

## Calculus Lab 13—Techniques of Differentiation

**Objective:** To further study differentiation, including the chain rule and implicit differentiation, using graphical methods as a complement to analytical techniques.

**Recall:**

`D(f);` Differentiates the function  $f$  with respect to its variable.  
`diff(e1,x);` Differentiates the expression  $e1$  with respect to  $x$ .  
`plot(expr,x=a..b);` Plots Maple expression  $expr$  on domain  $x \in [a,b]$ .  
`plot({expr1,expr2,...},x=a..b);` Plots multiple Maple expressions on a single set of axes.

The chain rule allows us to differentiate a function  $h(x) = (f \circ g)(x) = f(g(x))$  which is the composition of two other functions. It is often written as

$$h'(x) = f'(g(x))g'(x).$$

In words, this formula says, “To find the slope of  $h$  at  $x$ , multiply the slope of  $g$  at  $x$  by the slope of  $f$  at  $g(x)$ .”

*Exercise 1:* Plot  $f(x) = \sin x$ ,  $g(x) = x^2$ , and  $h(x) = \sin(x^2)$  (either separately or together, whichever you find most convenient). Copy down each plot. Sketch the tangent line to  $g(x)$  at  $x = 1.5$  and estimate its slope (you don't have to be careful about this estimate). Now consider the graph of  $f(x)$ . At what  $x$ -value should you draw a tangent on this graph in order that the product of its slope with the slope you obtained from the graph of  $g(x)$  equals the slope of the tangent to  $h(x)$  at  $x = 1.5$ ? Draw all these tangents, estimate their slopes, and check to see that the relationship between them implied by the chain rule holds.

We now turn our attention to implicit differentiation. Here we will use Maple expressions in a new way; we will write complete equations and use Maple's expression syntax to give the equation a label. For example:

```
e4:= x*y^2 + sin(y) - x = 0;
```

This isn't a very mathematically proper notation, but it is useful all the same. It says, “ $e4$  is a Maple variable in which we store the equation  $xy^2 + \sin(y) - x = 0$ .” We can refer to the equation by referring to the variable  $e4$ .

We might like to plot the function  $y = f(x)$ . However, it's “implicitly defined” (meaning that  $y$  is not isolated on one side of the equation). Maple can plot it anyway, but not with the standard plotting commands. We must first load into

memory the full `plots` package and then use the `implicitplot()` command, which plots implicitly defined functions.

**`with(plots):`** This loads the full `plots` package into memory.

**`implicitplot(e4,x=-3..3,y=-3..3);`** You must specify *both domain and range*.

Surprising graph. Notice  $y$  isn't even a function! (because there is more than one  $y$ -value belonging to each  $x$ -value.)

Maple does not yet know that  $y$  depends on  $x$ . Before continuing, we will have to tell it:

**`e5:=subs(y=y(x),e4);`**

The expression `e5` is the same as `e4` except that  $y$  is now dependent on  $x$ . Now we implicitly differentiate; in other words, we differentiate *both sides* of `e5`:

**`e6:=diff(e5,x);`**

Finally, we'd like to solve for  $y'$ .

**`yPrime:=diff(y(x),x);`** We'll let `yPrime` be our name for "derivative of  $y$ ."

**`solve(e6,yPrime);`**

*Exercise 2:* If  $x^4y^2 + x^2y = 1$ , plot its graph (copy down and submit it) and use implicit differentiation to find its derivative. Remember to isolate  $y'$ . How many distinct values do you expect this derivative to take at  $x = 1$ , and why? [Hint: Look at the graph. Also, you can improve the graph by asking Maple to sample more points when plotting it. To do this, try typing `grid=[100,100]` as an option in the `implicitplot()` command. Options are included after the domain and range, e.g. `implicitplot(expr,x=a..b,y=c..d, grid=[100,100])`].

These last examples illustrate one advantage in using Maple: we can work with complicated examples with relative ease, so we do not need to have contrived or overly simplified examples to practise with. Sometimes, however, it is easier to do some work in Maple and some work by hand:

*Exercise 3:* Find the value(s) at  $x = 1$  of the derivative you found in Exercise 2. You will probably find it useful to use Maple commands, a graph, and hand calculations in some convenient combination.