Two geometric flow problems arising from the physics of mass

Eric Woolgar

Dept of Mathematical and Statistical Sciences
University of Alberta
ewoolgar@math.ualberta.ca
http://www.math.ualberta.ca/~ewoolgar

Sept 2008

Collaborators

Collaborators

- Todd Oliynyk (Monash University)

Collaborators

- Todd Oliynyk (Monash University)
- Lily Gulcev (Alberta)
- Paul Mikula (Alberta)

Collaborators

- Todd Oliynyk (Monash University)
- Lily Gulcev (Alberta)
- Paul Mikula (Alberta)
- V Suneeta (Alberta)
- Mohammad Akbar (Alberta, PDF)

Geometric flows in physics

General Relativity:

Geometric flows in physics

General Relativity:

- Calabi flow.

Geometric flows in physics

General Relativity:

- Calabi flow. $\Bigg\}$ Arises from Einstein equations for Robinson-Trautman metric.

Geometric flows in physics

General Relativity:

- Calabi flow.
- Flow of static metrics.

} Arises from Einstein equations for Robinson-Trautman metric.

Geometric flows in physics

General Relativity:

- Calabi flow.
- Flow of static metrics.
- Flow of stationary metrics.

$\left.\right\}$ Arises from Einstein equations for Robinson-Trautman metric.

Geometric flows in physics

General Relativity:

- Calabi flow.

Arises from Einstein equations for Robinson-Trautman metric.

- Flow of static metrics.
- Flow of stationary metrics.

Parabolic flows from Ricci flow of Lorentzian metric.

Geometric flows in physics

General Relativity:

- Calabi flow.

  } Arises from Einstein equations for Robinson-Trautman metric.

- Flow of static metrics.
- Flow of stationary metrics.

  } Parabolic flows from Ricci flow of Lorentzian metric.

- Ricci flow on manifolds with boundary.

Geometric flows in physics

General Relativity:

- Calabi flow.

- Flow of static metrics.
- Flow of stationary metrics.
- Ricci flow on manifolds with boundary.

Arises from Einstein equations for Robinson-Trautman metric.

Parabolic flows from Ricci flow of Lorentzian metric.

Black hole thermodynamics.

Geometric flows in physics

General Relativity:

- Calabi flow.
- Flow of static metrics.
- Flow of stationary metrics.
- Ricci flow on manifolds with boundary.

Calabi flow: Arises from Einstein equations for Robinson-Trautman metric.

Flow of static metrics, flow of stationary metrics: Parabolic flows from Ricci flow of Lorentzian metric.

Ricci flow on manifolds with boundary: Black hole thermodynamics.

Quantum Gravity:

Geometric flows in physics

General Relativity:

- Calabi flow. } Arises from Einstein equations for Robinson-Trautman metric.
- Flow of static metrics.
- Flow of stationary metrics. } Parabolic flows from Ricci flow of Lorentzian metric.
- Ricci flow on manifolds with boundary. } Black hole thermodynamics.

Quantum Gravity:

- Renormalization group flow for nonlinear sigma models.

$$\frac{\partial g_{ij}}{\partial t} = -\alpha' R_{ij} - \frac{\alpha'}{2} R_{iklm} R_j{}^{klm} + \dots$$

The two problems:

The two problems:

- Tachyon condensation: the behaviour of mass under Ricci flow of asymptotically flat metrics.

The two problems:

- Tachyon condensation: the behaviour of mass under Ricci flow of asymptotically flat metrics.
    - Decay of tachyon: radiates closed string graviton modes.

The two problems:

- Tachyon condensation: the behaviour of mass under Ricci flow of asymptotically flat metrics.
  - Decay of tachyon: radiates closed string graviton modes.
  - RG flow emulates dynamics.

The two problems:

- Tachyon condensation: the behaviour of mass under Ricci flow of asymptotically flat metrics.
  - Decay of tachyon: radiates closed string graviton modes.
  - RG flow emulates dynamics.
  - This picture suggests that mass should decay in Ricci flow.

The two problems:

- Tachyon condensation: the behaviour of mass under Ricci flow of asymptotically flat metrics.
  - Decay of tachyon: radiates closed string graviton modes.
  - RG flow emulates dynamics.
  - This picture suggests that mass should decay in Ricci flow.
- Quasilocal mass: the behaviour of mass under the flow of static metrics. Two cases:

The two problems:

- Tachyon condensation: the behaviour of mass under Ricci flow of asymptotically flat metrics.
  - Decay of tachyon: radiates closed string graviton modes.
  - RG flow emulates dynamics.
  - This picture suggests that mass should decay in Ricci flow.
- Quasilocal mass: the behaviour of mass under the flow of static metrics. Two cases:
  - Asymptotically flat manifolds with inner boundary and Bartnik's boundary conditions.

The two problems:

- Tachyon condensation: the behaviour of mass under Ricci flow of asymptotically flat metrics.
  - Decay of tachyon: radiates closed string graviton modes.
  - RG flow emulates dynamics.
  - This picture suggests that mass should decay in Ricci flow.
- Quasilocal mass: the behaviour of mass under the flow of static metrics. Two cases:
  - Asymptotically flat manifolds with inner boundary and Bartnik's boundary conditions.
  - No inner boundary: complete manifold.

The Ricci flow derivative of mass is zero:

The Ricci flow derivative of mass is zero:

- Preservation of asymptotic flatness (short time existence): See Oliynyk [arxiv:math/0607438]; Dai and Ma [arxiv:math/0510083].

The Ricci flow derivative of mass is zero:

- Preservation of asymptotic flatness (short time existence): See Oliynyk [arxiv:math/0607438]; Dai and Ma [arxiv:math/0510083].
- ADM mass: For $\delta =$ flat background metric near infinity

$$M := \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i g_{jk} - \partial_k g_{ij}\right) d\Sigma^k(\delta)$$

The Ricci flow derivative of mass is zero:

- Preservation of asymptotic flatness (short time existence): See Oliynyk [arxiv:math/0607438]; Dai and Ma [arxiv:math/0510083].
- ADM mass: For $\delta =$ flat background metric near infinity

$$M := \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i g_{jk} - \partial_k g_{ij}\right) d\Sigma^k(\delta)$$

$$\frac{dM}{dt} = \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i \frac{\partial g_{jk}}{\partial t} - \partial_k \frac{\partial g_{ij}}{\partial t}\right) d\Sigma^k(\delta)$$

The Ricci flow derivative of mass is zero:

- Preservation of asymptotic flatness (short time existence): See Oliynyk [arxiv:math/0607438]; Dai and Ma [arxiv:math/0510083].
- ADM mass: For $\delta =$ flat background metric near infinity

$$M := \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i g_{jk} - \partial_k g_{ij}\right) d\Sigma^k(\delta)$$

$$\frac{dM}{dt} = \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i \frac{\partial g_{jk}}{\partial t} - \partial_k \frac{\partial g_{ij}}{\partial t}\right) d\Sigma^k(\delta)\ ;\ \frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

The Ricci flow derivative of mass is zero:

- Preservation of asymptotic flatness (short time existence): See Oliynyk [arxiv:math/0607438]; Dai and Ma [arxiv:math/0510083].
- ADM mass: For $\delta =$ flat background metric near infinity

$$M := \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i g_{jk} - \partial_k g_{ij} \right) d\Sigma^k(\delta)$$

$$\begin{aligned} \frac{dM}{dt} &= \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i \frac{\partial g_{jk}}{\partial t} - \partial_k \frac{\partial g_{ij}}{\partial t} \right) d\Sigma^k(\delta) \; ; \; \frac{\partial g_{ij}}{\partial t} = -2R_{ij} \\ &= -2 \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \underbrace{\left(\partial_i R_{jk} - \partial_k R_{ij} \right)} \underbrace{d\Sigma^k(\delta)} \end{aligned}$$

The Ricci flow derivative of mass is zero:

- Preservation of asymptotic flatness (short time existence): See Oliynyk [arxiv:math/0607438]; Dai and Ma [arxiv:math/0510083].
- ADM mass: For $\delta=$ flat background metric near infinity

$$M := \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i g_{jk} - \partial_k g_{ij}\right) d\Sigma^k(\delta)$$

$$\begin{aligned} \frac{dM}{dt} &= \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i \frac{\partial g_{jk}}{\partial t} - \partial_k \frac{\partial g_{ij}}{\partial t}\right) d\Sigma^k(\delta)\ ;\ \ \frac{\partial g_{ij}}{\partial t} = -2R_{ij} \\ &= -2 \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \underbrace{\left(\partial_i R_{jk} - \partial_k R_{ij}\right)}_{1/r^{n+1}} \cdot \underbrace{d\Sigma^k(\delta)}_{r^{n-1}} \end{aligned}$$

The Ricci flow derivative of mass is zero:

- Preservation of asymptotic flatness (short time existence): See Oliynyk [arxiv:math/0607438]; Dai and Ma [arxiv:math/0510083].
- ADM mass: For $\delta=$ flat background metric near infinity

$$M := \lim_{r\to\infty}\int_{\Sigma_r}\delta^{ij}\left(\partial_i g_{jk}-\partial_k g_{ij}\right) d\Sigma^k(\delta)$$

$$\begin{aligned}
\frac{dM}{dt} &= \lim_{r\to\infty}\int_{\Sigma_r}\delta^{ij}\left(\partial_i\frac{\partial g_{jk}}{\partial t}-\partial_k\frac{\partial g_{ij}}{\partial t}\right) d\Sigma^k(\delta)\ ;\ \frac{\partial g_{ij}}{\partial t}=-2R_{ij}\\
&= -2\lim_{r\to\infty}\int_{\Sigma_r}\delta^{ij}\underbrace{\left(\partial_i R_{jk}-\partial_k R_{ij}\right)}_{1/r^{n+1}}\cdot\underbrace{d\Sigma^k(\delta)}_{r^{n-1}}\\
&= 0
\end{aligned}$$

The Ricci flow derivative of mass is zero:

- Preservation of asymptotic flatness (short time existence): See Oliynyk [arxiv:math/0607438]; Dai and Ma [arxiv:math/0510083].
- ADM mass: For $\delta =$ flat background metric near infinity

$$M := \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i g_{jk} - \partial_k g_{ij}\right) d\Sigma^k(\delta)$$

$$\begin{aligned}
\frac{dM}{dt} &= \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \left(\partial_i \frac{\partial g_{jk}}{\partial t} - \partial_k \frac{\partial g_{ij}}{\partial t}\right) d\Sigma^k(\delta) \; ; \; \frac{\partial g_{ij}}{\partial t} = -2R_{ij} \\
&= -2 \lim_{r\to\infty} \int_{\Sigma_r} \delta^{ij} \underbrace{\left(\partial_i R_{jk} - \partial_k R_{ij}\right)}_{1/r^{n+1}} \cdot \underbrace{d\Sigma^k(\delta)}_{r^{n-1}} \\
&= 0
\end{aligned}$$

Short talk! ...What's the problem?

