Calculus Lab 29—Power Series Solutions

Objective: To use power series to solve first-order linear ordinary differential equations.

Maple Commands:

`taylor(expr,x=a,n);` Computes order *n-1* Taylor polynomial of `expr` about centre x=a.

`int(expr,x);` Integrates `expr` with respect to x.

`convert(expr,parfrac,x);` Finds partial fractions decomposition of `expr`. Useful when integrating rational functions.

`with(powseries):` Loads the `powseries` package. We will use several new commands from this package to solve differential equations by power series methods. These commands are explained below.

Consider the following differential equation:

$$\left(1+x^2\right)\frac{dy}{dx}+xy=0,$$

and the boundary condition $y(0)=1$. This equation is separable, but there are other techniques we can use to solve it as well. For example, we can try a series solution:

$$y=\sum_k a_k x^k.$$

Here the unknown coefficients a_k of the power series are to be determined using the differential equation and the initial condition.

Exercise 1: Use a power series solution to give the general solution of this differential equation. Give a recursive formula for the coefficients a_k. Apply the boundary condition to find the solution such that $y(0)=1$, and explicitly write out the first five non-zero terms of this series solution. [Hint: Try to do everything by hand but, if you get stuck, read further down the page to find out how to get Maple to help you with some of the work.]

Exercise 2: Solve the same problem by separation of variables. Write down the first five non-zero terms in the Taylor expansion (using Maple, if you wish) of your solution about $x=0$. Compare to your answer above.

This particular problem is much easier to solve by separation of variables than by series solution. Series solutions are more useful when methods like separation of variables cannot be used, or when such methods lead to integrals that we cannot do. But series solutions almost always require alot of tedious calculations, so there are plenty of opportunities to make calculational errors. Therefore, it is best to turn this work over to the computer. Let us return, therefore, to Exercise 1 and ask Maple to do it.

`with(powseries):` We first load the `powseries` package.

`eqn:=(1+x^2)*diff(y(x),x)+x*y(x)=0;` This creates our differential equation in Maple and names the equation `eqn`. Notice the `:=` sign is used only to assign the equation to the variable `eqn` in Maple's memory so we can refer to the equation using this name later, and is not the "equals sign" of the equation. Secondly, here Maple will automatically treat y as a function and $y(x)$ as the associated (unknown) expression we want Maple to treat everything here as an expression so we use $y(x)$.

`initvals:= y(0)=1;` This is the initial condition (or boundary condition).

`soln:=powsolve({eqn,initvals});` Here Maple solves the problem using power series and stores the solution in the variable `soln`, but it does not report the solution to us, since it first needs to know how many terms in the power series solution we would like to have displayed. By the way, to get a general solution, simply say `powsolve(eqn);` Maple will then ignore the boundary condition `initvals` when solving the problem.

`tpsform(soln,x,10);` "Transform to Power Series Form." This command displays the first few terms of the solution `soln`. Here we are asking it to display up to (but not including) the x^{10} term.

`soln(_k);` This gives us the recursive formula for the coefficients a_k. The answer that Maple returns is intended to be the right-hand side of an equation whose left-hand side is "a_k=". Maple writes subscripts in parentheses here, so for example "a_{k-2}" will be written as something like "`a(_k-2)`" in the Maple output.

Exercise 3: Find a series solution to the differential equation $\left(1+x^3\right)\dfrac{dy}{dx}-y=0$, subject to the condition $y(0)=2$. In your answer, explicitly write out the first five non-zero terms.

You might like to try to find an exact solution using other techniques. Your main challenge will be the integral you will encounter (which Maple can help with). If you can find such a solution, try to Taylor expand it and compare the resulting expansion to your series solution.