Calculus Lab 19—Integration by Substitution

Objectives: To gain understanding of and practice in integration by substitution.

Work along with the following Maple session to learn some new commands and recall some familiar ones.

`load(student):` We will need routines that are contained in the student package.

Ask Maple to compute the indefinite integral $\int 5x(5x^2+1)^6\,dx$.

`P:=int(5*x*(5*x^2+1)^6,x);` Maple's behaviour differs from one computer to the next, but very likely Maple returned a cumbersome-looking long expression. Let's check it:

`diff(P,x);` This should give back our original integrand $5x(5x^2+1)^6$. Most likely, however, your result looks much more complicated than the original integrand. Try multiplying out that integrand:

`expand(5*x*(5*x^2+1)^6);` Now this probably agrees. Maple first expanded the integrand and then integrated each term, which is why the answer looks complicated. There's a better way. Let's try doing the integral again.

`P:=Int(5*x*(5*x^2+1)^6,x);` Capital `I` in `Int`. This defines `P` to be the integral, but Maple doesn't actually do the integral (for that, we would use `int` with a small `i`).

`Q:=changevar(u=5*x^2+1,P,u);` This tells Maple to use the substitution $u=5x^2+1$ in `P`, and to use u as the variable of integration. Does this expression look simpler?

`value(Q);` Since we used `Int` instead of `int` above, the integral didn't get "done.'' We use `value()` to tell Maple it is now time to do the integral. Notice the answer has a u in it.

`subs(u=5*x^2+1,%);` Now we're done. We can check this with

`diff(",x);`

Exercise 1: Find a substitution that allows you to do each of the following integrals. You may use either hand calculations or Maple (Hint: If you use hand calculations, you can easily check your work by having Maple differentiate your answer). If you use Maple, in addition to writing down the final result, give the substitution you used and the new form of the integral that results from the substitution. If you use hand calculations, show your work.

a) $\int \sqrt{3-\frac{1}{2}t}\, dt$ b) $\int \frac{x+3}{\left(x^2+6x\right)^2}\, dx$ c) $\int x^5\sqrt{x^3+1}\, dx$

[It may help to remember here that Maple's square root function is `sqrt()`.]

The above exercise dealt with the substitution rule as applied to anti-derivatives. When you apply this rule to definite integrals, you must also take into account that the limits of integration are affected by the substitution. As we will see, a chief advantage of Maple is that the **changevar** command automatically handles limits of integration properly.

Exercise 2: Find a substitution that will allow you to calculate:

$$\int_0^2 \frac{2x+1}{\sqrt{6x^2+6x+1}}\, dx$$

Perform the calculation of the integral both by hand and by Maple (using the same techniques as for the anti-derivative on the last page). Submit the hand calculation on your answer sheet. Notice how the limits of integration are automatically taken care of by Maple's **changevar** command. In your hand calculation, did you take care to properly handle the limits of integration?

Define and plot the following expression:

```
f:=x^2/sqrt(x^6+1);
plot(f,x=0..2);
```

What is the area under this curve? It is $\int_0^2 \frac{x^2}{\sqrt{1+x^6}}\, dx$ which we write in Maple as:

`int(f,x=0..2);` This is the *definite* integral on the domain [0,2].

Maple was probably unable to compute this integral (if so, Maple will respond by simply writing out the integral without doing it, as if you had typed `Int` instead of `int`). How can we get around this problem? Here's one way:

Exercise 3: Define the function $g(u)=\frac{1}{3\sqrt{1+u^2}}$:

```
g:=1/(3*sqrt(1+u^2));
```

Plot g on the domain [0,8] and express the area under this curve as a definite integral with respect to the variable u. Use Maple to evaluate the integral exactly.

Copy this plot onto your answer sheet. Also copy down the plot of $f(x) = \frac{x^2}{\sqrt{1+x^6}}$ on the domain $[0,2]$ (you have already obtained this plot above). Use the substitution $u=x^3$ to prove that the areas under these curves on their respective domains actually are equal.

An important point about the above exercise is that neither Maple alone nor hand calculations alone would be very useful in computing the required area. Maple was probably unable to do the original integral of f with respect to x, and had to be told about the substitution converting the integral to the form of an integral of g with respect to u. Maple should have no trouble with this second form of the integral, which is not an easy form to do by hand (it can be done by a special type of substitution called trigonometric substitution, which we will study at another time). Hence, the optimal method in this case was to do some work by hand first, and then turn the rest over to Maple.