Calculus Lab 16—Infinite Sums and Partial Sums

Objective: To introduce Sigma notation and convergence behaviour of infinite sums, and to compute some convergent sums using partial sum methods.

New Maple Commands:

`Sum(expr,k=a..b);` returns Sigma notation for the sum from $k=a$ to $k=b$ of `expr`.

`sum(expr,k=a..b);` evaluates in closed form (where possible) the sum of `expr`, with limits $k=a$ to $k=b$.

Recent Maple Commands:

`taylor(expr,x=a,n+1);` computes n^{th}-order Taylor polynomial of `expr` about $x=a$.

Before proceeding with the lab exercises, we will need to understand how Maple treats sums, particularly infinite sums. Let's try a few examples:

`S1:=Sum(1/k^2,k=1..infinity);` The capital-S `Sum()` returns Sigma notation.

`evalf(S1);` The sum `S1` has infinitely many terms, yet it still adds up to a finite total.

`s2:=sum(1/k^2,k=1..infinity);` The small-s `Sum` returns a "closed form" answer when it can find one.

`evalf(s2);` Should give the same as `evalf(S1)`.

Maple cannot always find a sum. Sometimes this is because the sum does not converge, i.e. it does not add up to a finite number. But sometimes it is because of Maple's shortcomings. Try:

`s3:=sum(ln(k+1)/k^3,k=1..infinity);`

Even the small-s `sum()` returned only a Sigma-notation answer, because Maple was unable to find a nice "closed form" for the answer. But perhaps we might coax a real number from Maple:

`evalf(s3);`

Maple may simply not give any answer, or it might return the word `FAIL` followed by the number 0.9594129036. Maple's usual methods for computing the sum exactly have failed. However, it may have been able to attempt to approximate the sum, and if so, it will compute the approximation 0.9594129036 before giving up.

Exercise 1: Let's compute the sum `s3` using another method, sometimes called partial sums. Define the function:

```
mySum:=n->Sum(ln(1+k)/k^3,k=1..n);
```

This allows us to add up the first n terms in a convenient manner. What is the sum of the first 10 terms?

```
mySum(10);
evalf(%);
```

What is the sum of the first 100 terms? 500 terms? By how much does the sum increase in passing from 500 terms to 501 terms? What is

`limit(mySum(n),n=infinity);` ? (You may need `evalf()` here as well.)

Sometimes, however, when Maple fails to total up a sum, it is because the sum has no total or the total is infinite.

In the next exercise, we will use Taylor polynomials to partially deduce the formula (called the *geometric series formula*):

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ if —x—¡1.} \qquad (1)$$

To begin, test this formula. Compute $\sum_{k=0}^{\infty}(0.5)^k$ using Maple. Does the answer equal $\frac{1}{1-(0.5)}$? Now try $\sum_{k=0}^{\infty}(1.25)^k$. Notice Maple returns a huge number, because the sum you've tried to compute is diverging. For any x such that $|x| > 1$, the sum in (1) diverges.

Exercise 2: Compute the fourth-order Taylor polynomial of `1/(1-x)` about 0, copy down the result, and rewrite it by hand in Sigma notation:

```
taylor(1/(1-x),x=0,5);
```

Maple gives the result $1+x+x^2+x^3+x^4+O(x^5)$. How would you write this in Sigma notation? Repeat this exercise to obtain the tenth-order Taylor polynomial, and write it in Sigma notation. Based on your results, what would you propose would be the formula in Sigma notation for the n^{th}-order Taylor polynomial for $\frac{1}{1-x}$ about $x=0$? If $n \to \infty$, does your proposed formula reproduce formula (1)?

Exercise 3: A sum can do one of three different things. It can add up to a number, it can diverge to infinity, or it can do neither. Keeping in mind the third possibility, let's now study

$$\sum_{k=0}^{n} \cos(k\pi)$$

Again define a function that will allow us to conveniently vary the upper limit of this sum:

```
newsum:=n->sum(cos(k*Pi),k=0..n);
```

Try some consecutive values of n, e.g.

```
newsum(6);      newsum(7);      newsum(8);      newsum(9);
newsum(10),  ...
```

Notice a pattern? What is it? Can you explain the pattern? [Hint: What is the difference between the values of $\cos(k\pi)$ and $\cos((k+1)\pi)$?]

Now try to compute $\sum_{k=0}^{\infty} \cos(k\pi)$:

`limit(newsum(n),n=infinity);` What does the answer mean? Does this limit exist?

Does this sum add up to a number? If so, what number? If not, does it diverge to infinity?

You may wonder why in this exercise we didn't use

```
sum(cos(k*Pi),n=0..infinity);
```

Try it. Depending on the version of Maple installed on your computer, you may either get a wrong answer(!) or you will get an answer that is written in language that is not easy to understand. While Maple is a useful program, it is less than perfect.