Notions of Convergence

Notions of Convergence

- If $M = M_0 = const$ along the flow, $\lim_{t\to\infty} M(t) = M_0$.

- If $M = M_0 = const$ along the flow, $\lim_{t\to\infty} M(t) = M_0$.Initial data with nonzero mass cannot converge to flat space ... in a sufficiently strong topology (Dai and Ma).

Notions of Convergence

- If $M = M_0 = const$ along the flow, $\lim_{t\to\infty} M(t) = M_0$.Initial data with nonzero mass cannot converge to flat space ... in a sufficiently strong topology (Dai and Ma).
- Geometric convergence, however, requires only convergence on bounded subsets.

Notions of Convergence

- If $M = M_0 = const$ along the flow, $\lim_{t\to\infty} M(t) = M_0$.Initial data with nonzero mass cannot converge to flat space ... in a sufficiently strong topology (Dai and Ma).
- Geometric convergence, however, requires only convergence on bounded subsets.
- Analogously, in string scenario, one does not see the mass change at spatial infinity no matter how long one waits...mass change occurs at null infinity.

Notions of Convergence

- If $M = M_0 = const$ along the flow, $\lim_{t\to\infty} M(t) = M_0$.Initial data with nonzero mass cannot converge to flat space ... in a sufficiently strong topology (Dai and Ma).
- Geometric convergence, however, requires only convergence on bounded subsets.
- Analogously, in string scenario, one does not see the mass change at spatial infinity no matter how long one waits...mass change occurs at null infinity.
- Therefore, one should track the *quasilocal mass*.

Gutperle, Headrick, Minwalla, Schomerus (GHMS, 2002)

Gutperle, Headrick, Minwalla, Schomerus (GHMS, 2002)

- Expanding Ricci soliton on $\mathbb{R}^2$ given by

$$ds^2 = t\left(f^2(r)dr^2 + r^2\zeta^2 d\theta^2\right) \ , \ \zeta = const.$$

Gutperle, Headrick, Minwalla, Schomerus (GHMS, 2002)

- Expanding Ricci soliton on $\mathbb{R}^2$ given by

$$ds^2 = t\left(f^2(r)dr^2 + r^2\zeta^2 d\theta^2\right)\ ,\ \zeta = const.$$

- Changing coordinates to $\rho = r\sqrt{t}$, we get

$$ds^2 = f^2(\rho/\sqrt{t})d\rho^2 + \rho^2\zeta^2 d\theta^2\ .$$

Gutperle, Headrick, Minwalla, Schomerus (GHMS, 2002)

- Expanding Ricci soliton on $\mathbb{R}^2$ given by

$$ds^2 = t\left(f^2(r)dr^2 + r^2\zeta^2 d\theta^2\right)\ ,\ \zeta = const.$$

- Changing coordinates to $\rho = r\sqrt{t}$, we get

$$ds^2 = f^2(\rho/\sqrt{t})d\rho^2 + \rho^2\zeta^2 d\theta^2\ .$$

- $f(x)$ is given implicitly by

$$\left(\frac{1}{\zeta} - 1\right)\exp\left(\frac{1}{\zeta} - 1 - \frac{x^2}{2\alpha'}\right) = \left(\frac{1}{f(x)} - 1\right)\exp\left(\frac{1}{f(x)} - 1\right)$$

Gutperle, Headrick, Minwalla, Schomerus (GHMS, 2002)

- Expanding Ricci soliton on $\mathbb{R}^2$ given by

$$ds^2 = t\left(f^2(r)dr^2 + r^2\zeta^2 d\theta^2\right) \ , \ \zeta = const.$$

- Changing coordinates to $\rho = r\sqrt{t}$, we get

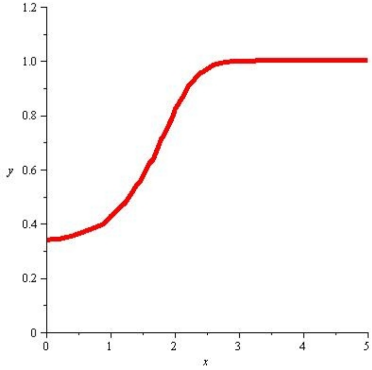
$$ds^2 = f^2(\rho/\sqrt{t})d\rho^2 + \rho^2\zeta^2 d\theta^2 \ .$$

- $f(x)$ is given implicitly by

$$\left(\frac{1}{\zeta} - 1\right)\exp\left(\frac{1}{\zeta} - 1 - \frac{x^2}{2\alpha'}\right) = \left(\frac{1}{f(x)} - 1\right)\exp\left(\frac{1}{f(x)} - 1\right)$$

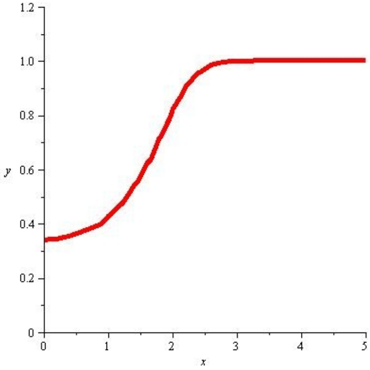
- $f(x) \to \zeta$ for $x \searrow 0$.
- $f(x) \to 1$ for $x \nearrow \infty$.
- f is monotonic on $(0, \infty)$.

Graph of $f(x)$ for $\zeta = 1/3$



$$\begin{aligned} ds^2 &= t\left(f^2(r)dr^2 + r^2\zeta^2 d\theta^2\right) \\ &= f^2(\rho/\sqrt{t})d\rho^2 + \rho^2\zeta^2 d\theta^2 \end{aligned}$$

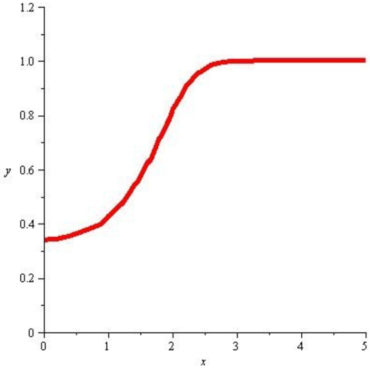
Graph of $f(x)$ for $\zeta = 1/3$



$$
\begin{aligned}
ds^2 &= t\left(f^2(r)dr^2 + r^2\zeta^2 d\theta^2\right) \\
&= f^2(\rho/\sqrt{t})d\rho^2 + \rho^2\zeta^2 d\theta^2
\end{aligned}
$$

- Asymptotic deficit angle is 2D mass (hint: think of isoperimetric deficit).

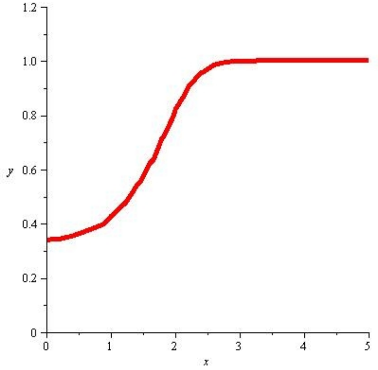
Graph of $f(x)$ for $\zeta = 1/3$



$$
\begin{aligned}
ds^2 &= t\left(f^2(r)dr^2 + r^2\zeta^2 d\theta^2\right) \\
&= f^2(\rho/\sqrt{t})d\rho^2 + \rho^2\zeta^2 d\theta^2
\end{aligned}
$$

- Asymptotic deficit angle is 2D mass (hint: think of isoperimetric deficit).
- At fixed t, the $\rho \to \infty$ limit is flat cone of deficit angle $\delta = 2\pi(1-\zeta)$.

