Calculus Lab 27—Convergence and Partial Sums

Objective: To study convergence of power series and to be able to interpret convergence behaviour in terms of the sequence of N^{th} partial sums.

Maple Commands:

`taylor(expr,x=a,n);` Computes order *n-1* Taylor polynomial of `expr` about centre $x=a$.

`sum(expr,k=a..N);` Computes the sum $\sum_{k=a}^{N}$`expr`, where `expr` is an expression that depends on k.

`Sum(expr,k=a..N);` Represents $\sum_{k=a}^{N}$`expr` in memory, but does not evaluate the sum. You can evaluate it later, using the `evalf` command.

Let's consider the geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad (*)$$

whenever this sum converges. To decide whether this sum converges, we form the N^{th} partial sum $s_N(x) = \sum_{k=0}^{N} x^k$. The geometric series converges to s if and only if $\lim_{N\to\infty} s_N = s$. Notice the N^{th} partial sum depends here on two quantities, x and N. Let's now define a quantity in Maple that will allow us to study the N^{th} partial sum.

`partsum:=(x,N)->sum(x^k,k=0..N);` We supply `partsum` with these two quantities and `partsum` then returns the required sum (you may have to apply `evalf(%)` to convert `partsum`'s output to a floating point number).

`partsum(1,4);` Does this make sense? How about

`partsum(1/4,infinity);` Is this what you expect, in view of formula (*)?

Exercise 1: We will test the validity of formula (*) above. Notice the right-hand side of this formula makes sense everywhere except at x=1, but what about the left-hand side? Choose some fixed non-zero value for x between -1 and 1. At this fixed value, evaluate the N^{th} partial sum of the geometric series $\sum_{k=0}^{N} x^k$ for several values of N (use `partsum` to do this). Now repeat the experiment, but this time fix some x>1 and try several values of N. Lastly, choose an x<-1 and repeat once more. For each case, describe how s_N behaves as N increases. What do your results suggest concerning the validity of formula (*)?

Use the ratio test to show that this geometric series converges whenever `-1<x<1` and diverges whenever x<-1 or x>1. What do you think happens when x=1? when x=-1? Try a few runs of the `partsum` procedure with x=-1 and various values of N. Make sure to test with both even and odd values of N.

<u>Remark:</u> Some earlier versions of Maple try to apply formula (*) in situations where it is not valid, with surprising (and wrong) results.

<u>*Exercise 2:*</u> The formula $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{x}{2}\right)^k = \ln\left(1+\frac{x}{2}\right)$ can be verified, whenever both sides of the formula make sense, by Taylor expansion of the right-hand side about `x=0`. Using the same technique as in the last exercise (with a suitably modified `partsum` procedure, of course), compute several N^{th} partial sums of the left-hand side for various values of x, say starting at x=0 and moving out in each direction by increments of 1/2 until the sum clearly exhibits divergent behaviour. What do you think is the radius of convergence of this sum? Verify your answer by applying the ratio test.

What about the points that lie at the boundary of the interval of convergence? Test these points using our Maple technique. Do you think the sum converges or diverges at each of these points? Apply an appropriate convergence test to confirm (or disprove) your predictions.