Calculus Lab 17—Numerical Integration

Objective: To approximate definite integrals using numerical techniques, and to begin to comprehend the error incurred when using these approximations.

New Maple Student Package Commands:

`with(student):` Loads the student package of Maple. Each of the following commands requires the student package to be loaded first.

`rightsum(expr,x=a..b,n);` Approximates integral of `expr` on interval [*a*,*b*] by a right-hand sum consisting of the areas of *n* rectangles. There are also `leftsum()` and `middlesum()` commands.

`rightbox(expr,x=a..b,n);` Draws graph, using *n* rectangles to illustrate right-hand sum. There are also `leftbox()` and `middlebox()` commands.

`trapezoid(expr,x=a..b,n);` Approximates integral by trapezoidal rule with *n* trapezoids.

`simpson(expr,x=a..b,n);` Approximates integral by Simpson's rule.

Other New Maple Commands:

`int(expr,x=a..b);` Definite integral of expression `expr` on interval [*a*,*b*].

`int(f(x),x=a..b);` Definite integral of function `f:=x->expr` on interval [*a*,*b*].

Consider the functions $f(x)=\dfrac{4}{1+x^3}$ and $g(x)=x^2\ln(x)$. We will attempt to determine the areas under both these functions over given domains, using both exact integration and numerical integration methods. If you haven't done so already, load the student package:

`with(student):`

Exercise 1: Begin with *f*(*x*). Plot the following graphs and copy down the results (correctly showing the displayed rectangles). Then compute and record the total area under the displayed rectangles. We will use 6 rectangles.

`leftbox(4/(1+x^3),x=0..2,6);` Interval [0,2] is divided up into 6 sub-intervals.

`evalf(leftsum(4/(1+x^3),x=0..2,6));` Total area under displayed rectangles.

```
rightbox(4/(1+x^3),  x=0..2,6);
evalf(rightsum(4/(1+x^3),  x=0..2,6));
```

Compare these approximations to the exact integral giving the area under this curve on the interval $[0,2]$:

```
int(4/(1+x^3),  x=0..2);
evalf(%);
```

Based on these sketches, would you expect the left sum to under-estimate or over-estimate the area under the curve of a decreasing positive function? Use the sketches to explain your answer. What do you expect in this regard for the right sum?

Does the result of the left sum increase or decrease as the number of rectangles n is increased? Why should you expect this?

What happens to the result of the right sum as the number of rectangles n is increased? If we use the same number n of rectangles for both left and right sums, how many rectangles do we need in order that the first two digits of the results obtained by each of these two sums agrees?

Exercise 2: Now consider $g(x) = x^2 \ln(x)$. We repeat the first part of the above exercise. For the interval $x \in [1,4]$, plot graphs showing, as above, the rectangles making up a left-sum approximation to the area under the graph. Repeat for a right-sum, using 10 rectangles in each case. Copy down the results, correctly showing the displayed rectangles. Compute and record the total area under the displayed rectangles.

Based on the plots you obtained in this exercise, do you expect the left sum to under-estimate or over-estimate the area under the curve of an increasing positive function? What about the right sum? Why?

Lastly, you might be wondering why you'd need to use numerical integration when you can use Maple's `int()` command to get an exact answer quickly. Here's why:

`int(1/sqrt(1+x^5),x=1..2);` Maple returns the integral in proper notation, but cannot actually do the integral.

`evalf(%);` This still gives a result, *but it uses numerical integration.* Maple itself decides on a numerical method and uses it to find an approximate answer, since it cannot do the integral exactly.

Exercise 3: Approximate $\int_1^2 \frac{1}{\sqrt{1+x^5}}\,dx$ using a left sum, a right sum, the trapezoidal rule, and Simpson's rule, each with $n = 10$. In this case, we do not know the exact answer (Maple's result, above, is very accurate, but in principle it is approximate and could have a large error, so let's agree not to trust it yet). We will therefore try to decide which of our four approximations is best by testing their "stability." The idea is to re-evaluate our four approximations with a slightly increased value of n.

Which approximation changes the least (is the "most stable") when you do this? This might be the one that is closest to the actual value of the integral, although we cannot be certain, the idea being that it changed very little because it was already closer than the others to the exact value and so had less room to improve. Better methods for establishing the error in these approximations to integrals are given in your textbook.

Is the result obtained from the left sum greater than or less than that obtained from the right sum? Does this make sense, in view of your answers to Exercises 1 and 2?