Calculus Lab 28—Direction Fields

<u>Maple Commands:</u>

`with(plots):` Loads the `plots` package of specialized plotting routines, such as:

`fieldplot([expr1,expr2],x=a..b,y=c..d,arrows=LINE);` We can use this to plot direction fields. The `arrows=LINE` part is optional, and is used merely to improve the appearance of the output.

`plot(expr,x=a..b);` Ordinary `plot` command. You can also specify range `y=c..d`.

`int(x*exp(x),x);` Computes $\int xe^x\,dx$, which you may find useful in Exercise 3(c).

`implicitplot(expr,x=a..b,y=c..d);` Plots y vs. x, where y is implicitly defined via the expression `expr`. For example, to plot a unit circle, type

`implicitplot(x^2+y^2=1,x=-1..1,y=-1..1);`

Consider a differential equation of the form

$$a(x,y)\frac{dy}{dx} - b(x,y) = 0 \qquad (1)$$

where $a(x,y)$ and $b(x,y)$ are given. We want to find all functions $y=f(x)$ for which this equation is true. Except when $a(x,y)=0$, this equation is equivalent to the equation

$$y' = \frac{b(x,y)}{a(x,y)} \qquad (2)$$

so we can regard the ratio $F(x,y)=b(x,y)/a(x,y)$ as the slope of the solution curves and use it to draw the direction field tangent to these curves. Maple permits this with the `fieldplot` command. It will not be available until we load the `plots` package:

```
with(plots):
```

Now let us use this to plot the direction field given by the equation

$$y' = -\frac{x}{y} \qquad (3)$$

Equation (3) is separable, and it is easily seen that the solutions obey $x^2 + y^2 = k^2$ where k^2 is a non-negative constant, so the solution curves are circles of radius $|k|$.

To plot the direction field, we supply a and b, surrounded by square brackets, as the first and second arguments, respectively, for the `fieldplot()` command:

```
fieldplot([1,-x/y],x=-2..2,y=-2..2,arrows=LINE);
```

From this picture, it might be hard to say with certainty that the solution curves are circles, so we'll use a trick to clear the nature of the solution curves. When choosing a and b, we have the freedom to multiply each by a common factor. For example, if we multiply equation (3) by $y/(x^2+y^2)^{1/2}$ we get

$$\frac{y}{\sqrt{x^2+y^2}}y' = -\frac{x}{\sqrt{x^2+y^2}} \qquad (4)$$

which is also in the form of (1), but this time we have $a(x,y)=y/(x^2+y^2)^{1/2}$ and $b(x,y)=-x/(x^2+y^2)^{1/2}$ (previously a was 1 and b was $-x/y$). Let's plot the field again:

```
fieldplot([y/sqrt(x^2+y^2),-x/(x^2+y^2)],x=-2..2,y=-
2..2,arrows=LINE);
```

Comparing our two plots, we see that the slope of the line Maple draws at each point is the same in each, but the lengths of these lines differ. The reason Maple does this is not relevant here, since it's only the slopes of these lines that matters.

Exercise 1: For each of the following differential equations, use Maple to help you sketch the direction field. [Hint: Make several plots, varying the range and domain, and perhaps even multiplying a and b by a common factor, until you have a clear picture of the behaviour.] On the sketch, "join up" the direction field to obtain some typical solution curves.

a) $y' = y + 1$

b) $\cos(y)\dfrac{dy}{dx} = \cos(x)$

Exercise 2: Using any convenient method, find the general solution of the differential equations in Exercise 1. For each solution, choose three different values of the constant of integration and in each case plot the resulting function using Maple's `plot` command. Try to choose values of the integration constant that illustrate the different behaviours of possible solution curves for these examples. In each case, make sure that one of your chosen values of the integration constant produces the solution curve that passes through the origin [i.e., arrange that one curve satisfies the condition $y(0)=0$.] Compare these curves to the ones you sketched in Exercise 1. [The `implicitplot()` command may be of some use for this question.]

Exercise 3:

a) Plot the direction field for the equation

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

and sketch some typical solution curves. Notice that these curves have the property that every solution curve approaches the origin. The curves you dealt with in Exercise 1 had somewhat analogous properties. In particular, the solution curves of 1(a) had the property that each curve approached $y=1$ in the limit as $x \to -\infty$.

Exercise 4:

a) Solution curves of the equation

$$y' = x - y$$

approach an "oblique asymptote" (a line $y = mx + b$) as $x \to \infty$. By plotting the direction field for this equation, can you decide what line this is (*i.e.*, what are the values of m and b)?

b) Can you find the general solution of this equation? From this solution, can you show that your answer in part (a) is correct?