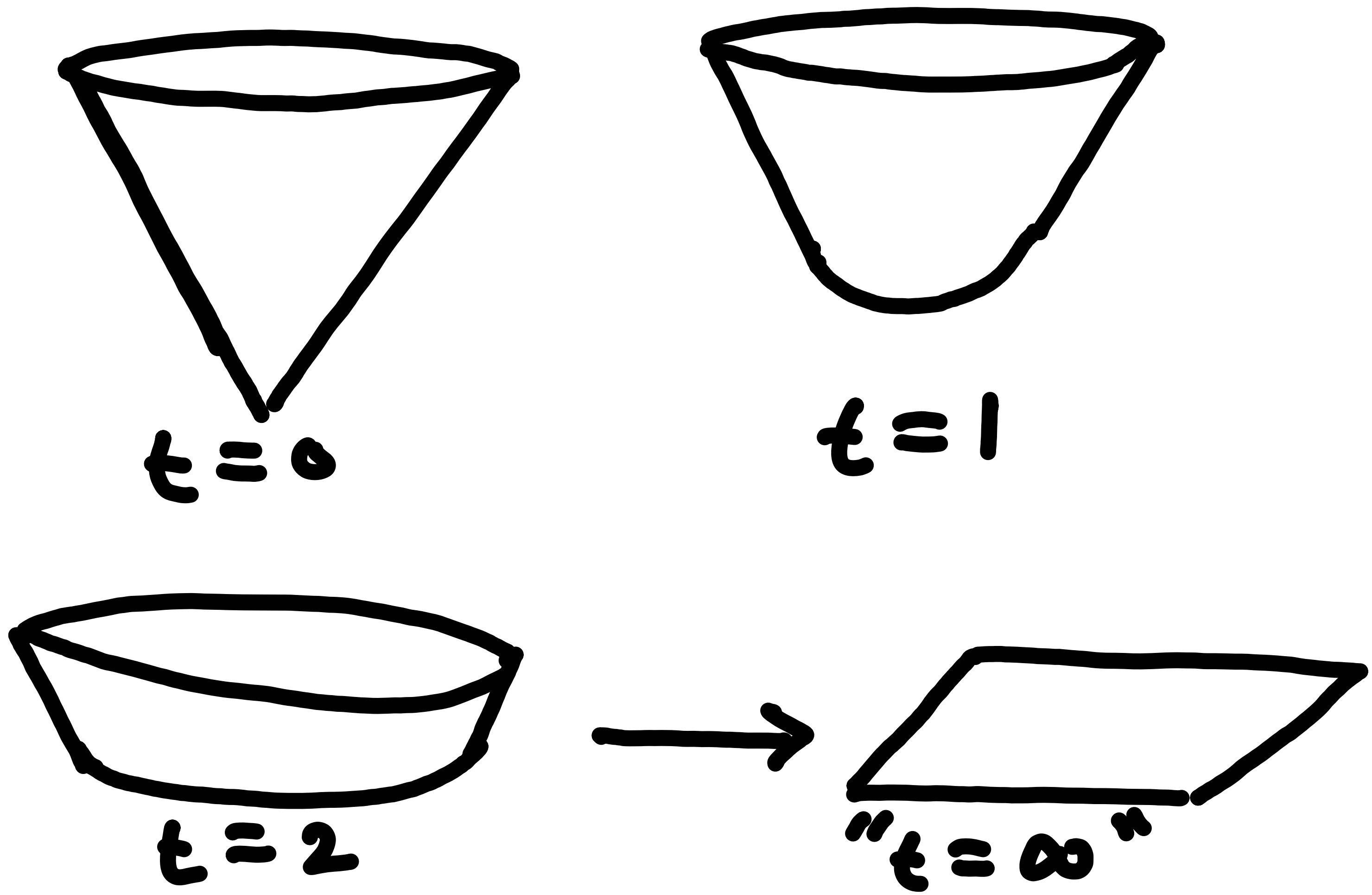
Graph of $f(x)$ for $\zeta = 1/3$



$$\begin{aligned} ds^2 &= t\left(f^2(r)dr^2 + r^2\zeta^2 d\theta^2\right) \\ &= f^2(\rho/\sqrt{t})d\rho^2 + \rho^2\zeta^2 d\theta^2 \end{aligned}$$

- Asymptotic deficit angle is 2D mass (hint: think of isoperimetric deficit).
- At fixed t, the $\rho \to \infty$ limit is flat cone of deficit angle $\delta = 2\pi(1-\zeta)$.
- At fixed ρ, the $t \to \infty$ limit is flat space.

GHMS soliton



Lessons Learned

Lessons Learned

- Mass (deficit angle) constant along flow.

Lessons Learned

- Mass (deficit angle) constant along flow.
- Geometric limit is flat space; mass jumps to zero.

Lessons Learned

- Mass (deficit angle) constant along flow.
- Geometric limit is flat space; mass jumps to zero.
- Quasi-local mass evaporates smoothly; evaporation occurs primarily on distance scales $r < \sqrt{t}$.

Lessons Learned

- Mass (deficit angle) constant along flow.
- Geometric limit is flat space; mass jumps to zero.
- Quasi-local mass evaporates smoothly; evaporation occurs primarily on distance scales $r < \sqrt{t}$.
- Agrees with qualitative "rolling tachyon" predictions in string theory.

Lessons Learned

- Mass (deficit angle) constant along flow.
- Geometric limit is flat space; mass jumps to zero.
- Quasi-local mass evaporates smoothly; evaporation occurs primarily on distance scales $r < \sqrt{t}$.
- Agrees with qualitative "rolling tachyon" predictions in string theory.

Does similar behaviour occur in higher dimensions, where mass is not a deficit angle?

Lessons Learned

- Mass (deficit angle) constant along flow.
- Geometric limit is flat space; mass jumps to zero.
- Quasi-local mass evaporates smoothly; evaporation occurs primarily on distance scales $r < \sqrt{t}$.
- Agrees with qualitative "rolling tachyon" predictions in string theory.

Does similar behaviour occur in higher dimensions, where mass is not a deficit angle?

- Long time existence for nontrivial data?

Lessons Learned

- Mass (deficit angle) constant along flow.
- Geometric limit is flat space; mass jumps to zero.
- Quasi-local mass evaporates smoothly; evaporation occurs primarily on distance scales $r < \sqrt{t}$.
- Agrees with qualitative "rolling tachyon" predictions in string theory.

Does similar behaviour occur in higher dimensions, where mass is not a deficit angle?

- Long time existence for nontrivial data? for realistic data?

Lessons Learned

- Mass (deficit angle) constant along flow.
- Geometric limit is flat space; mass jumps to zero.
- Quasi-local mass evaporates smoothly; evaporation occurs primarily on distance scales $r < \sqrt{t}$.
- Agrees with qualitative "rolling tachyon" predictions in string theory.

Does similar behaviour occur in higher dimensions, where mass is not a deficit angle?

- Long time existence for nontrivial data? for realistic data?
- Convergence to flat space for such data?

Higher dimensions

Oliynyk and Woolgar [CAG 15 (2007) 535; arxiv:math/0607438]:

Higher dimensions

Oliynyk and Woolgar [CAG 15 (2007) 535; arxiv:math/0607438]:

Assume

- Rotational symmetry.

Higher dimensions

Oliynyk and Woolgar [CAG 15 (2007) 535; arxiv:math/0607438]:

Assume

- Rotational symmetry.
- Asymptotic flatness.

Higher dimensions

Oliynyk and Woolgar [CAG 15 (2007) 535; arxiv:math/0607438]:

Assume

- Rotational symmetry.
- Asymptotic flatness.
- No minimal hyperspheres present initially.

Higher dimensions

Oliynyk and Woolgar [CAG 15 (2007) 535; arxiv:math/0607438]:

Assume

- Rotational symmetry.
- Asymptotic flatness.
- No minimal hyperspheres present initially.

Then

- No minimal hyperspheres ever form.

Higher dimensions

Oliynyk and Woolgar [CAG 15 (2007) 535; arxiv:math/0607438]:

Assume

- Rotational symmetry.
- Asymptotic flatness.
- No minimal hyperspheres present initially.

Then

- No minimal hyperspheres ever form.
- Ricci flow exists for all future time and preserves asymptotic flatness.

Higher dimensions

Oliynyk and Woolgar [CAG 15 (2007) 535; arxiv:math/0607438]:

Assume

- Rotational symmetry.
- Asymptotic flatness.
- No minimal hyperspheres present initially.

Then

- No minimal hyperspheres ever form.
- Ricci flow exists for all future time and preserves asymptotic flatness.
- Mass is constant along flow.

Higher dimensions

Oliynyk and Woolgar [CAG 15 (2007) 535; arxiv:math/0607438]:

Assume

- Rotational symmetry.
- Asymptotic flatness.
- No minimal hyperspheres present initially.

Then

- No minimal hyperspheres ever form.
- Ricci flow exists for all future time and preserves asymptotic flatness.
- Mass is constant along flow.
- Solution converges to flat space as $t \to \infty$.

Basic idea:

- Hamilton-DeTurck: $\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \mathcal{L}_X g_{ij}$; with $X = \chi(t,r)\frac{\partial}{\partial r}$.

Basic idea:

- Hamilton-DeTurck: $\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \mathcal{L}_X g_{ij}$; with $X = \chi(t,r)\frac{\partial}{\partial r}$.
- Fixed coordinates: $ds^2 = f^2(t,r)dr^2 + r^2 g(S^{n-1}, \mathrm{can})$

Basic idea:

- Hamilton-DeTurck: $\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \mathcal{L}_X g_{ij}$; with $X = \chi(t,r)\frac{\partial}{\partial r}$.
- Fixed coordinates: $ds^2 = f^2(t,r)dr^2 + r^2 g(S^{n-1}, \mathrm{can})$
- The $i = j > 1$ flow equation is a linear (algebraic) equation for $\partial\chi/\partial r$, so solve for $\partial\chi/\partial r$.

