Calculus Lab 10—Inverse Functions

Objective: To study inverses of functions from the point of view of the cancellation equations.

Recall:

`Tn:=taylor(expr,x=a,n+1);` `Tn` is the n^{th}-order Taylor polynomial of `expr` about a.

`Pn:=convert(Tn,polynom);` Converts `Tn` to a Maple polynomial (drops the $O(x^{n+1})$).

Recall that a function g is said to be the inverse of a function f on the domain $[a,b]$ if for all x in $[a,b]$ the following equations hold:

$$f(g(x)) = x$$

$$g(f(x)) = x$$

These are sometimes called the *cancellation equations*.

Given a function, how do you find the inverse function? First, you should make sure the function you are dealing with is one-to-one; that is, every y-value must belong to only one x-value.

> *Exercise 1:* Make three plots of $\cos(x)$, one plot for each of the following domains: `x=-10..10`, `x=-Pi/2..Pi/2`, and `x=0..Pi`. Use these plots to answer the following questions (you don't have to copy down the plots unless you feel it would help explain your answer):
>
> Does $\cos(x)$ have an inverse function, if x is allowed to be any number? if x can be any number between $-\pi/2$ and $\pi/2$? if x can be any number between 0 and π?

Now let us assume we have a one-to-one function. How do we find its inverse? The function itself takes an x-value and produces a y-value. According to the cancellation equations, the inverse function takes that y-value as input and gives back the original x-value.

That is, if $y = f(x)$, then to find the inverse function we must solve for x, which will of course depend on y. To make this dependence clear, we write $x = g(y)$. Then g is the inverse function for f. By the way, we usually write g in the form $y = g(x)$ instead of $x = g(y)$; in other words, once we find out what g is, we then use the symbol x to represent its independent variable and use y for its dependent variable.

How does one do this in Maple? Let's use an example:

`f:=x->4*x/(3+x);` This function is not defined at $x = -3$, but it is one-to-one.

`plot(f,-infinity..infinity);` This will produce a less-than-perfect graph, but perhaps it's good enough to confirm that `f` is one-to-one. By the way, if you see a vertical line running through $x = -3$, it's not part of the graph. It's an error produced by Maple's "join up the dots" plotting routine.

`solve(y=f(x),x);` Maple returns the answer $\frac{-3y}{y-4}$.

Now we define the inverse function for *f*. It is simply this answer, but with *x* replacing *y*:

```
g:=x->-3*x/(x-4);
```

Finally, check that *f* and *g* are inverses by applying the cancellation equations:

```
f(g(x));
simplify(%);
g(f(x));
simplify(%);
```

> *Exercise 2:* Using the above steps, find the inverse of the function $f(x) = \sqrt{x^3+1}$ on the domain $x \geq -1$. Check your answer using the cancellation equations.

Lastly, we will look at the inverse function for the tan function on the domain $-\pi < x < \pi$. It is usually called arctan. Try the following in Maple:

`tan(Pi/4);` Maple should answer: 1 (you might have to use the `simplify()` command).

`arctan(");` Maple should answer: $\pi/4$. This is what you would expect, using the cancellation equations. This tells us that

$$\pi = 4\arctan(1) .$$

But now here comes the really interesting part. Did you ever wonder how it happens that people know the value of the number π to many digits of accuracy? You could try to measure both the circumference and the diameter of a circle and divide the latter into the former. The answer should be π, but such measurements are bound to be inaccurate. Here's a way that involves no measurements and has nothing to do with circles(!):

> *Exercise 3:* Find the tenth-order taylor polynomial, expanded about `x=0`, for the function $f(x) = 4\arctan(x)$. Substitute $x = 1$ into this polynomial. What do you get? Did you get very close to the accepted value of π? Try increasing the order of

the polynomial, to see if you can do better. The first eight digits of the accepted value, by the way, is 3.1415926, and you will need an enormous taylor polynomial to get even a few digits correct. See how close you can come before Maple runs out of memory or you run out of time. Report the order of the highest-order Taylor polynomial you were able to try, and the approximate value for π that resulted.