Calculus Lab 33—Parametric Curves

Objective: To gain appreciation for the wide variety of behaviours that can be represented in graphs of simple parametric curves.

Maple Commands:

`int(expr,t=a..b);` Definite integral of an expression `expr` depending on t, which takes values in the domain of integration $[a,b]$.

`evalf(%);` If Maple can't do a definite integral exactly, apply `evalf` to the integral. Then Maple will use numerical integration to evaluate the integral approximately.

`plot([expr1,expr2,t=a..b]);` Plots the parametric curve $x=f(t)=$`expr1`, $y=g(t)=$`expr2`. Notice the positioning of the square brackets; they surround the parametric expressions and the parameter range.

`with(plots):` Loads the full `plots` package. Needed for the next command:

`spacecurve([expr1,expr2,expr3,t=a..b]);` Plots the space curve given parametrically by $x=f(t)=$`expr1`, $y=g(t)=$`expr2`, $z=h(t)=$`expr3`.

Note: This lab asks you to make many plots. Maple's plotting routines use up alot of memory. If you run out, or if your machine slows down considerably, simply quit Maple and restart it; this shouldn't take long.

Let's try a parametric curve:

```
plot([t*cos(2*Pi*t),t*sin(2*Pi*t),t=a..b],axes=none);
```

The basic spiral curve; the radius and angle both grow proportionally with the parameter t. Let's try another:

```
plot([sin(3*t),sin(4*t),t=0..2*Pi]);
```

Exercise 1: Find the arc length traversed in going exactly once around the last curve plotted above (you will probably have to rely on Maple to do the integrals).

Curves given by parametric equations of the form

$$x(t) = \sin(mt) \quad , \quad y(t) = \sin(nt)$$

are called *Lissajous figures.*

Exercise 2: Plot some Lissajous figures. [Hint: You can definea Maple function `liss:=(m,t)->sin(m*t)`. Then to plot the Lissajous figure used above, type `plot([liss(3,t),liss(4,t),t=0..2*Pi);` This should save typing when plotting many of these curves.]

What happens when m=n? when m=2n? What happens when the numbers `m` and `n` are consecutive members of the Fibonacci sequence 1, 2, 3, 5, 8, 13, 21,...? Sketch a good example of this case. Can you suggest any reason why this case leads to such complicated curves? The case m=3, n=5 leads to a curve that appears not to be a closed loop; why might this be somewhat misleading?

Another interesting family of curves is given parametrically by the equations

$$x(t) = a\cos(t) - \sin(bt) \quad , \quad y(t) = c\sin(t) - \sin(dt)$$

where a, b, c, and d are numbers. A simple way to define this family and plot a few members using Maple is to define the functions

```
x:=(a,b,t)->a*cos(t)-sin(b*t);
y:=(c,d,t)->c*sin(t)-sin(d*t);
```

Now let's plot the case a=1, b=3, c=1, d=2:

```
plot([x(1,3,t),y(1,2,t),t=0..10]);
```

Let's now adopt an experimental approach and look for interesting behaviour within this family of curves.

Exercise 2: Explore the behaviour of the family of curves described by the above parametric equations. Try several different values of the parameters less than or equal to 5. Describe the range of behaviour you observe in words, and copy down one or two typical examples for illustration purposes.

Maple can also plot "space curves" (parametric curves in 3-dimensional space). To do this, we need to load the full `plots` package.

`with(plots):` From here onward, if you quit Maple and restart it, make sure to execute this instruction again.

`spacecurve([t*cos(t),t*sin(t),t,t=0..8*Pi]);` A type of spiral.

`spacecurve([sin(t),cos(t),sin(5*t),t=0..2*Pi], axes=normal);` A 3d version of a Lissajous figure. As with all 3d plots in Maple, you may optionally specify the type of axes (place the option outside the square brackets but inside the parentheses) or have no axes at all.

Exercise 3: Consider the intersection of the cylinder $x^2 + y^2 = 9$ with the surface $z = 4x^2$. Sketch the curve that represents the intersection of these two surfaces. [Hint: Start with the parametrization $x = 3\cos t$ and then use the above equations to obtain parametric equations for y and z.]