Basic idea:

- Hamilton-DeTurck: $\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \mathcal{L}_X g_{ij}$; with $X = \chi(t,r)\frac{\partial}{\partial r}$.
- Fixed coordinates: $ds^2 = f^2(t,r)dr^2 + r^2 g(S^{n-1}, \mathrm{can})$
- The $i = j > 1$ flow equation is a linear (algebraic) equation for $\partial\chi/\partial r$, so solve for $\partial\chi/\partial r$.
- Substitute result into $i = j = 1$ equation to get a single parabolic equation for f:

$$\frac{\partial f}{\partial t} = \frac{1}{f^2}\frac{\partial^2 f}{\partial r^2} - \frac{2}{f^3}\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{(n-2)}{r} - \frac{1}{rf^2}\right)\frac{\partial f}{\partial r} - \frac{(n-2)}{r^2 f}\left(f^2 - 1\right)$$

Basic idea:

- Hamilton-DeTurck: $\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \mathcal{L}_X g_{ij}$; with $X = \chi(t,r)\frac{\partial}{\partial r}$.
- Fixed coordinates: $ds^2 = f^2(t,r)dr^2 + r^2 g(S^{n-1}, \mathrm{can})$
- The $i = j > 1$ flow equation is a linear (algebraic) equation for $\partial\chi/\partial r$, so solve for $\partial\chi/\partial r$.
- Substitute result into $i = j = 1$ equation to get a single parabolic equation for f:

$$
\begin{aligned}
\frac{\partial f}{\partial t} \quad = \quad & \frac{1}{f^2}\frac{\partial^2 f}{\partial r^2} - \frac{2}{f^3}\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{(n-2)}{r} - \frac{1}{rf^2}\right)\frac{\partial f}{\partial r} \\
& -\frac{(n-2)}{r^2 f}\left(f^2 - 1\right)
\end{aligned}
$$

- Mean curvature of spheres $r = a$ is $H = \frac{n-1}{rf(t,r)} \geq const > 0 \forall t \geq 0$: No minimal spheres form, coordinates good for all $t \geq 0$.

Basic idea:

- Hamilton-DeTurck: $\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \mathcal{L}_X g_{ij}$; with $X = \chi(t,r)\frac{\partial}{\partial r}$.
- Fixed coordinates: $ds^2 = f^2(t,r)dr^2 + r^2 g(S^{n-1}, \mathrm{can})$
- The $i = j > 1$ flow equation is a linear (algebraic) equation for $\partial\chi/\partial r$, so solve for $\partial\chi/\partial r$.
- Substitute result into $i = j = 1$ equation to get a single parabolic equation for f:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{1}{f^2}\frac{\partial^2 f}{\partial r^2} - \frac{2}{f^3}\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{(n-2)}{r} - \frac{1}{rf^2}\right)\frac{\partial f}{\partial r} \\ &\quad - \frac{(n-2)}{r^2 f}\left(f^2 - 1\right)\end{aligned}$$

- Mean curvature of spheres $r = a$ is $H = \frac{n-1}{rf(t,r)} \geq const > 0 \forall t \geq 0$: No minimal spheres form, coordinates good for all $t \geq 0$.
- Indeed, easy to show that $f(t,r) \sim 1 + const/(1+t)$.

Brown-York quasilocal mass and curvature

Brown-York quasilocal mass and curvature

- f is equivalent to Brown-York quasilocal mass

$$\mu[\Sigma] := \int_{\Sigma} (H_0 - H)\, d\Sigma$$

Brown-York quasilocal mass and curvature

- f is equivalent to Brown-York quasilocal mass

$$\mu[\Sigma] := \int_{\Sigma} (H_0 - H)\, d\Sigma = c_n r^{n-2} \left(1 - \frac{1}{f(t,r)}\right)$$

$\Sigma =$ sphere of coordinate radius r, $c_n = \mathrm{vol}(\mathbb{S}^{n-1}, \mathrm{can})$.

Brown-York quasilocal mass and curvature

- f is equivalent to Brown-York quasilocal mass

$$\mu[\Sigma] := \int_\Sigma (H_0 - H)\, d\Sigma = c_n r^{n-2} \left(1 - \frac{1}{f(t,r)}\right)$$

$\Sigma =$ sphere of coordinate radius r, $c_n = \mathrm{vol}(\mathbb{S}^{n-1}, \mathrm{can})$.

- f and $\frac{\partial f}{\partial r}$ give the sectional curvatures:

Brown-York quasilocal mass and curvature

- f is equivalent to Brown-York quasilocal mass

$$\mu[\Sigma] := \int_\Sigma (H_0 - H)\, d\Sigma = c_n r^{n-2} \left(1 - \frac{1}{f(t,r)}\right)$$

$\Sigma =$ sphere of coordinate radius r, $c_n = \mathrm{vol}(\mathbb{S}^{n-1}, \mathrm{can})$.

- f and $\frac{\partial f}{\partial r}$ give the sectional curvatures:
  - $\kappa_1 = \frac{1}{rf^3}\frac{\partial f}{\partial r} = \frac{1}{c_n rf}\frac{\partial}{\partial r}\frac{\mu}{r^{n-2}}$: curvature in planes containing $\frac{\partial}{\partial r}$.

Brown-York quasilocal mass and curvature

- f is equivalent to Brown-York quasilocal mass

$$\mu[\Sigma] := \int_{\Sigma} (H_0 - H)\, d\Sigma = c_n r^{n-2} \left(1 - \frac{1}{f(t,r)}\right)$$

$\Sigma =$ sphere of coordinate radius r, $c_n = \mathrm{vol}(\mathbb{S}^{n-1}, \mathrm{can})$.

- f and $\frac{\partial f}{\partial r}$ give the sectional curvatures:
  - $\kappa_1 = \frac{1}{rf^3}\frac{\partial f}{\partial r} = \frac{1}{c_n rf}\frac{\partial}{\partial r}\frac{\mu}{r^{n-2}}$: curvature in planes containing $\frac{\partial}{\partial r}$.
  - $\kappa_2 = \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right)$: curvature in planes tangent to symmetry spheres.

Brown-York quasilocal mass and curvature

- f is equivalent to Brown-York quasilocal mass

$$\mu[\Sigma] := \int_\Sigma (H_0 - H)\, d\Sigma = c_n r^{n-2} \left(1 - \frac{1}{f(t,r)}\right)$$

$\Sigma =$ sphere of coordinate radius r, $c_n = \mathrm{vol}(\mathbb{S}^{n-1}, \mathrm{can})$.

- f and $\frac{\partial f}{\partial r}$ give the sectional curvatures:
  - $\kappa_1 = \frac{1}{rf^3}\frac{\partial f}{\partial r} = \frac{1}{c_n rf}\frac{\partial}{\partial r}\frac{\mu}{r^{n-2}}$: curvature in planes containing $\frac{\partial}{\partial r}$.
  - $\kappa_2 = \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right)$: curvature in planes tangent to symmetry spheres.
  - Bianchi: $\frac{\partial \kappa_2}{\partial r} = \frac{2}{r}(\kappa_1 - \kappa_2)$.

Brown-York quasilocal mass and curvature

- f is equivalent to Brown-York quasilocal mass

$$\mu[\Sigma] := \int_\Sigma (H_0 - H)\, d\Sigma = c_n r^{n-2} \left(1 - \frac{1}{f(t,r)}\right)$$

  $\Sigma =$ sphere of coordinate radius r, $c_n = \mathrm{vol}(\mathbb{S}^{n-1}, \mathrm{can})$.

- f and $\frac{\partial f}{\partial r}$ give the sectional curvatures:
  - $\kappa_1 = \frac{1}{rf^3}\frac{\partial f}{\partial r} = \frac{1}{c_n r f}\frac{\partial}{\partial r}\frac{\mu}{r^{n-2}}$: curvature in planes containing $\frac{\partial}{\partial r}$.
  - $\kappa_2 = \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right)$: curvature in planes tangent to symmetry spheres.
  - Bianchi: $\frac{\partial \kappa_2}{\partial r} = \frac{2}{r}\left(\kappa_1 - \kappa_2\right)$.

- PDE for f implies $f \sim 1 + \frac{const}{(1+t)}$, $t \to \infty$, so $\mu(t,r) \to 0\ \forall r$.

Brown-York quasilocal mass and curvature

- f is equivalent to Brown-York quasilocal mass

$$\mu[\Sigma] := \int_\Sigma (H_0 - H)\, d\Sigma = c_n r^{n-2}\left(1 - \frac{1}{f(t,r)}\right)$$

$\Sigma =$ sphere of coordinate radius r, $c_n = \mathrm{vol}(\mathbb{S}^{n-1}, \mathrm{can})$.

- f and $\frac{\partial f}{\partial r}$ give the sectional curvatures:
  - $\kappa_1 = \frac{1}{rf^3}\frac{\partial f}{\partial r} = \frac{1}{c_n rf}\frac{\partial}{\partial r}\frac{\mu}{r^{n-2}}$: curvature in planes containing $\frac{\partial}{\partial r}$.
  - $\kappa_2 = \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right)$: curvature in planes tangent to symmetry spheres.
  - Bianchi: $\frac{\partial \kappa_2}{\partial r} = \frac{2}{r}(\kappa_1 - \kappa_2)$.
- PDE for f implies $f \sim 1 + \frac{const}{(1+t)}$, $t \to \infty$, so $\mu(t,r) \to 0\ \forall r$.
- Likewise κ_1, κ_2 and all derivatives $\to 0$; flow converges geometrically to flat space.

Rotational symmetry with a minimal sphere

Rotational symmetry with a minimal sphere

- Garfinkle and Isenberg
  - Corseted 3-spheres, normalized Ricci flow.

Rotational symmetry with a minimal sphere

- Garfinkle and Isenberg
  - Corseted 3-spheres, normalized Ricci flow.
  - Collapse if corseting ratio < 0.17. Blow-up limit of critical case is Bryant soliton.

Rotational symmetry with a minimal sphere

- Garfinkle and Isenberg
  - Corseted 3-spheres, normalized Ricci flow.
  - Collapse if corseting ratio < 0.17. Blow-up limit of critical case is Bryant soliton.
- Husain and Seahra
  - Schwarzschild throat.

