Calculus Lab 25—Improper Integrals

Objective: We will study a function defined by an improper integral, and interpret the integral as a limit. We will use the definition to compute values of the function.

1. Consider the function $f(t) = t^3 e^{-t}$. Plot this function, say on the domains [0,5] and again on [0,20]. Notice the function never actually crosses the t-axis (except at t=0). Why not?

Because $f(t)$>0 for all t>0, we can interpret the integral of $f(t)$ over any interval for which t>0 as an area. Now consider the area of the region R under the graph of $f(t)$, above the t-axis, and between t=0 and t=a, where a is just some number. Since the area of this region obviously depends on A, let's denote it by $A(a)$.

$$A(a) = \int_0^a t^3 e^{-t}\, dt \qquad (1)$$

We now claim three things:

a. The area $A(a)$ of region R increases as a increases.

b. The area $A(a)$ of region R is never larger than 6, no matter how large a is.

c. If N is any number smaller than 6, we can arrange that $A(a)$ is bigger than N provided we choose a large enough.

We will now check these claims by having Maple compute $A(a)$ for several values of a. Maple cannot compute the integral in (1) analytically, but it can compute it by numerical integration. We can force it to use numerical integration by combining both `int()` and `evalf()` in one command.

`evalf(int(t^3*exp(-t),t=0..2));` In this example, a=2.

2. What is the area under the graph between `t=0` and `t=a` where:
   - (i) a=5?
   - (ii) a=10?
   - (iii) a=20?
   - (iv) What is the smallest value of a such that $A(a)$ is greater than 5.99995?
   - (v) Can you make $A(a)$ greater than 6 if you choose a large enough? Try several large values of a to see.

Do your results support the claims made above? These claims justify the statement

$$A(a)=\int_0^{\infty} t^3 e^{-t}\, dt \qquad (2)$$

The integral in (2) is called improper because, in this case, its upper limit is not a real number. Maple can compute improper integrals. Let's see what it gets for this one:

```
int(t^3*exp(-t),t=0..infinity);
```

The answer should not be a surprise.

Now let's look at a whole class of similar functions, $f(t)=t^n e^{-t}$, where n is for now any positive integer. Let's write a function that allows us to vary n in the above expression:

```
g(n):=n->(t^n)*exp(-t);
```

Let's plot one of these, say the familiar one where n=3:

```
plot(g(3),t=0..10);
```

Look familiar?

3. Plot several specific examples of *g(n)* for different positive integers *n*. Do the graphs look similar? What features change with *n*?

   Use Maple to compute the areas *I(n)* under *g(n)* and above the positive *x*-axis for each of the cases *n*=1, 2, 3, 4, and 5. Does it surprise you that these answers are integers? that they are even finite? By examining these answers, can you guess a simple rule (without doing an integral) for *I(*8*)*, the area under the curve of *g(*8*)*? Check your guess with Maple.

4. Using hand calculations and integration by parts, show that

$$\int_0^{\infty} t^n e^{-t}\, dt = n\int_0^{\infty} t^{n-1} e^{-t}\, dt \qquad (3)$$

   or, in other words,

$$I(n)=nI(n-1), \qquad (4)$$

   where

$$I(n)=\int_0^{\infty} t^n e^{-t}\, dt\,. \qquad (5)$$

5. Show that

$$I(0)=\int_0^{\infty} e^{-t}\, dt = 1 \qquad (6)$$

Using only equations (4) and (6), can you write a simple formula or rule for computing $I(n)$ that uses no integrals? Check your formula by computing $I(8)$ and comparing the answer to the one you obtained in part 3.

6. Using Maple, compute $I(1/2)$. Use equations (4) and (6) to compute $I(3/2)$ and $I(5/2)$. Check the answers by having Maple compute these integrals numerically.