its own derivative.

In the last lab, we saw that the exponential function grew more rapidly than any power of x. The reason for this rapid growth is that the rate at which the exponential function increases (i.e. the derivative) is in proportion to the size of the function itself. In mathematical language, if $y(t)$ is an exponential function of t, then

$$\frac{dy}{dt} = ky\ .$$

Here `k` is a constant of proportionality. This equation can be rewritten as

$$\frac{1}{y}\frac{dy}{dt} = k\ .$$

This says that the ratio of (dy/dt) to y is always equal to the constant value k, no matter at what value of t we choose to compute y and (dy/dt). We will confirm this graphically.

Finding Slopes of Tangent Lines:

We will have to find some slopes of tangent lines. To do this, we first need to load Maple's student package:

```
with(student):
```

We can now graph both an expression and its tangent line at some chosen point by using the student package's `showtangent()` command. For example, to graph $f(u) = \cos(u)$ and its tangent at $u = 13\pi/6$, use

```
showtangent(cos(u),u=13*Pi/6);
```

You must now compute the slope of the tangent line by hand (although you can use Maple to do the arithmetic). Find the coordinates of two (well-separated) points on the tangent line by positioning the cursor at each point, clicking (see the note below), and reading the coordinates from the box on the upper left of the screen. Call these (x_1,y_1) and (x_2,y_2). Then the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1}\ .$$

This slope is of course equal to the value of the derivative of the curve at the point of tangency.

Remember: When finding the coordinates of a point on the graph, click once anywhere in the graph to "select" the graph (its border will change slightly when you do this) and then click once more on the point whose coordinates you wish to know. Thus, the process really requires you to click twice in total. If you only click once, the coordinates Maple displays may not be properly updated.

Exercise 1: Apply the `showtangent()` command to display the tangent line to the curve of $y = e^{7t/3}$ at the point along the curve where $t = 1$. Compute the slope of the tangent line. Repeat this exercise at four other points of your choosing along the curve. Report your results in a table showing at each chosen point the values of t, y, dy/dt, and lastly the ratio $(dy/dt)/y$. What value should you expect to find for this ratio in each case? How close did you come to what you expected?

If you were to repeat this exercise, but instead of using the function $y = e^{7t/3}$ you were to use instead the function $y = t^n$ with n some fixed number? Would you expect that the ratio $(dy/dt)/y$ would be constant? (You don't have to actually repeat the exercise to answer this, but you can if you wish.) Differentiate $y = t^n$ and divide the result by y (which you can replace by t^n of course) to see.

Radium 228 is a radioactive chemical substance. Radioactive substances decay according to the law of exponential decay. Let y_0 be the amount of Radium 228 present at the initial time and let $y(t)$ be the amount present a time t later. Then the law of radioactive decay is

$$y(t) = y_o e^{-0.1t} \qquad (*)$$

where `t` is measured in years.

For example, if 10 g. of Radium 228 are present initially, the equation will read

$$y(t) = 10e^{-0.1t}$$

which we can write as a Maple expression:

```
y:=10*exp(-0.1*t);
```

Exercise 2: Assume you are studying a radioactive ore sample containing 40 g. of Radium 228 at the present time. Graph equation (*) above, using 40 in this case, and on the same set of axes graph the straight line $y = 20$. By locating the point of intersection of these two curves, read off the time (in years) that you will have to wait for 20 g. of Radium 228 to decay. This is called the "half-life," since it is the time for half the initial quantity of Radium 228 to decay.

Repeating this graphical technique, determine the time at which all but 10 g. of Radium 228 will have decayed and the time at which all but 5 g. will have decayed. (The trick here is to choose an appropriate domain for each graph.) Will there be a time when all the Radium initially present will have decayed? If so, what is that time?