Rotational symmetry with a minimal sphere

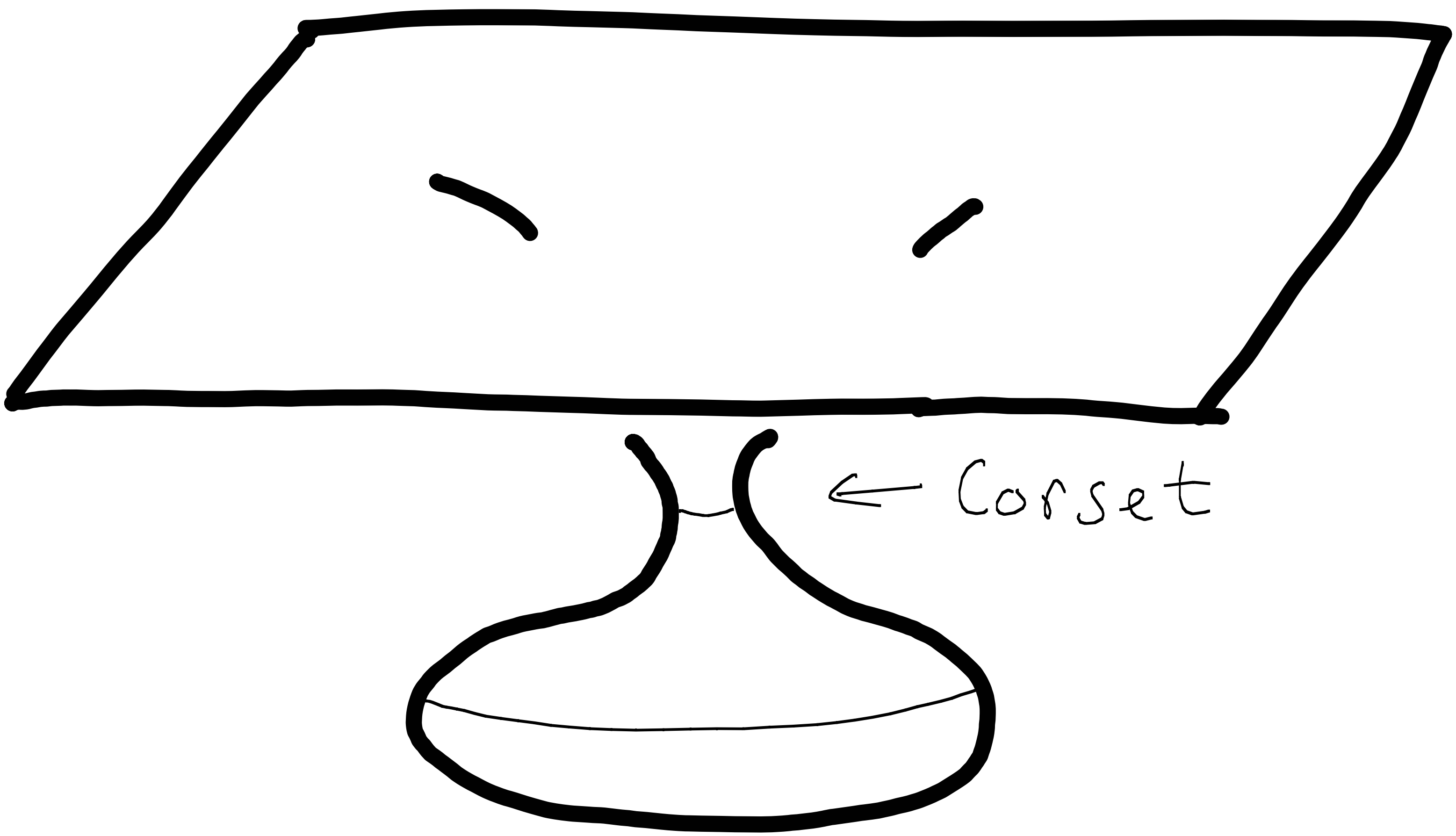
- Garfinkle and Isenberg
  - Corseted 3-spheres, normalized Ricci flow.
  - Collapse if corseting ratio < 0.17. Blow-up limit of critical case is Bryant soliton.
- Husain and Seahra
  - Schwarzschild throat.
  - Also see critical behaviour.

Rotational symmetry with a minimal sphere

- Garfinkle and Isenberg
  - Corseted 3-spheres, normalized Ricci flow.
  - Collapse if corseting ratio < 0.17. Blow-up limit of critical case is Bryant soliton.
- Husain and Seahra
  - Schwarzschild throat.
  - Also see critical behaviour.
- Mikula and Woolgar
  - Bubble nucleation.

Rotational symmetry with a minimal sphere

- Garfinkle and Isenberg
  - Corseted 3-spheres, normalized Ricci flow.
  - Collapse if corseting ratio < 0.17. Blow-up limit of critical case is Bryant soliton.
- Husain and Seahra
  - Schwarzschild throat.
  - Also see critical behaviour.
- Mikula and Woolgar
  - Bubble nucleation.
  - Preliminary: No critical behaviour: Flows tends to flat space.


← Corset

Summary

Results:

Summary

Results:

- Ricci flow preserves asymptotic flatness.

Summary

Results:

- Ricci flow preserves asymptotic flatness.
- Mass is constant.

Summary

Results:

- Ricci flow preserves asymptotic flatness.
- Mass is constant.
- Rotationally symmetric flow:

Summary

Results:

- Ricci flow preserves asymptotic flatness.
- Mass is constant.
- Rotationally symmetric flow:
  - If no initial minimal sphere is present, none forms.

Summary

Results:

- Ricci flow preserves asymptotic flatness.
- Mass is constant.
- Rotationally symmetric flow:
  - If no initial minimal sphere is present, none forms.
  - Flow then exists for all $t > 0$ and converges to flat space as $t \to \infty$.

Summary

Results:

- Ricci flow preserves asymptotic flatness.
- Mass is constant.
- Rotationally symmetric flow:
  - If no initial minimal sphere is present, none forms.
  - Flow then exists for all $t > 0$ and converges to flat space as $t \to \infty$.
  - String theory picture qualitatively confirmed.

Summary

Results:

- Ricci flow preserves asymptotic flatness.
- Mass is constant.
- Rotationally symmetric flow:
  - If no initial minimal sphere is present, none forms.
  - Flow then exists for all $t > 0$ and converges to flat space as $t \to \infty$.
  - String theory picture qualitatively confirmed.
  - If minimal sphere is present initially, numerical work suggests it evaporates.

Summary

Results:

- Ricci flow preserves asymptotic flatness.
- Mass is constant.
- Rotationally symmetric flow:
  - If no initial minimal sphere is present, none forms.
  - Flow then exists for all $t > 0$ and converges to flat space as $t \to \infty$.
  - String theory picture qualitatively confirmed.
  - If minimal sphere is present initially, numerical work suggests it evaporates.

Questions:

Summary

Results:

- Ricci flow preserves asymptotic flatness.
- Mass is constant.
- Rotationally symmetric flow:
  - If no initial minimal sphere is present, none forms.
  - Flow then exists for all $t > 0$ and converges to flat space as $t \to \infty$.
  - String theory picture qualitatively confirmed.
  - If minimal sphere is present initially, numerical work suggests it evaporates.

Questions:

- Can we perturb about rotational symmetry?

Summary

Results:

- Ricci flow preserves asymptotic flatness.
- Mass is constant.
- Rotationally symmetric flow:
  - If no initial minimal sphere is present, none forms.
  - Flow then exists for all $t > 0$ and converges to flat space as $t \to \infty$.
  - String theory picture qualitatively confirmed.
  - If minimal sphere is present initially, numerical work suggests it evaporates.

Questions:

- Can we perturb about rotational symmetry?
- What other highly symmetric cases are physically interesting, and can we find similar results in such cases?

Bartnik's Quasi-Local Mass

[1] R Bartnik, Tsing Hua Lectures on Geometry and Analysis (Int Press 1995) p 1–27

Special case—Static spacetime:

[1] R Bartnik, Tsing Hua Lectures on Geometry and Analysis (Int Press 1995) p 1–27

Bartnik's Quasi-Local Mass

Special case—Static spacetime:

- "Blob of matter": Bounded region (B, G) with $R_G \geq 0$ and boundary (dB, h).

[1] R Bartnik, Tsing Hua Lectures on Geometry and Analysis (Int Press 1995) p 1–27

Bartnik's Quasi-Local Mass

Special case—Static spacetime:

- "Blob of matter": Bounded region (B, G) with $R_G \geq 0$ and boundary (dB, h).
- Consider an asymptotically flat manifold (M, g) that has (i) $R_g \geq 0$ and (ii) inner boundary dM, such that the induced boundary metric and mean curvature matches that of $dB \subset B$.

