Calculus Lab 15—Taylor Polynomials

Objective: To approximate functions by their Taylor polynomials and to gain a graphical and numerical understanding of the error incurred in so doing.

New Commands:

`taylor(e1,x=a,n);` Computes the degree $n-1$ Taylor polynomial of `e1` about x = a.

`convert(%,polynom);` Convert the last expression (`%`) to a polynomial.

Reminder: functions versus expressions:

`f:=x->(x^2)*sin(x);` Defines the *function* $f(x)=x^2\sin(x)$.

`f(Pi);` Evaluates f at $x=\pi$.

`D(f);` Differentiates the function f.

`plot(f,-2..2);` Plots this function on $-2<x<2$.

`e1:= (1/x)tan(x);` Defines the *expression* $(1/x)\tan(x)$ and calls it `e1`.

`subs(x=Pi,e1);` Evaluates `e1` at $x=\pi$.

`diff(e1,x);` Differentiates `e1` with respect to x.

`plot(e1,x=-2..2);` Plots `e1`; must say "`x=`" when specifying the domain.

To begin, let's compute a familiar Taylor polynomial of degree 4:

`T1:=taylor(sin(x),x=0,5);` (Notice no x^4 term: is this unexpected?)

Notice the large $\mathrm{O}(x^5)$ symbol. This means that Maple isn't exactly giving us what we asked for. Maple uses this notation to tell us that $\sin(x)$ is equal to the polynomial we see displayed plus terms containing x^5, and possibly higher powers of x as well. We only want the Taylor polynomial, not the extra $\mathrm{O}(x^5)$ symbol. If we try to treat this answer as if it were a true polynomial, our efforts will fail. For example, try:

`subs(x=0.1,T1);` Here we try to evaluate our polynomial at $x=0.1$.

`expand(x*T1);` This is an attempt to multiply our Taylor polynomial by x.

These commands won't work properly. To fix this, we must first convert `T1` into a proper Maple polynomial using

`P1:=convert(T1,polynom);`

Now try the above commands using `P1`, which has no $\mathrm{O}(x^5)$ symbol. Compare the result of `expand(x*P1);` with

```
taylor(x*sin(x),x=0,5);
```

Useful Hint:

When completing the following exercises, we will often have to compute Taylor polynomials of several different orders for the same function. We can save some typing by combining the `convert()` and `taylor()` commands. To give an example, say that we want to compute a Taylor polynomial about $a = 0$ of $\frac{1}{1-x}$, but we are uncertain what order Taylor polynomial will do, so we'd like to vary the order, but do not want to have to type the `convert()` and `taylor()` commands more than once. Then we can define the function

```
Tpoly:=n->convert(taylor(1/(1-x),x=0,n+1),polynom);
```

Now let's find the Taylor polynomial for a few different choices of the order n.

```
Tpoly(6);
Tpoly(12);
```

`Tpoly(20);` Good thing you don't have to compute this one by hand!

`subs(x=0.5,Tpoly(6));` Evaluates the sixth-order polynomial at $x = 0.5$.

```
plot({Tpoly(6),Tpoly(12),1/(1-x)},x=0..1,y=0..20);
```

Notice how these three curves agree well for small x but disagree near $x = 1$.

Exercise 1: Plot $\cos(x)$ together with the second order, fourth order, and sixth order Taylor polynomials for $\cos(x)$ about $a = 0$ all on the same set of axis. A good domain to start with is [-4,4], but try a few graphs with slightly different domains until you are certain you understand what you are seeing. Sketch one such graph to pass in, indicating which curve is which. By "second order" we mean include the term containing x^2.

Next, plot the second order Taylor polynomial for $\cos x$ on the same set of axes as the graphs of $(\cos x) + 0.1$ and $(\cos x) - 0.1$. [Hint: Don't choose a very large domain for the x-values.] Using this graph (which you don't have to copy down or hand in), find the largest interval of x-values on which the difference between $\cos x$ and its second order Taylor polynomial is not more than 0.1. Repeat for the fourth and sixth order polynomials.

Exercise 2: Consider the second, fourth, sixth, and twelfth order Taylor polynomials for $\cos(x)$ about $a = 0$. Evaluate each of them at $x = \pi/3$ and report the values (you don't have to report the polynomials themselves), using Maple's default ten digits of accuracy (you can use the `evalf()` command to make Maple report the answer in ten-digit decimal form). Compare to the actual value of $\cos(\pi/3)$. Remember, to make a π in Maple, type `Pi`.

Exercise 3: For this exercise, we will use the expression $\frac{1}{1-x}$ and compare its exact value to the values of its Taylor polynomials of various orders.

Use Maple's `evalf()` command (or any other handy method) to evaluate the expression $\frac{1}{1-x}$ when $x = 0.75$. Compare this to the values obtained from substituting $x = 0.75$ into the fifth-order, fiftieth order, and hundredth order Taylor polynomials.

Now repeat this experiment, using different values of x. In particular, choose at least one other value between 0.75 and 1.0, and one value larger than 1.0. What do you notice happening? Do you have an explanation for it?