Calculus Lab 5—Exponentials and Logarithms

Objective: To understand the statement,

"The exponential function e^x grows more rapidly than any power function x^n,"

and the complementary statement,

"The logarithm function grows more slowly than any positive power of x."

New Maple Commands:

`evalf(E);` The number e (the "natural base") is denoted by `E` in Maple.

`exp(expression);` Takes the exponential. The output of this is $e^{\text{expression}}$.

`E^(expression);` Same as `exp(expression);`.

`ln(expression);` Takes the natural logarithm of `expression`.

`log(expression);` Same as `ln(expression);`.

`log10(expression);` Takes the base 10 logarithm of `expression`.

Some Recent Maple Commands:

`taylor(e1,x=a,n);` Computes the degree $n-1$ Taylor polynomial of `e1` about $x = a$.

`convert(%,polynom);` Converts the last expression (`%`) to a true Maple polynomial.

`limit(expr,x=a);` Takes the limit as $x \to a$.

`limit(cos(x)/x,x=infinty);` You can use `infinity` to denote ∞ in Maple.

The inverse function to the exponential function is the logarithm. To see what this means, notice the outcome of the following commands:

```
exp(ln(x));
simplify(");
ln(exp(x));
simplify(");
```

Now let's plot $y = e^x$, and $y = x^n$ on the same set of axes, for various values of n:

```
plot({exp(x),x,x^2,x^3,x^4},x=0..1);
```

Which curve is which? Hint: What is the y-intercept of e^x? of x^n?

Exercise 1: Repeat the above plot command. Each time, increase the upper limit of the domain by one (so plot using `x=0..2`, then plot again with `x=0..3`, and so on). As you increase the domain, you will eventually find a point where $e^x = x^3$ and beyond that point $e^x > x^3$. Find the (x,y) coordinates of this point. Continue to incrementally increase the domain, stopping when you are able to display the point where $e^x = x^4$ and find the the (x,y) coordinates of this point as well. Give a rough sketch of this last graph, labelling the curves.

Now ask Maple to compute the limits as $x \to \infty$ of e^x, e^x / x, e^x / x^2, e^x / x^3, and e^x / x^4. Are the results expected from the graphs? What would you expect the limit as $x \to \infty$ of e^x / x^{100} to be? Confirm your expectation by checking this limit in Maple.

Exercise 2: Now plot the natural logarithm $\ln(x)$ together with $x^{1/2}$, $x^{1/3}$, $x^{1/4}$, and $x^{1/5}$ on the same set of axes, beginning with the domain `x=0..50`. (If there are problems, try plotting with the domain `x=1..50` instead. The point $x = 0$ might cause minor trouble because $\ln(x)$ is not defined at $x = 0$.)

```
plot({ln(x),x^(1/2),x^(1/3),x^(1/4),x^(1/5)},x=0..50);
```

Now increase the domain incrementally until you reach the point where the curve of $x^{1/3}$ overtakes that of $\ln(x)$. What are the coordinates of this point? What are the coordinates of the point where the curve of $x^{1/4}$ overtakes that of $\ln(x)$? where the curve of $x^{1/5}$ overtakes that of $\ln(x)$? [Hint: Don't increase the domain by only small amounts. You will need to use truly enormous values for the upper limit of the domain. Also, don't bother to continue to plot $x^{1/3}$ and $x^{1/4}$ when looking for $x^{1/5}$ to overtake $\ln(x)$, as they will only get in the way.]

Ask Maple to compute the limits as $x \to \infty$ of $\ln(x)$, $\ln(x)/x^{1/2}$, $\ln(x)/x^{1/3}$, $\ln(x)/x^{1/4}$, and $\ln(x)/x^{1/5}$. Are the results expected from the graphs?