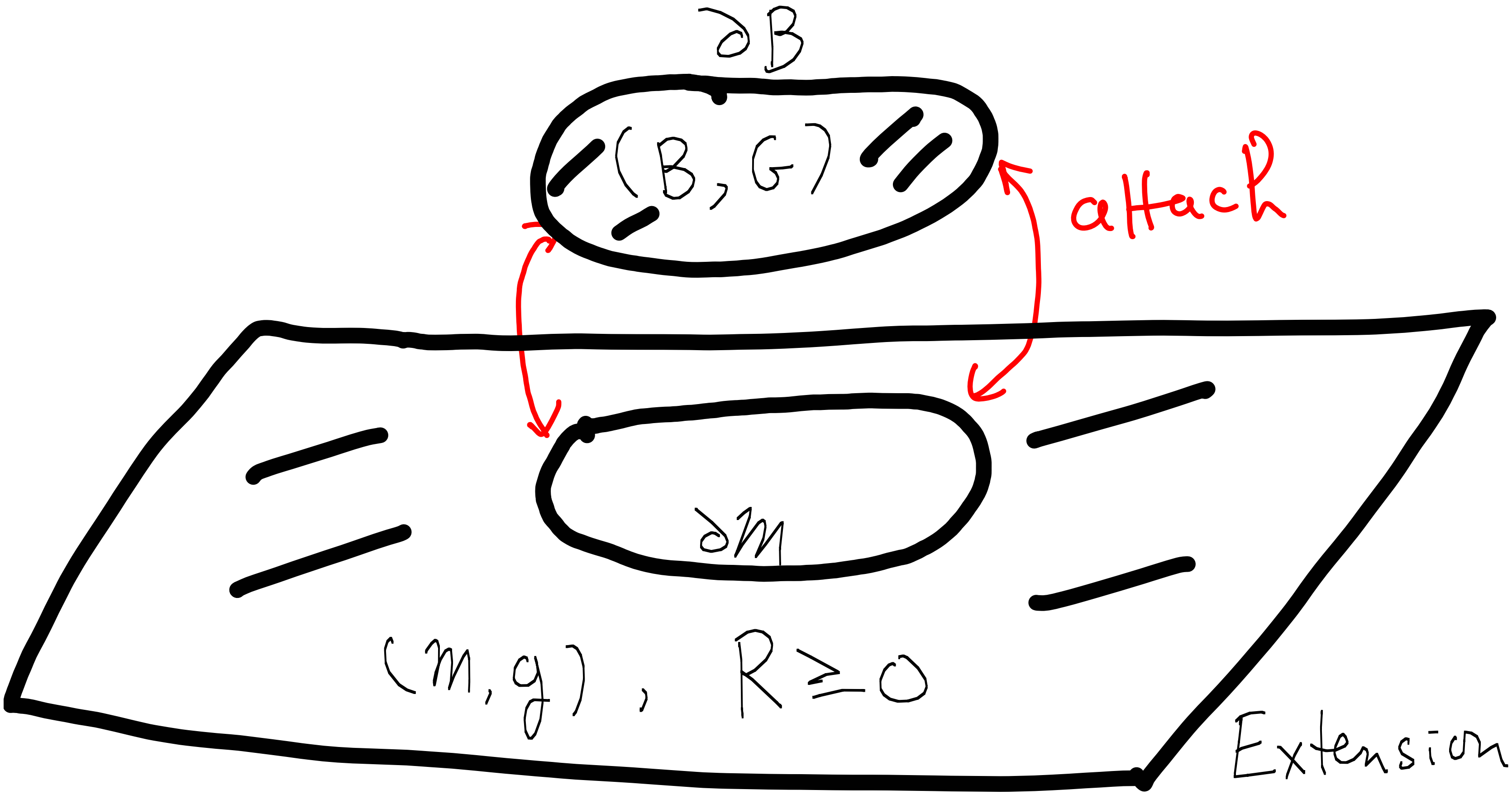
[1] R Bartnik, Tsing Hua Lectures on Geometry and Analysis (Int Press 1995) p 1–27

Bartnik's Quasi-Local Mass

Special case—Static spacetime:

- "Blob of matter": Bounded region (B, G) with $R_G \geq 0$ and boundary (dB, h).
- Consider an asymptotically flat manifold (M, g) that has (i) $R_g \geq 0$ and (ii) inner boundary dM, such that the induced boundary metric and mean curvature matches that of $dB \subset B$.
- Glue dB to dM. Call (M, g) an *extension* of B.

[1] R Bartnik, Tsing Hua Lectures on Geometry and Analysis (Int Press 1995) p 1–27


∂B
(B, G)
attach
∂M
(M, g), $R \geq 0$
Extension

Bartnik's Quasi-Local Mass

Special case—Static spacetime:

- "Blob of matter": Bounded region (B, G) with $R_G \geq 0$ and boundary (dB, h).
- Consider an asymptotically flat manifold (M, g) that has (i) $R_g \geq 0$ and (ii) inner boundary dM, such that the induced boundary metric and mean curvature matches that of $dB \subset B$.
- Glue dB to dM. Call (M, g) an *extension* of B.
- Compute ADM mass of (M, g).

[1] R Bartnik, Tsing Hua Lectures on Geometry and Analysis (Int Press 1995) p 1–27

Bartnik's Quasi-Local Mass

Special case—Static spacetime:

- "Blob of matter": Bounded region (B, G) with $R_G \geq 0$ and boundary (dB, h).
- Consider an asymptotically flat manifold (M, g) that has (i) $R_g \geq 0$ and (ii) inner boundary dM, such that the induced boundary metric and mean curvature matches that of $dB \subset B$.
- Glue dB to dM. Call (M, g) an *extension* of B.
- Compute ADM mass of (M, g).
- Infimum of the ADM mass, taken over all such extensions, is the Bartnik mass of (B, G).[1]

[1] R Bartnik, Tsing Hua Lectures on Geometry and Analysis (Int Press 1995) p 1–27

Bartnik's Quasi-Local Mass

Special case—Static spacetime:

- "Blob of matter": Bounded region (B, G) with $R_G \geq 0$ and boundary (dB, h).
- Consider an asymptotically flat manifold (M, g) that has (i) $R_g \geq 0$ and (ii) inner boundary dM, such that the induced boundary metric and mean curvature matches that of $dB \subset B$.
- Glue dB to dM. Call (M, g) an *extension* of B.
- Compute ADM mass of (M, g).
- Infimum of the ADM mass, taken over all such extensions, is the Bartnik mass of (B, G).[1]

Question: Is the infimum ever nonzero?

[1] R Bartnik, Tsing Hua Lectures on Geometry and Analysis (Int Press 1995) p 1–27

Bartnik's Quasi-Local Mass

Special case—Static spacetime:

- "Blob of matter": Bounded region (B, G) with $R_G \geq 0$ and boundary (dB, h).
- Consider an asymptotically flat manifold (M, g) that has (i) $R_g \geq 0$ and (ii) inner boundary dM, such that the induced boundary metric and mean curvature matches that of $dB \subset B$.
- Glue dB to dM. Call (M, g) an *extension* of B.
- Compute ADM mass of (M, g).
- Infimum of the ADM mass, taken over all such extensions, is the Bartnik mass of (B, G).[1]

Question: Is the infimum ever nonzero?

Conjecture: The infimum is realized as the mass of a solution of the static Einstein equations.

[1] R Bartnik, Tsing Hua Lectures on Geometry and Analysis (Int Press 1995) p 1–27

Static metrics

Static metrics

Static Einstein Equations:

$$\begin{aligned} ds^2 &= -e^{2u} d\tau^2 + e^{\frac{2u}{n-2}} g_{ij} dx^i dx^j \\ R_{ij} &= \left(\frac{n-1}{n-2}\right) \nabla_i u \nabla_j u \end{aligned}$$

Static metrics

Static Einstein Equations:

$$\begin{aligned} ds^2 &= -e^{2u} d\tau^2 + e^{\frac{2u}{n-2}} g_{ij} dx^i dx^j \\ R_{ij} &= \left(\frac{n-1}{n-2}\right) \nabla_i u \nabla_j u \\ \Rightarrow \Delta u &= 0 \end{aligned}$$

Static metrics

Static Einstein Equations:

$$
\begin{aligned}
ds^2 &= -e^{2u} d\tau^2 + e^{\frac{2u}{n-2}} g_{ij} dx^i dx^j \\
R_{ij} &= \left(\frac{n-1}{n-2}\right) \nabla_i u \nabla_j u \\
\Rightarrow \Delta u &= 0
\end{aligned}
$$

Flow of Static Metrics (B List; PhD thesis under G Huisken):

$$
\begin{aligned}
\frac{\partial g_{ij}}{\partial \lambda} &= -2\left(R_{ij} - k_n^2 \nabla_i u \nabla_j u\right) \\
\frac{\partial u}{\partial \lambda} &= \Delta u
\end{aligned}
$$

Static metrics

Static Einstein Equations:

$$\begin{aligned} ds^2 &= -e^{2u}d\tau^2 + e^{\frac{2u}{n-2}}g_{ij}dx^i dx^j \\ R_{ij} &= \left(\frac{n-1}{n-2}\right)\nabla_i u \nabla_j u \\ \Rightarrow \Delta u &= 0 \end{aligned}$$

Flow of Static Metrics (B List; PhD thesis under G Huisken):

$$\begin{aligned} \frac{\partial g_{ij}}{\partial \lambda} &= -2\left(R_{ij} - k_n^2 \nabla_i u \nabla_j u\right) \\ \frac{\partial u}{\partial \lambda} &= \Delta u \end{aligned}$$

Flow parameter is λ.

Static metrics

Static Einstein Equations:

$$\begin{aligned} ds^2 &= -e^{2u}d\tau^2 + e^{\frac{2u}{n-2}}g_{ij}dx^i dx^j \\ R_{ij} &= \left(\frac{n-1}{n-2}\right)\nabla_i u \nabla_j u \\ \Rightarrow \Delta u &= 0 \end{aligned}$$

Flow of Static Metrics (B List; PhD thesis under G Huisken):

$$\begin{aligned} \frac{\partial g_{ij}}{\partial \lambda} &= -2\left(R_{ij} - k_n^2 \nabla_i u \nabla_j u\right) \\ \frac{\partial u}{\partial \lambda} &= \Delta u \end{aligned}$$

Flow parameter is λ.Take $k_n^2 = \frac{n-1}{n-2}$ if $n \neq 2$, so in this case fixed points give static Ricci-flat $(n+1)$-metrics.

Static metrics

Static Einstein Equations:

$$\begin{aligned} ds^2 &= -e^{2u}d\tau^2 + e^{\frac{2u}{n-2}} g_{ij}dx^i dx^j \\ R_{ij} &= \left(\frac{n-1}{n-2}\right)\nabla_i u \nabla_j u \\ \Rightarrow \Delta u &= 0 \end{aligned}$$

Flow of Static Metrics (B List; PhD thesis under G Huisken):

$$\begin{aligned} \frac{\partial g_{ij}}{\partial \lambda} &= -2\left(R_{ij} - k_n^2 \nabla_i u \nabla_j u\right) \\ \frac{\partial u}{\partial \lambda} &= \Delta u \end{aligned}$$

Flow parameter is λ.Take $k_n^2 = \frac{n-1}{n-2}$ if $n \neq 2$, so in this case fixed points give static Ricci-flat $(n+1)$-metrics. Flow is defined for $n = 2$ as well, since k_n is arbitrary.

