Calculus Lab 14—Max/Min Theory and Practice

Objective: To explore some of the theory underlying max/min analysis.

Maple Commands:

`plot(expr,x=a..b,y=c..d);` Plots expression `expr`; restricts both domain (x-values) and range (y-values).

`plot({expr1,expr2},x=a..b);` Plots both `expr1` and `expr2` on one set of axes.

`diff(expr,x);` Differentiates expression `expr` with respect to x.

Let's consider a theorem that you may or may not have heard of in your lectures. We will give it in the language of graphs of functions. This language can be made just as precise as the more formal language often used in textbooks, but is more convenient for our purposes.

Rolle's Theorem (Graphical Form): Assume that there is a horizontal line which intersects the graph of $f(x)$ twice, say at $x = a$ and $x = b$. If the graph of $f(x)$ is an unbroken curve (in precise language, if f is continuous) from a to b, then either there is a point between $x = a$ and $x = b$ where the tangent line to the graph is horizontal, or there's a point where the tangent line does not exist.

So let's see what this means with some simple examples.

Exercise 1: For each case below, graph the given function and the given horizontal line on a single set of axes. Copy the graphs onto paper to be handed in. For each graph, answer the questions listed below.

a) $f(x) = -x^2 + 4x + 1$ and the line $y = 3$.

b) $f(x) = \dfrac{1}{(x-2)^2}$ and the line $y = 5$. [You may have trouble getting Maple to draw a decent graph. If so, try graphing just $f(x)$ alone and using a restriction on the range, such as $y = 0..10$. You can draw in the horizontal line $y = 5$ by hand.]

c) $f(x) = 2x^3 - 7x^2 + 2x + 3$ and the line $y = -10$. (Be careful to choose a useful domain.)

d) $f(x) = (x-1)^3 + 2$ and the lines $y = 1$, $y = 2$, and $y = 3$.

e) $f(x) = \left|\dfrac{x^2 - 1}{x^2 + 1}\right|$ and the line $y = 1/2$.

Q1. For each graph, list all values of x for which the graph of the function and the horizontal line intersect. At each point of intersection, what is the sign of the slope of the function?

Q2. For each graph, give the values of x where the tangent line to the graph of the function appears to be horizontal? Are there any points where the tangent line cannot be defined? The points at which the tangent line is either horizontal or undefined are called critical points.

Q3. Are any of the above examples not in accord with Rolle's theorem? If so, explain.

Q4. If a point on a graph is higher than its immediate neighbours, it is called a local maximum; if it's lower than its immediate neighbours then it's called a local mimimum. For each graph that has local maximum and/or minimum points, what are the x-values of these points? Do these points coincide with the answers to `Q2`? Can you state what relationship exisits between the local maximum and minimum points and the critical points (take into account example `(d)` above).

Q5. Can you relate the signs of the slope of the function before and after the critical point (answers to `Q1`) to the type of the critical point (local maximum, local minimum, or neither)?

Mean Value Theorem for Derivatives: Assume that there is a line with slope m which intersects the graph of $f(x)$ twice, say at $x = a$ and $x = b$. If the graph of `f(x)` is an unbroken curve from `a` to `b`, then either there is a point between $x = a$ and $x = b$ where the tangent line to the graph has slope m, or there's a point where the tangent line does not exist.

This is like Rolle's Theorem "with a tilt'' (to understand the theorem, just take one of your graphs from the the last exercise and tilt it—try to imagine tilting everything except the axes—at an angle so that the horizontal line now acquires a slope m).

Exercise 2: Graph the function $f(x) = x^{2/3}$ and the line $y = \frac{1}{3}x$ on the same set of axes (use `x=0..50` for the domain in the Maple `plot()` command) and note the x-values where the graphs intersect. Now try to produce a new plot containing a line of the same slope as the given line but tangent to the graph of the function $f(x)$ (what part of the equation of the line should you change in order to achieve this?). Note the equation of the tangent line and the x-value of the point of tangency. Find this tangent line and x-value by analytical means (i.e. set up and solve equations to find the tangent line and point of tangency).