Calculus Lab 26—Error Estimates

Objective: To understand how to estimate the error when approximating a function by a Taylor polynomial.

Maple Commands:

`diff(expr,x);` Differentiates expression `expr` with respect to x.

`subs(x=a,");` Evaluates the previous expression (here `diff(expr,x);`) at $x=a$.

`T:=taylor(expr,x=a,n+1);` Returns a taylor series. Precisely, the n^{th}-order Taylor polynomial for `expr` about centre $x=a$ is displayed, plus a symbol $O(x^{n+1})$, indicating that terms in the Taylor series containing at least $n+1$ powers of x have been dropped.

`convert(T,polynom);` When applied to `T` as above, gets rid of that $O(x^{n+1})$ symbol. What's left is a true Taylor polynomial (not an infinite Taylor series), which can be used in subsequent calculations.

Taylor's formula states that the error incurred when one uses an n^{th}-order Taylor polynomial with centre a to approximate a function $f(x)$ is given by the "remainder term"

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

where z is a number between a and x and $f^{(n+1)}(z)$ is the $(n+1)^{th}$ derivative of f, evaluated at z. Students often find this baffling, because the formula does not tell us exactly what number z is. We are told only that it is *some number* lying between a and x.

If we knew z exactly, then there would be *no error* incurred when using a Taylor polynomial to replace a function. We would simply calculate the value of the Taylor polynomial at x, evaluate the remainder term exactly (since we would know z exactly), and add the two together. This would be exactly equal to $f(x)$.

Usually we will not be able to determine z exactly. Then the idea is to find the largest possible value for $|R_n(x)|$ for all z between a and x. This is the maximum possible error.

Exercise 1:

a) Write down the Taylor series for $\ln(x)$ expanded about centre $x=1$.

b) What is the fifth-order Taylor polynomial $T_5(x)$ for $f(x)=\ln(x)$ with centre $x=1$? Evaluate this polynomial for $x=1.5$. [Hint: If you use Maple to do this, you must compute `taylor(ln(x),x=1,6);` noticethe last argument is 6, not 5. Then you must `convert` the result before using `subs`.]

c) Write an expression for the remainder term $R_5(1.5)$ for this case. Notice that your result should be a function of the variable z. What is the interval in which z must lie?

d) Plot $R_5(1.5)$ as a function of z on the interval you just determined. Use this plot to read off the maximum value of $|R_5(1.5)|$ on this interval. What, therefore, is the maximum error you will incur when using $T_5(1.5)$ as an approximation to $\ln(1.5)$?

e) What is the exact error incurred (what is the exact difference between $T_5(1.5)$ and $\ln(1.5)$)?

Often it is useful to state what sort of error will be considered tolerable and to use this to bound the remainder term. One can then decide how many terms a Taylor polynomial must have in order that the remainder be smaller than this bound. For example, you might be writing a program to be used by a pocket calculator chip to compute values of trig functions. The chip can only perform some simple operations such as addition and multiplication. Therefore, a natural way to program the chip to compute trig functions is to have it use Taylor polynomials to approximate trig functions, since calculations with polynomials involve only operations built from addition and multiplication.

Exercise 2: What order Taylor polynomial should you program into the aforementioned calculator chip if you want the calculator to compute $\arctan(x)$ correctly to five digits after the decimal place (more precisely, errors should be no greater than 0.000 005)? Assume that x lies in the domain $[0,0.1]$.

Hint: One strategy (among many) for attacking this problem is to create a table. The first column contains values of n. The second column contains the maximum value of $|f^{(n)}|$ on the domain $[0,0.1]$. (To get this, use Maple to compute and plot $f^{(n)}$ and read off the bound on $|f^{(n)}|$ from the graph. You actually need to bound the absolute value of $f^{(n)}$. Why?) Plug this into the Remainder Formula to estimate the maximum error if you use a Taylor polynomial of order n to estimate $\arctan(x)$ and record this in the third column. Stop when the third column entry is less than the error requirement stated above.