Static Metric Flow is Ricci Flow (+ Diffeo)

Static Metric Flow is Ricci Flow (+ Diffeo)

- Begin with metric

$$dS^2 = G_{\mu\nu} dx^\mu dx^\nu = e^{2k_n u} d\tau^2 + g_{ij} dx^i dx^j$$

with

$$\frac{\partial u}{\partial \tau} = 0 \; , \; \frac{\partial g_{ij}}{\partial \tau} = 0.$$

Static Metric Flow is Ricci Flow (+ Diffeo)

- Begin with metric

$$dS^2 = G_{\mu\nu} dx^\mu dx^\nu = e^{2k_n u} d\tau^2 + g_{ij} dx^i dx^j$$

  with

$$\frac{\partial u}{\partial \tau} = 0 \ , \ \frac{\partial g_{ij}}{\partial \tau} = 0.$$

- Define vector field

$$X := -k_n \nabla^i u \frac{\partial}{\partial x^i}.$$

Static Metric Flow is Ricci Flow (+ Diffeo)

- Begin with metric

$$dS^2 = G_{\mu\nu} dx^\mu dx^\nu = e^{2k_n u} d\tau^2 + g_{ij} dx^i dx^j$$

  with

$$\frac{\partial u}{\partial \tau} = 0 \ , \ \frac{\partial g_{ij}}{\partial \tau} = 0.$$

- Define vector field

$$X := -k_n \nabla^i u \frac{\partial}{\partial x^i}.$$

- Apply Hamilton-DeTurck flow

$$\frac{\partial G_{\mu\nu}}{\partial \lambda} = -2R^G_{\mu\nu} + \mathcal{L}_X G_{\mu\nu}.$$

Static Metric Flow is Ricci Flow (+ Diffeo)

- Begin with metric

$$dS^2 = G_{\mu\nu} dx^\mu dx^\nu = e^{2k_n u} d\tau^2 + g_{ij} dx^i dx^j$$

with

$$\frac{\partial u}{\partial \tau} = 0 \ , \ \frac{\partial g_{ij}}{\partial \tau} = 0.$$

- Define vector field

$$X := -k_n \nabla^i u \frac{\partial}{\partial x^i}.$$

- Apply Hamilton-DeTurck flow

$$\frac{\partial G_{\mu\nu}}{\partial \lambda} = -2R^G_{\mu\nu} + \mathcal{L}_X G_{\mu\nu}.$$

- This yields the static metric flow equations of the last slide.

Rotationally Symmetric Case

Rotationally Symmetric Case

- What are the rotationally symmetric solutions of List's flow?

Rotationally Symmetric Case

- What are the rotationally symmetric solutions of List's flow?
- Special case: no minimal spheres:

$$ds^2 = f^2(\lambda, r)dr^2 + r^2 d\Omega^2 \ , \ u = u(\lambda, r) \ .$$

Rotationally Symmetric Case

- What are the rotationally symmetric solutions of List's flow?
- Special case: no minimal spheres:

$$ds^2 = f^2(\lambda, r)dr^2 + r^2 d\Omega^2 \ , \ u = u(\lambda, r) \ .$$

- Above condition implies manifold is noncompact, so assume asymptotic flatness.

Rotationally Symmetric Case

- What are the rotationally symmetric solutions of List's flow?
- Special case: no minimal spheres:

$$ds^2 = f^2(\lambda, r)dr^2 + r^2 d\Omega^2 \ , \ u = u(\lambda, r) \ .$$

- Above condition implies manifold is noncompact, so assume asymptotic flatness.
- Assume complete manifold with no inner boundary: limiting case of Bartnik's problem.

Problem:

Say $(M, g(0))$ is asymptotically flat. Are there solutions $(M, g(t), u(t))$ of

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{1}{f^2}\frac{\partial^2 f}{\partial r^2} - \frac{2}{f^3}\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{(n-2)}{r} - \frac{1}{rf^2}\right)\frac{\partial f}{\partial r} \\
&\quad - \frac{(n-2)}{r^2 f}\left(f^2 - 1\right) + \frac{k_n^2}{f}\left(\frac{\partial u}{\partial r}\right)^2 \\
\frac{\partial u}{\partial t} &= \Delta u + \mathcal{L}_X u \\
&= \frac{1}{f^2}\frac{\partial^2 u}{\partial r^2} + \left[\frac{1}{rf^2} - \frac{1}{f^3}\frac{\partial f}{\partial r} + \frac{n-2}{r}\right]\frac{\partial u}{\partial r}
\end{aligned}
$$

for $t \in [0, \infty)$ and, if so, does $(M, g(t))$ converge geometrically to some (M_∞, g_∞), and does $u(t)$ converge?

Long time existence?

Long time existence?

- Bounds on $|\nabla u|$:

Long time existence?

- Bounds on $|\nabla u|$:
  - List: $|\nabla u|$ decays at least as $1/\sqrt{t}$.

Long time existence?

- Bounds on $|\nabla u|$:
  - List: $|\nabla u|$ decays at least as $1/\sqrt{t}$.
  - u stays smooth at origin: $\frac{1}{r}|\nabla u| < const$.

Long time existence?

- Bounds on $|\nabla u|$:
  - List: $|\nabla u|$ decays at least as $1/\sqrt{t}$.
  - u stays smooth at origin: $\frac{1}{r}|\nabla u| < const$.
- Bounds on f:

Long time existence?

- Bounds on $|\nabla u|$:
  - List: $|\nabla u|$ decays at least as $1/\sqrt{t}$.
  - u stays smooth at origin: $\frac{1}{r}|\nabla u| < const$.
- Bounds on f:
  - Bound below: $f(t,r) \geq const$ depending only on initial data $f(0,r)$ and, for $n = 2$, the limit of f as $r \to \infty$.

Long time existence?

- Bounds on $|\nabla u|$:
  - List: $|\nabla u|$ decays at least as $1/\sqrt{t}$.
  - u stays smooth at origin: $\frac{1}{r}|\nabla u| < const$.
- Bounds on f:
  - Bound below: $f(t,r) \geq const$ depending only on initial data $f(0,r)$ and, for $n=2$, the limit of f as $r \to \infty$.
  - Bound above: $f(t,r) < const \cdot (1+t)^{p/2}$, $p > 0$ depends on $|\nabla u|_{(0,r)}$. In particular, $f < e^t$.

Long time existence?

- Bounds on $|\nabla u|$:
  - List: $|\nabla u|$ decays at least as $1/\sqrt{t}$.
  - u stays smooth at origin: $\frac{1}{r}|\nabla u| < const$.
- Bounds on f:
  - Bound below: $f(t,r) \geq const$ depending only on initial data $f(0,r)$ and, for $n = 2$, the limit of f as $r \to \infty$.
  - Bound above: $f(t,r) < const \cdot (1+t)^{p/2}$, $p > 0$ depends on $|\nabla u|_{(0,r)}$. In particular, $f < e^t$.
- Thus expect existence for all $t > 0$, but not always convergence on $\mathbb{R}^n$.

Long time existence?

- Bounds on $|\nabla u|$:
  - List: $|\nabla u|$ decays at least as $1/\sqrt{t}$.
  - u stays smooth at origin: $\frac{1}{r}|\nabla u| < const$.
- Bounds on f:
  - Bound below: $f(t,r) \geq const$ depending only on initial data $f(0,r)$ and, for $n = 2$, the limit of f as $r \to \infty$.
  - Bound above: $f(t,r) < const \cdot (1+t)^{p/2}$, $p > 0$ depends on $|\nabla u|_{(0,r)}$. In particular, $f < e^t$.
- Thus expect existence for all $t > 0$, but not always convergence on $\mathbb{R}^n$.

Curvature bounds

Curvature bounds

Theorem (List): Say solution exists for $t < T$. If solution fails to exist at T, then $\lim_{t \nearrow T} \sup_{x \in M} |\mathrm{Riem}(t,x)| \to \infty$.

Curvature bounds

Theorem (List): Say solution exists for $t < T$. If solution fails to exist at T, then $\lim_{t\nearrow T} \sup_{x\in M} |\mathrm{Riem}(t,x)| \to \infty$.

- $n > 2$: Rotational symmetry: Only two sectional curvatures:

$$\kappa_1 = \frac{1}{rf^3}\frac{\partial f}{\partial r}\ ,\ \kappa_2 = \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right)\ ,$$
$$\frac{\partial \kappa_2}{\partial r} = \frac{2}{r}\left(\kappa_1 - \kappa_2\right)\ ,\ \mathrm{Weyl} = 0.$$

Curvature bounds

Theorem (List): Say solution exists for $t < T$. If solution fails to exist at T, then $\lim_{t \nearrow T} \sup_{x \in M} |\mathrm{Riem}(t,x)| \to \infty$.

- $n > 2$: Rotational symmetry: Only two sectional curvatures:

$$\kappa_1 = \frac{1}{rf^3}\frac{\partial f}{\partial r} \ , \ \kappa_2 = \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right) \ ,$$
$$\frac{\partial \kappa_2}{\partial r} = \frac{2}{r}\left(\kappa_1 - \kappa_2\right) \ , \ \mathrm{Weyl} = 0.$$

- $\kappa_2 > -const/(1+t)$.

Curvature bounds

Theorem (List): Say solution exists for $t < T$. If solution fails to exist at T, then $\lim_{t \nearrow T} \sup_{x \in M} |\mathrm{Riem}(t,x)| \to \infty$.

