Calculus Lab 31—Radioactive Waste

Objective: To model a real engineering problem by solving a differential equation.

Maple Commands:

`dsolve(expr,y(x));` Invokes Maple's **d**ifferential equation **solve**r to solve `expr` for $y(x)$. To use this, first define `expr` to be some ordinary differential equation.

`diff(expr,x);` Differentiates the expression `expr` with respect to x.

`plot(expr,x=a..b);` Ordinary `plot` command. You can also specify range `y=c..d`.

`plot({expr1,expr2},x=a..b);` Plots both `expr1` and `expr2` on one set of axes.

`implicitplot(expr,x=a..b,y=c..d);` Plots y vs. x, where y is implicitly defined via the expression `expr`. For example, to plot a unit circle, type `implicitplot(x^2+y^2=1,x=-1..1,y=-1..1);`

If the following problem interests you, you can read more in Section 1.7 of *Differential Equations and Their Applications*, 3rd Ed., M. Braun (Springer-Verlag, New York, 1983), which was the source for the material below.

An important and controversial unsolved problem concerning nuclear power is the question of how to safely dispose of the radioactive waste materials from a nuclear plant. For several years, the US Atomic Energy Commission (or AEC, now called the Nuclear Regulatory Commission) had made a practice of sealing waste in drums and dumping them in the sea in not less than 50 fathoms (about 100 metres) of water. The reasons for covering the drums with this much water may have been non-technical.

Tests showed that the drums were not susceptible to corrosion or leakage, but engineers argued that the impact that occurred when a drum fell 100 m to the sea floor could crack open the drums. Further tests determined that impacts at speeds of greater than about 12 m/s could result in cracking. Hence, the problem is to determine what is the speed of a drum when it hits bottom.

From the physics of falling objects, we can obtain a differential equation for the velocity as a function of time. It is

$$m\frac{dv}{dt} = mg - B - cv \qquad (1)$$

where m is the mass of the drum and g is a constant with value about g=9.8 m/s^2, but we can safely use g=10 m/s^2 here. The product mg actually represents the force of gravity pulling the drum down. B is a constant and represents the buoyant force of the water on the drum. Its value is about B=14.6 kg-m/s^2. The drums are quite heavy (about 200 kg), so mg will be about 2000 kg-m/s^2. Since this number is so much greater than B, we will simply drop B and write

$$m\frac{dv}{dt} = mg - cv \qquad (2)$$

Finally, the product cv represents the viscosity (friction) of the water on the drum. Since c depends on the detailed shape of the drums, it was measured experimentally by measuring the force needed to tow drums through water. The measured value was c=1.2 kg/s.

There is one slight problem. If we solve equation (2), we will obtain the speed v of the falling drum as a function of time t. But that's no good since we need to know the speed as a function of depth y through which the drum has fallen, and in particular we need to know v when the depth is 100 m. The chain rule can help us. We let v depend on y, and in turn the depth y through which the drum has fallen will depend on t. Then:

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v \qquad (3)$$

where in the last step we use that $\frac{dy}{dt} = v$. Plugging (3) into (2) gives the equation we need:

$$mv\frac{dv}{dy} = mg - cv \qquad (4)$$

Notice that this is a separable equation for $v = v(y)$.

---

1. By any convenient method (hand calculations, Maple, or a combination), integrate equation (4) to prove that the general solution is:

$$y = C_0 - \frac{m}{c}v - \frac{m^2 g}{c^2}\ln(mg - cv) \qquad (5)$$

   where C_0 is a constant of integration.

2. Find the particular form of equation (5) that describes drums that are disposed of by placing them gently on the water's surface; that is, apply the initial condition $v(0)=0$ to determine and replace the constant C_0. Now plug in the values given for c and g above to get an equation whose only unknowns are the mass m of the drum, the depth y, and the speed $v(y)$.

By the way, equation (5) is in a somewhat unusual form. We've isolated the independent variable y on the left, whereas usually we would want to isolate the dependent variable $v=v(y)$. It is in fact impossible to isolate v in (5), but that won't prevent us from extracting the information we need.

3. The drums disposed of in this way by the AEC were typically over 200 kg in mass. For several values of the mass between 50 and 500 kg, plot graphs with depth y on the vertical axis and velocity v on the horizontal axis. Based on these graphs, is it safe to drop radioactive drums of these masses into 100 m of water, given that the drums could crack if the impact velocity of the drum with the sea floor exceeds 12 m/s? Support your conclusion with sketches of one or two typical graphs and with numbers extracted from the graphs. Would it make any difference to your conclusions if either more or less material was placed in each drum?

4. What happens to the logarithm in equation (5) when $v=mg/c$? Can you explain the significance of this particular value of the velocity?