Calculus Lab 18—The Fundamental Theorem

Objective: To explore the Fundamental Theorem of Calculus in the form

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Maple Commands:

`int(expr,x);` Indefinite integral of `expr` with respect to variable x.

`int(expr,x=a..b);` Definite integral of `expr` on interval $[a,b]$.

`Int(expr,x=a..b);` Capital `I` means that Maple doesn't actually compute the integral; it just "represents" it in memory (which is sometimes useful since this representation can be used in subsequent calculations).

`with(student):` Loads `student` package containing numerical integration procedures.

`rightbox(expr,x=a..b,n);` Graphical depiction of right sum using n rectangles to approximate integral of `expr` on $[a,b]$. There is also a `leftbox()` command.

`rightsum(expr,x=a..b,n);` Computes right sum. There is also a `leftsum()` command.

`simpson(expr,x=a..b,n);` Simpson's rule approximation to definite integral of `expr` on interval $[a,b]$, using n sub-intervals.

`diff(expr,x);` Differentiates expression `expr` with respect to variable x.

We begin by attempting to compute the integral of $f(t) = \frac{\sin t}{t}$ on $[0,x]$.

`Int(sin(t)/t,t);` This just displays the integral we want to do.

`int(sin(t)/t,t);` Maple did the integral, but the answer isn't helpful. After all, what does `Si(t)` mean?

It means that Maple doesn't know how to do this integral, so it merely gives it a name, `Si(t)`. Notice the value is a function of the upper limit of the integral:

$$Si(x) = \int_0^x \frac{\sin(t)}{t}dt$$

Exercise 1: We will interpret the function Si as a signed area on a graph. To do this, plot the integrand $(\sin(t))/t$ on the interval $[0,1]$ and draw 10 rectangles underneath it to represent area. We will use right-hand rectangles, since left-hand rectangles would require special care at $t=0$.

with(student): The next command will work only if we load the `student` package.

```
rightbox(sin(t)/t,t=0..1,10);
```

Sketch this (you don't need to draw the little rectangles!) and explain the meaning of $Si(1)$ in terms of areas on the graph.

Use a right sum with 10 rectangles and the `evalf()` command to numerically approximate $\int_0^1 \frac{\sin(t)}{t}\,dt$. Do you think this right sum under-estimates or over-estimates the integral? [Hint: Is $(\sin(t))/t$ increasing or decreasing on this interval?]

What about a left sum?

```
leftsum(sin(t)/t,t=0..1,10);
evalf(%);
```

What goes wrong here? Instead, try the following. Take a left sum of $\sin(t)/t$ on the interval $[0.1,1]$, using 9 rectangles. What value does this give? Now we have omitted one rectangle, the one above the interval $[0,0.1]$. This rectangle was the source of our earlier trouble. How high is this rectangle? [Hint: $\lim\left(\frac{\sin t}{t}\right)=?$]. How wide is this rectangle? Use this information to compute the complete left sum on the interval $[0,1]$, using all 10 rectangles. Is the left sum an over-estimate or under-estimate for $Si(1)$?

Even though Maple cannot compute an exact form for the integral that defines $Si(x)$, if you ask Maple for the value of $Si(1)$, it will give you an answer (the same way we just got an answer: by numerical integration):

evalf(Si(1)); Notice the effect of the "small error" discussed above.

Exercise 2: Use the instruction `evalf(Si(x));` to have Maple evaluate $\int_0^{1+h} \frac{\sin(t)}{t}\,dt$ where h takes the values 0.1, 0.01, 0.001, -0.1, -0.01, and -0.001. Record the results in a table (using Maple's default 10 digit accuracy). [If you wish, Maple can keep the table for you. Under the `Insert` menu, choose `spreadsheet`. You can enlarge the resulting table by dragging its border with the mouse.] For each value of h, record the difference quotient:

$$\frac{Si(1+h)-Si(1)}{h}=\frac{\int_0^{1+h}\frac{\sin(t)}{t}\,dt-\int_0^{1}\frac{\sin(t)}{t}\,dt}{h} \qquad (*)$$

Now compute the limit of this quotient as $h \to 0$:

```
evalf(limit((Si(1+h)-Si(1))/h,h=0));
```

- What name do we usually give to the limit of the left-hand side of equation (*)?
- Now consider the right-hand side of (*). The numerator represents an area—what area? The denominator represents a width. The quotient of these is a height—the height of what? [Hint: Compute the value of $(\sin(t))/t$ at $t = 1$ and compare to the values for the difference quotient that you obtained above.] What theorem is at work here?

Exercise 3: Have Maple find the indefinite integrals of each of the following, and then differentiate the answers. Compare the result in each case to the original function. Are they always the same? (Careful: Two things that are the same may not always look the same. It may help to apply the `simplify(%)` command to your results.)

1. $x\sqrt{1+x^2}$
2. $x\tan(x^2)$
3. $\dfrac{\ln(1+x)}{x^2}$
4. $\sin^2(x)$ (Careful not to input $\sin(x^2)$ inadvertently.)