- $n > 2$: Rotational symmetry: Only two sectional curvatures:

$$\kappa_1 = \frac{1}{rf^3}\frac{\partial f}{\partial r}\ ,\ \kappa_2 = \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right)\ ,$$
$$\frac{\partial \kappa_2}{\partial r} = \frac{2}{r}\left(\kappa_1 - \kappa_2\right)\ ,\ \mathrm{Weyl} = 0.$$

- $\kappa_2 > -const/(1+t)$.
- $n = 2$: Curvature reduces to scalar curvature R.

Curvature bounds

Theorem (List): Say solution exists for $t < T$. If solution fails to exist at T, then $\lim_{t \nearrow T} \sup_{x \in M} |\mathrm{Riem}(t,x)| \to \infty$.

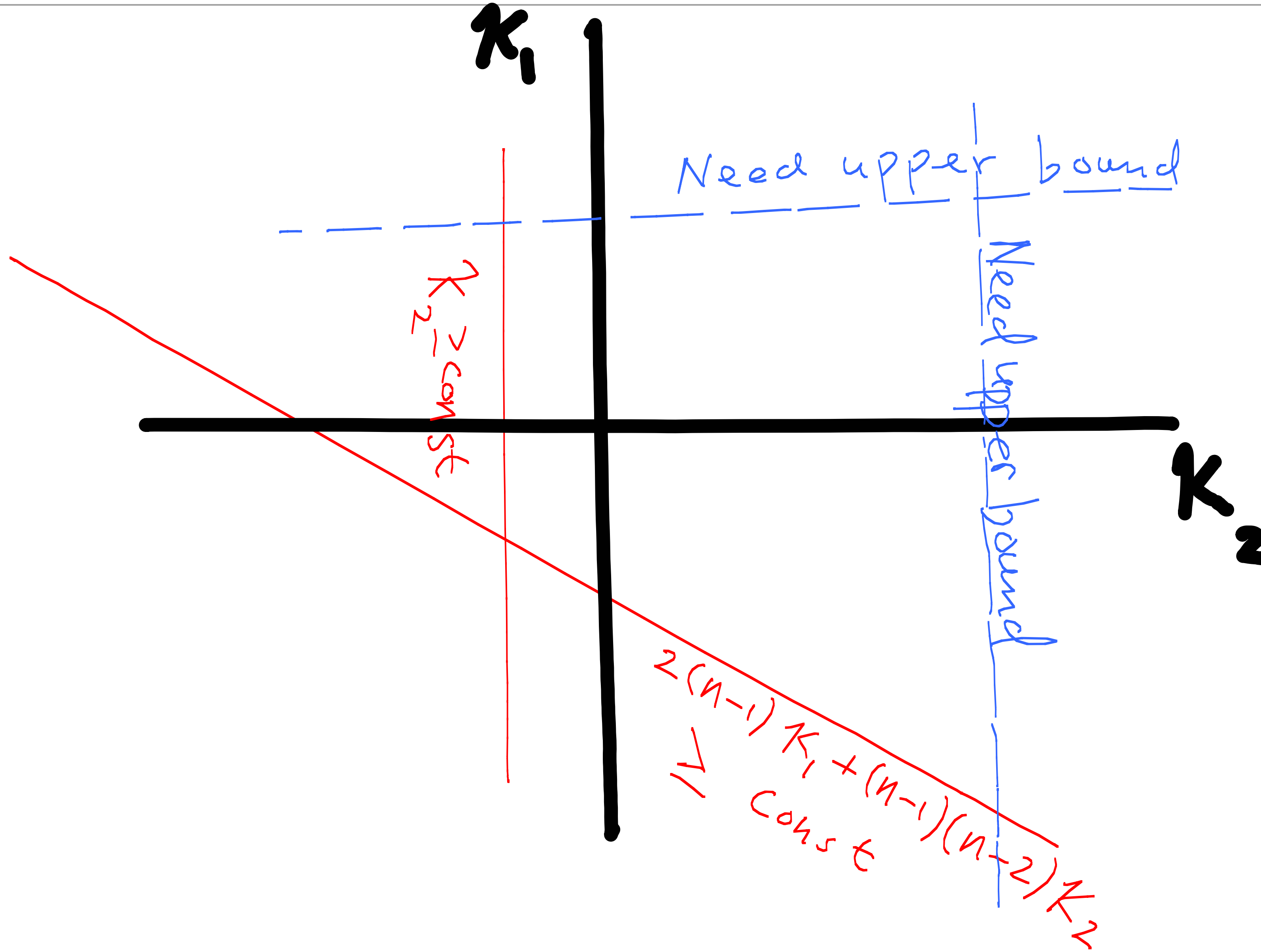
- $n > 2$: Rotational symmetry: Only two sectional curvatures:

$$\kappa_1 = \frac{1}{rf^3}\frac{\partial f}{\partial r}\ ,\ \kappa_2 = \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right)\ ,$$
$$\frac{\partial \kappa_2}{\partial r} = \frac{2}{r}\left(\kappa_1 - \kappa_2\right)\ ,\ \mathrm{Weyl} = 0.$$

- $\kappa_2 > -const/(1+t)$.
- $n = 2$: Curvature reduces to scalar curvature R.
- Either case: R is bounded below, tends to positive.

$$R \equiv 2(n-1)\kappa_1 + (n-1)(n-2)\kappa_2 \geq -const/(1+t).$$

Note Title



K_1
K_2
Need upper bound
Need upper bound
$K_2 \geq$ const
$2(n-1)K_1 + (n-1)(n-2)K_2$
$\geq$ const

Flow of stationary metrics

Flow of stationary metrics

Hamilton-DeTurck flow of metric

$$ds^2 = -e^{2\sqrt{\frac{n-1}{n-2}}u}\left(dt + A_i dx^i\right)^2 + g_{ij}dx^i dx^j.$$

Flow of stationary metrics

Hamilton-DeTurck flow of metric

$$ds^2 = -e^{2\sqrt{\frac{n-1}{n-2}}u}\left(dt + A_i dx^i\right)^2 + g_{ij}dx^i dx^j.$$

Following List, pull back by vector field

$$X := -\sqrt{\frac{n-1}{n-2}}\nabla u.$$

Flow of stationary metrics

Hamilton-DeTurck flow of metric

$$ds^2 = -e^{2\sqrt{\frac{n-1}{n-2}}u}\left(dt + A_i dx^i\right)^2 + g_{ij}dx^i dx^j.$$

Following List, pull back by vector field

$$X := -\sqrt{\frac{n-1}{n-2}}\nabla u.$$

Also convenient to apply gauge transformation to A.

Flow of stationary metrics

Hamilton-DeTurck flow of metric

$$ds^2 = -e^{2\sqrt{\frac{n-1}{n-2}}u}\left(dt + A_i dx^i\right)^2 + g_{ij}dx^i dx^j.$$

Following List, pull back by vector field

$$X := -\sqrt{\frac{n-1}{n-2}}\nabla u.$$

Also convenient to apply gauge transformation to A. Get:

$$\begin{aligned}
\frac{\partial u}{\partial \lambda} &= \Delta u + \frac{1}{4}\sqrt{\frac{n-2}{n-1}}e^{2\sqrt{\frac{n-1}{n-2}}u}|F|^2 \ ,\\
\frac{\partial B_i}{\partial \lambda} &= -\nabla^j F_{ij} - 2\sqrt{\frac{n-1}{n-2}}F_{ij}\nabla^j u \ ,\\
\frac{\partial g_{ij}}{\partial \lambda} &= -2R_{ij} + 2\frac{n-2}{n-2}\nabla_i u \nabla_j u - e^{2\sqrt{\frac{n-1}{n-2}}u}g^{kl}F_{ik}F_{jl} \ ,\\
F_{ij}[A] &:= \nabla_i A_j - \nabla_j A_i \ .
\end{aligned}$$

Alternative flow for stationary metrics

Alternative flow for stationary metrics

- Fixed points of previous system do not obey the stationary metric equations.

Alternative flow for stationary metrics

- Fixed points of previous system do not obey the stationary metric equations.
- Propose instead the new flow

$$
\begin{aligned}
\frac{\partial u}{\partial \lambda} &= \Delta u + \frac{1}{4} e^{2\left(\frac{n-1}{n-2}\right)u} |F|^2 \ , \\
\frac{\partial B_i}{\partial \lambda} &= -\nabla^j F_{ij} - 2\left(\frac{n-1}{n-2}\right) F_{ij} \nabla^j u \ , \\
\frac{\partial g_{ij}}{\partial \lambda} &= -2R_{ij} + 2\left(\frac{n-1}{n-2}\right) \nabla_i u \nabla_j u \\
&\quad - e^{2\left(\frac{n-1}{n-2}\right)u} \left[F_{ik} F_j{}^k - \frac{1}{2(n-2)} g_{ij} |F|^2 \right] \ .
\end{aligned}
$$

Alternative flow for stationary metrics

- Fixed points of previous system do not obey the stationary metric equations.
- Propose instead the new flow

$$
\begin{aligned}
\frac{\partial u}{\partial \lambda} &= \Delta u + \frac{1}{4} e^{2\left(\frac{n-1}{n-2}\right)u} |F|^2 \ , \\
\frac{\partial B_i}{\partial \lambda} &= -\nabla^j F_{ij} - 2\left(\frac{n-1}{n-2}\right) F_{ij} \nabla^j u \ , \\
\frac{\partial g_{ij}}{\partial \lambda} &= -2R_{ij} + 2\left(\frac{n-1}{n-2}\right) \nabla_i u \nabla_j u \\
&\quad - e^{2\left(\frac{n-1}{n-2}\right)u} \left[F_{ik} F_j{}^k - \frac{1}{2(n-2)} g_{ij} |F|^2 \right] \ .
\end{aligned}
$$

- Fixed points correspond to stationary metrics.