Calculus Lab 4—The Number *e*

Objective: To investigate different ways in which the natural base e arises.

Maple Commands:

`plot(expr,x=a..b);` Plots `expr` (a Maple expression) on domain $x \in [a,b]$.

`plot({expr1,expr2},x=a..b);` Plots both `expr1` and `expr2` on same set of axes.

`evalf(expr);` Forces Maple to evaluate (approximately, if necessary) `expr` as a 10 digit floating point number.

`f:=x->expr;` Defines `f` to be the Maple function that takes `x` as input and returns `expr`, an expression involving `x`, as output.

You are probably well familiar with the irrational constant π which occurs throughout trigonometry and geometry. It is defined in an easily understood way; it's just the ratio of the circumference of a circle to its diameter.

In algebra and calculus, another irrational constant often appears, the so-called *natural base*, denoted by e. Like π, the decimal expansion of e requires an infinite number of decimal places. In this lab,we will explore several different (but interrelated) ways in which this number arises, and in the process determine its approximate value.

We will begin with the definition of e as a limit:

Definition: If $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$, then $a = e$.

Then we can determine the value of e by the following method:

1. Choose a specific value of a and plot the function $f(h) = \frac{a^h - 1}{h}$ on an interval about $h = 0$.
2. Read off the y-intercept of this function—that gives us the function's value at $h = 0$, which in this case is equivalent to taking the limit $h \to 0$ since the function is continuous.
3. If this value is not 1, change a and repeat the process. The closer the y-intercept gets to 1, the closer our chosen a will be to e.

Exercise 1: Plot the function $f(h) = \frac{a^h - 1}{h}$ for the specific value $a = 2$. For all plots in this exercise, use the domain $h \in [-1,1]$; you are not required to pass in copies of the plots. What is the value of the y-intercept? Now repeat for $a=3$ and for $a=2.5$. Again find the y-intercept in both cases. Now try to choose a such that the y-intercept appears to be 1. What value of a do you obtain by this method? [You can improve your accuracy by using a much smaller domain for your plots, *e.g.* $h \in [-0.1, 0.1]$.)

There is another way to use limits to define the number e, which is

Definition: $$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

We won't concern ourselves with showing that this definition is equivalent to our other one. Instead, we will use this to try to get a value for e.

Exercise 2: Use Maple to compute $f(n)=\left(1+\frac{1}{n}\right)^n$ for several values of n including $n=10$, $n=100$, and $n=1000$. Record the results in a table. What value does $f(n)$ appear to approach as n gets very large?

The last definition can be generalized, allowing us to define a whole function:

Definition: $$e^x = \exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Maple knows about this function. It calls it `exp()`.

`exp(1);` This should be close to the values you obtained in the above exercises.

```
plot(exp(x),x=-2..2);
```

In addition, there is an inverse function to $\exp(x)$, called the *natural logarithm* and denoted by $\ln(x)$. Maple knows about it as well and calls it `ln()` or `log()`.

```
ln(1);
```

`evalf(ln(7.5));` Remember `evalf()` converts the answer to a decimal number.

```
ln(exp(1));
plot(exp(x),x=-2..5);
```

Because $\ln(x)$ and $\exp(x)$ are inverse functions for each other, they obey *cancellation equations* which read:

$$\ln(\exp(x)) = x$$
$$\exp(\ln(x)) = x$$

Exercise 3: Solve each of the following equations by plotting both the function on the left-hand side and the function on the right-hand side on a single set of axes and finding *all* points where the two graphs intersect. Copy down a plot in each case, and indicate the points of intersection.

a) $e^x = x^3$ Vary the domain of the plot to ensure you find *all* solutions. Also, be sure you know which graph is which (hint: these functions behave differently at $x = 0$).

b) $e^{-x} = -x^2 + 3x$

c) $x = 3\ln(x)$

d) $e^x = e^{-x}$

e) $x^4 - 5x^2 + 1 = e^{-x}$

The roots in parts (a) and (c) should have been the same. Explain why, using the calcellation equations.